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Characteristic Numbers of Upper Triangular One-Band Block Operator Matrices

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Abstract. In this work the boundedness and compactness properties of upper triangular one-band block operator matrices in the infinite direct sum of Hilbert spaces have been studied. We also obtain the necessary and sufficient conditions when these operators belong to Schatten-von Neumann classes.

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1. Introduction

It is known that the traditional infinite direct sum of Hilbert spaces $H_n,\ n \geq 1$ is defined as

$$H = \bigoplus_{n=1}^{\infty} H_n = \left\{ u = (u_n) : x_n \in H_n,\ n \geq 1,\ \sum_{n=1}^{\infty} \|u_n\|^2_{H_n} < +\infty \right\}.$$

Recall that H is a Hilbert space with norm induced by the inner product (see [7])

$$(u, v)_H = \sum_{n=1}^{\infty} (u_n, v_n)_{H_n},\ u, v \in H.$$

It is known that a lot of many physical problems of today arising in the modelling of processes of multiparticle quantum mechanics, quantum field theory and in the physics of rigid bodies support to study the theory of linear operators in the direct sum of Hilbert spaces (see [13, 19] and references in it).

Investigation of these problems with the view of spectral analysis of finite or infinite dimensional real and complex entires special matrices (upper and lower triangular double-band or third-band or Toeplitz types) in sequences spaces $\omega,\ c,\ c_0,\ bs,\ b\omega_p,\ l_p$ have been widely studied in current literature (for example, see [1–6, 8, 12, 18].

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On the other hand some spectral analysis of 2×2 and 3×3 types block operator matrices have been provided in [11, 14, 15, 17]. Note that