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On Some Connections Between Suborbital Graphs and Special Sequences

Ümmügülsün Akbaba[a], Ali Hikmet Değer[a,*], Tuğba Tuylu[a]

[a] *Department of Mathematics, Faculty of Karadeniz Technical University, 61080, Trabzon, Turkey.*

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Abstract. In this work, we used the terms of identity alternate sequence and also the even terms of alternate sequences of Fibonacci and Lucas, the famous number sequences, to establish connections with the special vertex values of the paths of minimal length in the suborbital graphs. These types of vertices also give rise to special continued fractions, hence from recurrence relations for continued fractions, values of these vertices and values of these special sequences were associated.

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1. Introduction

In 1967, Sims described the suborbital graphs by the charecteristics of a permutation group action on a set [10]. In 1991, Jones, Singerman and Wicks worked on the special suborbital graphs $\mathbf{F}_{u,N}$ by using Sims's ideas [7]. They used the modular group Γ and it's congruence subgroup $\Gamma_0(N)$. The action is on the extended rational set $\hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$.

These suborbital graphs are Γ-invariant directed graphs with vertex set $\hat{\mathbb{Q}}$ and their edge-sets being the orbits of Γ on the cartesian square $\hat{\mathbb{Q}}^2$. For details see [1, 3, 7].

The simplest example of such a graph is $\mathbf{F}_{1,1}$, known as Farey graph shown in the Figure 1. The graph has extended rational set $\hat{\mathbb{Q}}$ as its vertex set; it is self-paired, so we can regard it as an undirected graph. Also vertices r/s and x/y are adjacent if and only if $ry - sx = \pm 1$; for instance, the vertices adjacent to ∞ are the integers.

For the suborbital graph $\mathbf{F}_{u,N}$ we can give edge conditions as above Theorem:

Theorem 1.1. $\frac{r}{s} \to \frac{x}{y} \in \mathbf{F}_{u,N}$ *if and only if* $x \equiv \mp ur\,(mod\,N)$, $ry - sx = \mp N$.

So, there is an obvious edge $\infty = \frac{1}{0} \to \frac{u}{N} \in \mathbf{F}_{u,N}$ by this Theorem.

Thus, with this edge conditions, the following question comes to mind as "What is the farthest vertex, where the vertex u/N can be connected and create an edge in the graph $\mathbf{F}_{u,N}$ for both direction right and left?"

Answer of this problem was given in [4]. See Figure 2.

It turns out that these graphs give rise to a special continued fraction which is a special case of very famous fraction coming out from the following Pringsheim's theorem.

*Corresponding Author
Email addresses: ummugulsun.akbaba@ktu.edu.tr (Ü. Akbaba), ahikmetd@ktu.edu.tr (A.H. Değer), tugba.tuylu.61@gmail.com (T. Tuylu)

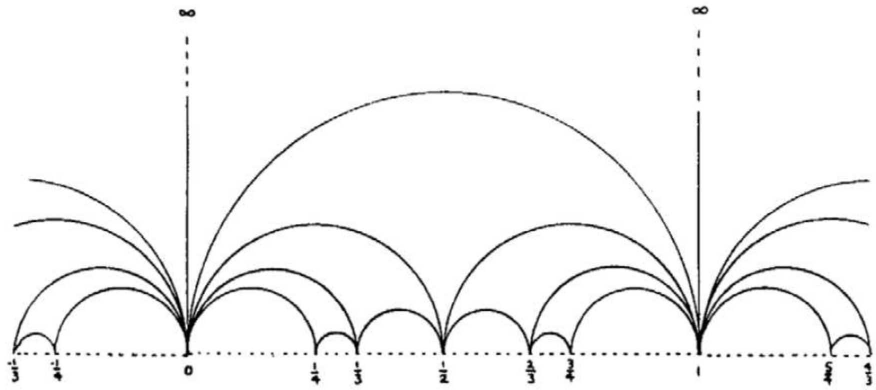

Figure 1. Farey graph $\mathbf{F}_{1,1}$

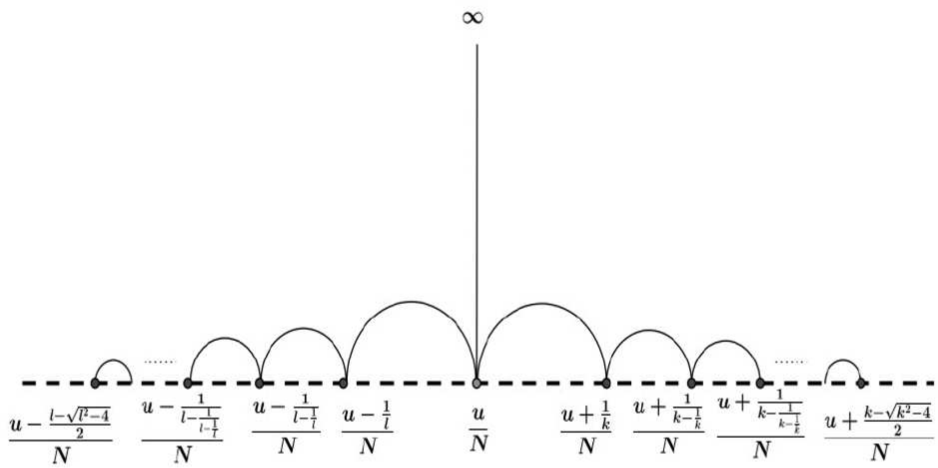

Figure 2. The farthest vertices on the path of minimal length in the suborbital graph $\mathbf{F}_{u,N}$

Theorem 1.2. *If $|b_m| \geq 1 + |a_m|$ for all m, the continued fraction* $\cfrac{1}{k-\cfrac{1}{k-\cfrac{1}{k-\ddots}}}$ *convergent to some value v with $|v| \leq 1$.*

In previous works, suborbital graphs for congruence subgroup $\Gamma_0(N)$ of the modular group Γ to have vertices of the graph $\mathbf{F}_{u,N}$ and hyperbolic paths of minimal length with recurrence relations give rise to a special continued fraction.

Let us give the definitions used in our paper;

Definition 1.3. (a) Let $v_0, v_1, \ldots, v_m$ be a sequence of different vertices of the graph $\mathbf{F}_{u,N}$. If $m \geq 2$ then the configuration $v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_m \rightarrow v_0$ is called a *directed circuit* (*closed path*). If at least one arrow (not all) is reversed in this configuration, it is called an *undirected* (*anti-directed*) *circuit*. If $m = 2$ then the circuit, directed or not, is called a *triangle*. If $m = 1$ then we will call the configuration $v_0 \rightarrow v_1 \rightarrow v_0$ a *self paired edge*.

(b) The configurations $v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_m$ and $v_0 \rightarrow v_1 \rightarrow \ldots$ are called a *path* and an *infinite path* in $\mathbf{F}_{u,N}$, respectively.

(c) If $\frac{r}{s} \overset{<}{\longrightarrow} \frac{x}{y} \in \mathbf{F}_{u,N}$ $\left(\text{or } \frac{x}{y} \overset{<}{\longleftarrow} \frac{r}{s} \in \mathbf{F}_{u,N}\right)$, the *farthest vertex* means that there is no vertex which has greater (or smaller) value than $\frac{x}{y}$ joined with the vertex $\frac{r}{s}$ in the suborbital graph $\mathbf{F}_{u,N}$ by the conditions from Theorem 1.1.

(d) The path $v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_i \rightarrow v_{i+1} \rightarrow \ldots \rightarrow v_j \rightarrow \ldots \rightarrow v_m$ is called of *minimal length* if and only if $v_i \nleftrightarrow v_j$, where $i < j-1$, $i \in \{0, 1, 2, 3, ..., m-2\}$, $j \in \{2, 3, ..., m\}$ and v_{i+1} must be the farthest vertex which can be joined with the vertex v_i in $\mathbf{F}_{u,N}$.

(e) If $\mathbf{F}_{u,N}$ does not contain any circuits it is called a *forest*. If $\mathbf{F}_{u,N}$ is a connected non-empty graph without circuits it is called a *tree*.

1.1. **Continued Fractions With Recurrence Relations.** We know any continued fraction can be expressed as the symbol $b_0 + \mathrm{K}_{m=1}^{\infty}(a_m/b_m)$ by [2]. Using the terminology in [2], the n^{th} numerator A_n and the n^{th} denominator B_n of a continued fraction $b_0 + \mathrm{K}(a_m/b_m)$ are defined by the recurrence relations (second order linear difference equations)

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} := b_n \begin{bmatrix} A_{n-1} \\ B_{n-1} \end{bmatrix} + a_n \begin{bmatrix} A_{n-2} \\ B_{n-2} \end{bmatrix},$$

where $n = 1, 2, 3, \ldots$ with initial conditions $A_{-1} := 1, B_{-1} := 0, A_0 := b_0, B_0 := 1$. The modified approximant $T_n(z_n)$ can then be written as $T_n(z_n) = \frac{A_n + A_{n-1}z_n}{B_n + B_{n-1}z_n}$, where $n = 0, 1, 2, 3, \ldots$ and hence for the n^{th} approximant f_n we have $f_n = T_n(0) = \frac{A_n}{B_n}, f_{n-1} = T_n(\infty) = \frac{A_{n-1}}{B_{n-1}}$.

In the present study, the action of a congruence group of $SL(2,\mathbb{Z})$ on $\hat{\mathbb{Q}}$ is examined.From this action and its properties, vertices of paths of minimal length on the suborbital graph $\mathbf{F}_{u,N}$ give rise to some special sequences values, that are alternate sequences such as identity, Fibonacci and Lucas sequences.

2. Special Vertices of $\mathbf{F}_{u,N}$ and Some Sequences

Lemma 2.1 ([4]). *If $(u, N) = 1$, then exist an integer k such that $u^2 + ku + 1 \equiv 0\,(mod\,N)$.*

Corollary 2.2 ([4]). *If $(u, N) = 1$, then exist an integer l such that $u^2 - lu + 1 \equiv 0\,(mod\,N)$.*

Corollary 2.3 ([4]). *Let $u^2 + ku + 1 \equiv 0\,(mod\,N)$ and $u^2 - lu + 1 \equiv 0\,(mod\,N)$ with $1 < k, l \leq N$. If $\mathbf{F}_{u,N}$ is self-paired, then $k = l = N$ and otherwise $l = N - k$.*

Corollary 2.4. *Let $\omega = \begin{pmatrix} -u & (u^2 - lu + 1)/N \\ -N & u - l \end{pmatrix} \in \Gamma_0(N)(= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0\,(mod\,N))$. Hence, for left direction, if $\frac{u - \frac{x}{y}}{N}$ is the vertex on the path of minimal length in $\mathbf{F}_{u,N}$, then the farthest vertex which can be joined with it is $\frac{u - \frac{y}{ly - x}}{N}$, where $\omega \begin{pmatrix} u - \frac{x}{y} \\ N \end{pmatrix} = \begin{pmatrix} u - \frac{y}{ly-x} \\ N \end{pmatrix}$ So, for positive integers q, $\overleftarrow{v_q} = \varphi^q(v_0)$, where $v_0 = \frac{u}{N}$.*

Corollary 2.5 ([4]). *If $u^2 + ku + 1 \equiv 0\,(mod\,N)$ and $1 < k \leq N$, then there is an infinite path of the minimal length to right direction in $\mathbf{F}_{u,N}$ as*

$$\frac{1}{0} \longrightarrow \frac{u}{N} \longrightarrow \frac{u + \frac{1}{k}}{N} \longrightarrow \frac{u + \frac{1}{k - \frac{1}{k}}}{N} \longrightarrow \frac{u + \frac{1}{k - \frac{1}{k - \frac{1}{k}}}}{N} \longrightarrow \cdots$$

whose vertices are in the set

$$M := \bigcup_{m=0}^{\infty} \left\{ \frac{u + T_m(0)}{N} : T_m = t_0 t_1 t_2 \ldots t_m,\ t_0(z) = z, t_m(z) := t(z) = \frac{-1}{-k + z} \right\} \cup \{\infty\}.$$

Corollary 2.6 ([4]). *If $u^2 - lu + 1 \equiv 0\,(mod\,N)$ and $1 < l \leq N$, then there is an infinite path of the minimal length to left direction in $\mathbf{F}_{u,N}$ as*

$$\cdots \longleftarrow \frac{u - \frac{1}{l - \frac{1}{l - \frac{1}{l}}}}{N} \longleftarrow \frac{u - \frac{1}{l - \frac{1}{l}}}{N} \longleftarrow \frac{u - \frac{1}{l}}{N} \longleftarrow \frac{u}{N} \longleftarrow \frac{1}{0}$$

whose vertices are in the set

$$P := \bigcup_{m=0}^{\infty} \left\{ \frac{u - T_m(0)}{N} : T_m = t_0 t_1 t_2 \ldots t_m,\ t_0(z) = z, t_m(z) := t(z) = \frac{-1}{-l + z} \right\} \cup \{\infty\}.$$

Corollary 2.7. *Let $\varphi = \begin{pmatrix} -u & (u^2 + ku + 1)/N \\ -N & u + k \end{pmatrix} \in \Gamma_0(N)(= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0\,(mod\,N))$. Hence, for right direction, if If $\frac{u + \frac{x}{y}}{N}$ is the vertex on the path of minimal length in $\mathbf{F}_{u,N}$, then the farthest vertex which can be joined with it is $\frac{u + \frac{y}{ky - x}}{N}$, where $\varphi \begin{pmatrix} u + \frac{x}{y} \\ N \end{pmatrix} = \begin{pmatrix} u + \frac{y}{ky-x} \\ N \end{pmatrix}$ So, for positive integers q, $\overrightarrow{v_q} = \varphi^q(v_0)$, where $v_0 = \frac{u}{N}$.*

Since $a_n := -1$ and $b_n := -k$, for all $n \geq 1$, then from recurrence relations we get $B_n = -A_{n+1}$ and so for the right directional path of minimal length in suborbital graph $\mathbf{F}_{u,N}$, n^{th} vertex is given by

$$\overrightarrow{v_n} = \frac{u + T_n(0)}{N} = \frac{u + \frac{A_n}{B_n}}{N} = \frac{A_{n+1}u - A_n}{A_{n+1}N}. \tag{2.1}$$

Similarly for left direction n^{th} vertex is

$$\overleftarrow{v_n} = \frac{u - T_n(0)}{N} = \frac{u - \frac{A_n}{B_n}}{N} = \frac{A_{n+1}u + A_n}{A_{n+1}N} \tag{2.2}$$

where $a_n := -1$ and $b_n := -l$, for all $n \geq 1$.

Corollary 2.8 ([4]). *From recurrence relations we have $kA_{n+1} + A_{n+2} + A_n = 0$.*

Corollary 2.9 ([4]). *From solving the characteristic equation$A_n = -kA_{n-1} - A_{n-2}$, if $k = 2$ then $A_n = (-1)^n n$ and if $k > 2$ then*

$$A_n = (-1)^n 2^{1-n} \sum_{t=1}^{n} (k + \sqrt{k^2 - 4})^{n-t} (k - \sqrt{k^2 - 4})^{t-1}.$$

For left direction in the suborbital graph $\mathbf{F}_{u,N}$, this is exactly same if we change k to l.

Corollary 2.10. *If $k = 2$ then for the right direction the n^{th} vertex on the path of minimal length starting with the vertex u/N in the suborbital graph $\mathbf{F}_{u,N}$ is*

$$\overrightarrow{v_n} = \frac{(n+1)u + n}{(n+1)N} = \frac{u + \frac{n}{n+1}}{N}.$$

Proof. For all $n \geq 0$, since $b_n = -k = -2$ and $a_n = -1$, then $A_n = (-1)^n n$. So, n^{th} vertex on the path of minimal length to right direction is

$$\frac{A_{n+1}u - A_n}{A_{n+1}N} = \frac{(-1)^{n+1}(n+1)u - (-1)^n}{(-1)^{n+1}(n+1)N} = \frac{(n+1)u + n}{(n+1)N}.$$

□

Corollary 2.11. *If $l = 2$ then for the right direction the n^{th} vertex on the path of minimal length starting with the vertex u/N in the suborbital graph $\mathbf{F}_{u,N}$ is*

$$\overleftarrow{v_n} = \frac{(n+1)u - n}{(n+1)N} = \frac{u - \frac{n}{n+1}}{N}.$$

Proof. For all $n \geq 0$, since $b_n = -l = -2$ and $a_n = -1$, then $A_n = (-1)^n n$. So, n^{th} vertex on the path of minimal length to left direction is

$$\frac{A_{n+1}u + A_n}{A_{n+1}N} = \frac{(-1)^{n+1}(n+1)u + (-1)^n}{(-1)^{n+1}(n+1)N} = \frac{(n+1)u - n}{(n+1)N}.$$

□

Example 2.12.

$$\begin{array}{c} \infty \\ \downarrow \\ \ldots \longleftarrow \frac{1}{8} = \frac{1-\frac{3}{4}}{2} \longleftarrow \frac{1}{6} = \frac{1-\frac{2}{3}}{2} = \frac{1-\frac{1}{2-\frac{1}{2}}}{2} \longleftarrow \frac{1}{4} = \frac{1-\frac{1}{2}}{2} \longleftarrow \frac{1}{2} \longrightarrow \frac{3}{4} = \frac{1+\frac{1}{2}}{2} \longrightarrow \frac{5}{6} = \frac{1+\frac{2}{3}}{2} = \frac{1+\frac{1}{2-\frac{1}{2}}}{2} \longrightarrow \frac{7}{8} = \frac{1+\frac{3}{4}}{2} \longrightarrow \ldots \end{array}$$

For example, for the suborbital graph $\mathbf{F}_{1,2}$, which is self-paired graph, so $k = l = 2$ from the congruence $u^2 + ku + 1 \equiv 0\ (modN)$ and $u^2 - lu + 1 \equiv 0\ (modN)$. So, above path give rise to vertices on the path of minimal length to both direction.

Theorem 2.13 ([4]). *From recurrence relations, if A_n is the n^{th} numerator of a continued fraction $\mathbf{K}\left(\frac{-1}{-3}\right) = \cfrac{1}{3-\cfrac{1}{3-\cfrac{1}{3-\ddots}}}$, then $A_n = (-1)^n F_{2n}$, where F_n is the n^{th} Fibonacci number.*

Lemma 2.14. *If F_n is the n^{th} Fibonacci number, then*

$$\begin{pmatrix} (-1)^{n-1}F_{2n-2} & (-1)^n F_{2n} \\ (-1)^{n+1}F_{2n} & (-1)^n F_{2n+2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}^n.$$

Proof. From the matrix relation of the continued fractions, it is known that

$$\begin{pmatrix} A_{n-1} & A_n \\ -A_n & -A_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -k \end{pmatrix}^n.$$

Also if $k = 3$ than $A_n = (-1)^n F_{2n}$, where A_n is the n^{th} numerator of a continued fraction $\mathbf{K}\left(\frac{-1}{-3}\right) = \cfrac{1}{3-\cfrac{1}{3-\cfrac{1}{3-\ddots}}}$.

From the equation $A_n = (-1)^n F_{2n}$ we can write $A_{n-1} = (-1)^{n-1}F_{2n-2}, A_{n+1} = (-1)^{n+1}F_{2n+2}, -A_n = (-1)^{n+1}F_{2n}$ and $-A_{n+1} = (-1)^n F_{2n+2}$. Hence,

$$\begin{pmatrix} A_{n-1} & A_n \\ -A_n & -A_{n+1} \end{pmatrix} = \begin{pmatrix} (-1)^{n-1}F_{2n-2} & (-1)^n F_{2n} \\ (-1)^{n+1}F_{2n} & (-1)^n F_{2n+2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}^n.$$

□

Lemma 2.15 ([5]). *Let us give some lemmas without proof below, where A_n is the n^{th} numerator of a continued fraction*

$$\mathbf{K}\left(\frac{-1}{-3}\right) = \cfrac{1}{3-\cfrac{1}{3-\cfrac{1}{3-\ddots}}}$$

a) $F_{2n+1} = (-1)^{n+1}(A_{n+1} + A_n)$, where $n \geq 0, n \in \mathbb{Z}$.
b) $F_{2n}A_{n+1} + F_{2n+2}A_n = 0$, where $n \geq 1, n \in \mathbb{Z}$.
c) $F_{2n+1} = \frac{(-1)^{n+1}}{3}(2A_{n+1} - A_{n-1})$,where $n \geq 0, n \in \mathbb{Z}$.
d) $F_{2n} = \frac{1}{3}(F_{2n+2} + F_{2n-2})$, where $n \geq 1, n \in \mathbb{Z}$.
e) $F_{2n-2} = \frac{1}{2}(F_{2n+2} - 3F_{2n-1})$, where $n \geq 2, n \in \mathbb{Z}$.
f) $F_{2n-1} = \frac{1}{3}(F_{2n+1} + F_{2n-3})$, where $n \geq 2, n \in \mathbb{Z}$.
g) $2F_{2n} - 3F_{2n-1} = F_{2n-2} - F_{2n-3}$, where $n \geq 2, n \in \mathbb{Z}$.
h) $F_{4n+2} = -A_{2n+1}$, where $n \geq 0, n \in \mathbb{Z}$.

Lemma 2.16. *Let*

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -1 \end{pmatrix}.$$

$$P^n = \begin{pmatrix} (-1)^n F_{n-1} & 0 & (-1)^{n+1}F_n \\ 3[(-1)^{n+1}F_{n-2} - 1] & 1 & 3[(-1)^n F_{n-1} - 1] \\ (-1)^{n+1}F_n & 0 & (-1)^n F_{n+1} \end{pmatrix}$$

where $n > 1$, F_n is Fibonacci sequence with $F_0 = 0, F_1 = 1, F_2 = 2$ initial conditions and $F_n = F_{n-1} + F_{n-2}$ recurrence relation.

Proof. Let $P^n = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$. Now for p_{11}, we will use the mathematical induction method;

for $n = 2$, $P^2 = \begin{pmatrix} 1 & 0 & -1 \\ -3 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$ and $p_{11} = (-1)^2 F_{2-1} = 1$ holds.

Assume that the statement is true for a particular value $n = k$,that is $P^k = \begin{pmatrix} (-1)^k F_{k-1} & 0 & (-1)^{k+1}F_k \\ 3[(-1)^{k+1}F_{k-2} - 1] & 1 & 3[(-1)^k F_{k-1} - 1] \\ (-1)^{k+1}F_k & 0 & (-1)^k F_{k+1} \end{pmatrix}$.

Prove that the statement is true for $n = k + 1$;

$$P^{k+1} = PP^k = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} (-1)^k F_{k-1} & 0 & (-1)^{k+1}F_k \\ 3[(-1)^{k+1}F_{k-2}-1] & 1 & 3[(-1)^k F_{k-1}-1] \\ (-1)^{k+1}F_k & 0 & (-1)^k F_{k+1} \end{pmatrix} = \begin{pmatrix} (-1)^{k+1}F_k & 0 & * \\ * & 1 & * \\ * & 0 & * \end{pmatrix}$$

Thus, the other elements of the matrix can be found by using a similar method.

□

Lemma 2.17 ([4]). *Let A_n be $\mathbf{K}(\frac{-1}{-3})$ continued fractions n^{th} numerator. Then $F_{2n}A_{n+1} + F_{2n+2}A_n = 0$.*

Corollary 2.18. *If $k = 3$ then $u^2 + 3u + 1 \equiv 0\,(mod\,N)$. For right direction, since the value of n^{th} vertex on the path of minimal length in the suborbital graph $\mathbf{F}_{u,N}$ is*

$$\overrightarrow{v_n} = \frac{u + T_n(0)}{N} = \frac{u - \frac{A_n}{A_{n+1}}}{N}$$

and $A_n = (-1)^n F_{2n}$ then this vertex can be given by

$$\overrightarrow{v_n} = \frac{u + \frac{F_{2n}}{F_{2n+2}}}{N}$$

where $v_0 = u/N$.

Proof. For all $n \geq 0$, since $b_n = -k = -3$ and $a_n = -1$, from the Theorem 2.13.

$$A_n = (-1)^n F_{2n}.$$

So by (2.1), the n^{th} vertex on the path of the minimal length of $\mathbf{F}_{u,N}$ for the right direction is

$$\overrightarrow{v_n} = \frac{A_{n+1}u - A_n}{A_{n+1}N} = \frac{(-1)^{n+1}F_{2n+2}u - (-1)^n F_{2n}}{(-1)^{n+1}F_{2n+2}N} = \frac{u + \frac{F_{2n}}{F_{2n+2}}}{N}.$$

□

Corollary 2.19. *If $l = 3$ then $u^2 - 3u + 1 \equiv 0\,(mod\,N)$. For left direction, since the value of n^{th} vertex on the path of minimal length in the suborbital graph $\mathbf{F}_{u,N}$ is*

$$\overleftarrow{v_n} = \frac{u - T_n(0)}{N} = \frac{u + \frac{A_n}{A_{n+1}}}{N}$$

and $A_n = (-1)^n F_{2n}$ then this vertex can be given by

$$\overleftarrow{v_n} = \frac{u - \frac{F_{2n}}{F_{2n+2}}}{N}$$

where $v_0 = u/N$.

Proof. For all $n \geq 0$, since $b_n = -l = -3$ and $a_n = -1$, from the Theorem 2.13.

$$A_n = (-1)^n F_{2n}.$$

So by (2.2), the n^{th} vertex on the path of the minimal length of $\mathbf{F}_{u,N}$ for the left direction is

$$\overleftarrow{v_n} = \frac{A_{n+1}u + A_n}{A_{n+1}N} = \frac{(-1)^{n+1}F_{2n+2}u + (-1)^n F_{2n}}{(-1)^{n+1}F_{2n+2}N} = \frac{u - \frac{F_{2n}}{F_{2n+2}}}{N}.$$

□

Example 2.20. Let $u := 1$ and $N := 5$. Then minimal positive integers are $k = 3$ and $l = 2$, where for the congruences $u^2 + ku + 1 \equiv 0\,(mod\,N)$ and $u^2 - lu + 1 \equiv 0\,(mod\,N)$ respectively. So, for the suborbital graph $\mathbf{F}_{1,5}$, for right and left direction, the farthest vertices which can be joined with the vertex $v_0 = 1/5$ are $\overrightarrow{v_1} = \frac{1+\frac{1}{3}}{5} = \frac{4}{15}$ and $\overleftarrow{v_1} = \frac{1-\frac{1}{2}}{5} = \frac{1}{10}$ respectively (See Figure 3). Similarly for right and left direction, the farthest vertices which can be joined with vertices $\overrightarrow{v_1} = \frac{4}{15}$ and $\overleftarrow{v_1} = \frac{1}{10}$ are $\overrightarrow{v_2} = \frac{1+\frac{1}{3-\frac{1}{3}}}{5} = \frac{1+\frac{3}{8}}{5} = \frac{11}{40}$ and $\overleftarrow{v_2} = \frac{1-\frac{1}{2-\frac{1}{2}}}{5} = \frac{1-\frac{2}{3}}{5} = \frac{1}{15}$ respectively.

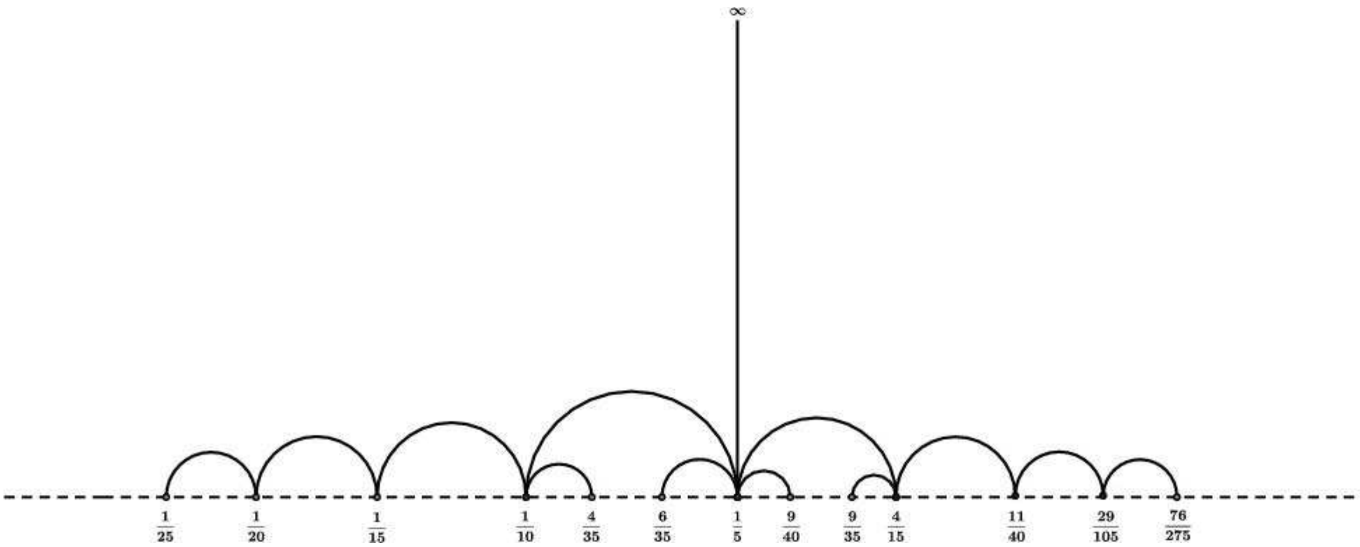

FIGURE 3. Some vertices in the suborbital graph $\mathbf{F}_{1,5}$

Also, we know that for right and left direction the n^{th} vertices are $\overrightarrow{v_n} = \frac{u - \frac{F_{2n}}{F_{2n+2}}}{N}$ and $\overleftarrow{v_n} = \frac{1-\frac{n}{n+1}}{5}$, respectively. For the value $n = 4$, since $\begin{pmatrix} (-1)^{n-1}F_{2n-2} & (-1)^n F_{2n} \\ (-1)^{n+1}F_{2n} & (-1)^n F_{2n+2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}^n$, then$\begin{pmatrix} -F_6 & F_8 \\ -F_8 & F_{10} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}^4 = \begin{pmatrix} -8 & 21 \\ -21 & 55 \end{pmatrix}$. So by using these equations, 3^{th} and 4^{th} vertices to right direction are

$$\overrightarrow{v_3} = \frac{1 + \frac{1}{3-\frac{1}{3-\frac{1}{3}}}}{5} = \frac{1 + T_3(0)}{5} = \frac{1 + \frac{F_6}{F_8}}{5} = \frac{1 + \frac{8}{21}}{5} = \frac{29}{105}$$

and

$$\overrightarrow{v_4} = \frac{1 + \frac{1}{3-\frac{1}{3-\frac{1}{3-\frac{1}{3}}}}}{5} = \frac{1 + T_4(0)}{5} = \frac{1 + \frac{F_8}{F_{10}}}{5} = \frac{1 + \frac{21}{55}}{5} = \frac{76}{275},$$

Similarly, 3^{th} and 4^{th} vertices to left direction are $\overleftarrow{v_3} = \frac{1-\frac{3}{4}}{5}$ and $\overleftarrow{v_4} = \frac{1-\frac{4}{5}}{5} = \frac{1}{25}$

Lemma 2.21. *Let A_n be $\mathbf{K}(\frac{-1}{-3})$ continued fractions n^{th} numerator. Then ${A_n}^2(L_{4n+4} - 2) - A_{n+1}^2(L_{4n} - 2) = 0$.*

Proof. From Lemma 2.17. we can write $F_{2n}A_{n+1} + F_{2n+2}A_n = 0$. So, here $\frac{F_{2n}}{F_{2n+2}} = -\frac{A_n}{A_{n+1}}$. If we use that identity in [8]; $L_{4n} = 5{F_{2n}}^2 + 2$, $F_{2n} = \sqrt{\frac{L_{4n}-2}{5}}$ is obtained. Therefore we have that

$$\frac{F_{2n}}{F_{2n+2}} = \frac{\sqrt{L_{4n} - 2}}{\sqrt{L_{4n+4} - 2}} = -\frac{A_n}{A_{n+1}}$$

and so ${A_n}^2(L_{4n+4} - 2) - A_{n+1}^2(L_{4n} - 2) = 0$. □

Definition 2.22 ([6]**).** The Fibonacci numbers defined by

$$F_n : \ F_0 = 0, F_1 = 1, \ F_{n+1} = F_n + F_{n-1} \ (n \geq 1).$$

Then, the continued fraction expansion of F_n/F_{n+1} is given by

$$\frac{F_n}{F_{n+1}} = [0, \underbrace{1, \cdots, 1}_{n-2}, 2]$$

By this definition also we get the continued fraction expansion of F_{2n}/F_{2n+2} by following corollary.

Corollary 2.23. *The continued fraction expansion of F_{2n}/F_{2n+2} is*

$$\frac{F_{2n}}{F_{2n+2}} = [0, 2, \underbrace{1, \cdots, 1}_{2n-3}, 2].$$

3. An Example of Algorithm for $\mathcal{F}_{1,3}$- Continued Fraction

In this section, we have established a connection between an algorithm for obtaining the $\mathcal{F}_{1,3}$- continued fraction and a formula for obtaining the corresponding vertex for the suborbital graph $\mathbf{F}_{1,3}$.

Suppose $u, N \in \mathbb{N}$ such that $1 \leq u \leq N$ and $(u, N) = 1$. Then $F_{u,N}$ is the graph whose set of vertices is

$$\mathcal{X}_N = \left\{ \frac{x}{y} : x, y \in \mathbb{Z}, y > 0, (x, y) = 1 \textit{ and } N|y \right\} \cup \{\infty\}.$$

Definition 3.1 ([9]). A finite continued fraction of the form

$$\frac{1}{0+}\,\frac{3}{b+}\,\frac{\epsilon_1}{a_1+}\,\frac{\epsilon_2}{a_2+}\cdots\frac{\epsilon_n}{a_n} \quad (n \geq 0)$$

or an infinite continued fraction of the form

$$\frac{1}{0+}\,\frac{3}{b+}\,\frac{\epsilon_1}{a_1+}\,\frac{\epsilon_2}{a_2+}\cdots\frac{\epsilon_n}{a_n}\cdots$$

where b is an integer co-prime to $3, a_1, a_2, \cdots$ are positive integers, and $\epsilon_1, \epsilon_2, \cdots \in \{\pm 1\}$, such that for $i \geq 1$, $a_i + \epsilon_{i+1} \geq 1$ and $(p_i, q_i) = 1$ with $p_i a_i p_{i-1} + \epsilon_i p_{i-2}$, $g_i = a_i q_{i-1} + \epsilon_i q_{i-2}$, $(p_{-1}, q_{-1} = (1, 0)$ and $(p_0, q_0) = (b, 3)$, is called an $\mathcal{F}_{1,3}$-continued fraction.

For $i \geq 1$, the expression

$$\frac{1}{0+}\,\frac{3}{b+}\,\frac{\epsilon_1}{a_1+}\,\frac{\epsilon_2}{a_2+}\cdots\frac{\epsilon_i}{a_i}$$

is called the $i-th$ $\mathcal{F}_{1,3}$-convergent of the continued fraction. The continued fraction

$$\frac{\epsilon_i}{a_i+}\,\frac{\epsilon i+1}{a_{i+1}+}\cdots\frac{\epsilon_n}{\epsilon_n}\cdots$$

is referred to as the $i-th$ fin of the $\mathcal{F}_{1,3}$- continued fraction. The rational number p_i/q_i is the $i-th$ convergent of the continued fraction. The sequence $\{\frac{p_i}{q_i}\}_{i\geq 1}$ is called the sequence of $\mathcal{F}_{1,3}$-convergents corresponding to the given $\mathcal{F}_{1,3}$-continued fraction.

Theorem 3.2 ([9]). *Given any $x \in \mathcal{X}_\ni$, the $\mathcal{F}_{1,3}$-continued fraction expansion*

$$x = \frac{1}{0+}\,\frac{3}{b+}\,\frac{\epsilon_1}{a_1+}\,\frac{\epsilon_2}{a_2+}\cdots\frac{\epsilon_n}{a_n},$$

of x is obtained as follows:

$$b = \begin{cases} 3\lfloor x \rfloor + 1, & \lfloor x \rfloor < x < \lfloor x \rfloor + \frac{1}{2}, \\ 3\lfloor x \rfloor + 2, & \lfloor x \rfloor + \frac{1}{2} < x < \lfloor x \rfloor + 1 \end{cases}$$

Set $y_1 = 3x - b$. Then

(1) $\epsilon_i = sign(y_i)$,
(2) $a_i = \left\lfloor (\frac{1}{|y_i|} + 1) \right\rfloor$ *or* $\left\lceil (\frac{1}{|y_i|} - 1) \right\rceil$ *or* $\frac{1}{|y_i|}$, *such that* $a_i \not\equiv -\epsilon_i p_{i-1} p_{i-2} \mod 3$ *and* $a_i + \epsilon_i \geq 1$,
(3) $y_{i+1} = \frac{1}{|y_i|} - a_i$.

In fact, n is the smallest non-negative integer for which $y_{n+1} = 0$.

Example 3.3. Let $x = \frac{1}{27} \in \mathcal{X}_3$. By Theorem 3.2.
$b = 3\lfloor x \rfloor + 1 = 1$
$y_1 = 3x - b = 3\frac{1}{30} - 1 = -\frac{8}{9}$
$\epsilon_1 = sign(y_1) = -1$
$a_1 = \lceil \left(\frac{1}{|-\frac{8}{9}|} + 1 \right) \rceil = 2, 2 \not\equiv -(-1) \cdot 1 \cdot 1 \mod 3$ and $2 + (-1) \geq 1$

$y_2 = \frac{1}{|-\frac{8}{9}|} - 2 = -\frac{7}{8}$
$\epsilon_2 = sign(y_2) = -1$
$a_2 = \lceil\left(\frac{1}{|-\frac{7}{8}|} + 1\right)\rceil = 2, 2 \not\equiv -(-1)\cdot 1\cdot 1 \mod 3$ and $2+(-1) \geq 1$
$y_3 = \frac{1}{|-\frac{7}{8}|} - 2 = -\frac{6}{7}$
$\epsilon_3 = sign(y_3) = -1$
$a_3 = \lceil\left(\frac{1}{|-\frac{6}{7}|} + 1\right)\rceil 25, 2 \not\equiv -(-1)\cdot 1\cdot 1 \mod 3$ and $2+(-1) \geq 1$
$y_4 = \frac{1}{|-\frac{6}{7}|} - 2 = -\frac{5}{6}$
$\epsilon_4 = sign(y_4) = -1$
$a_4 = \lceil\left(\frac{1}{|-\frac{5}{6}|} + 1\right)\rceil = 2, 2 \not\equiv -(-1)\cdot 1\cdot 1 \mod 3$ and $2+(-1) \geq 1$
$y_5 = \frac{1}{|-\frac{5}{6}|} - 2 = -\frac{-4}{5}$
$\epsilon_5 = sign(y_5) = -1$
$a_5 = \lceil\left(\frac{1}{|-\frac{4}{5}|} + 1\right)\rceil = 2, 2 \not\equiv -(-1)\cdot 1\cdot 1 \mod 3$ and $2+(-1) \geq 1$
$y_6 = \frac{1}{|-\frac{4}{5}|} - 2 = -\frac{3}{4}$
$\epsilon_6 = sign(y_6) = -1$
$a_6 = \lceil\left(\frac{1}{|-\frac{3}{4}|} + 1\right)\rceil = 2, 2 \not\equiv -(-1)\cdot 1\cdot 1 \mod 3$ and $2+(-1) \geq 1$
$y_7 = \frac{1}{|\frac{3}{4}|} - 2 = -\frac{2}{3}$
$\epsilon_7 = sign(y_7) = -1$
$a_7 = \lceil\left(\frac{1}{|-\frac{2}{3}|} + 1\right)\rceil = 2{,}2 \not\equiv -(-1)\cdot 1\cdot 1 \mod 3$ and $2+(-1) \geq 1$
$y_8 = \frac{1}{|-\frac{2}{3}|} - 2 = -\frac{1}{2}$
$\epsilon_8 = sign(y_8) = -1$
$a_8 = \frac{1}{|y_8|} = 2{,}2 \not\equiv -(-1)\cdot 1\cdot 1 \mod 3$ and $2+(-1) \geq 1$
$y_9 = \frac{1}{|-\frac{1}{2}|} - 2 = 0$
is obtained. So we can write , the $\mathcal{F}_{1,3}$-continued fraction expansion of x is as follows;

$$x = \frac{1}{27} = \frac{1}{0+}\,\frac{3}{1+}\,\frac{-1}{2+}\,\frac{-1}{2+}\,\frac{-1}{2+}\,\frac{-1}{2+}\,\frac{-1}{2+}\,\frac{-1}{2+}\,\frac{-1}{2+}\,\frac{-1}{2}.$$

From this continued fraction, we can write the vertices of the minimal length path of suborbital graph $\mathbf{F}_{1,3}$;

$$\cdots \leftarrow \frac{1}{27} \leftarrow \frac{1}{24} \leftarrow \frac{1}{21} \leftarrow \frac{1}{18} \leftarrow \frac{1}{15} \leftarrow \frac{1}{12} \leftarrow \frac{1}{9} \leftarrow \frac{1}{6} \leftarrow \frac{1}{3} \leftarrow \frac{1}{0}.$$

On the other hand from (2.2); for left direction, we can write 8^{th} vertex of the minimal length path of suborbital graph $\mathbf{F}_{1,3}$ as,

$$\overleftarrow{v_8} = \frac{(n+1)u-n}{(n+1)N} = \frac{u-\frac{n}{n+1}}{N} = \frac{1-\frac{8}{9}}{3} = \frac{1}{27}$$

is obtained.

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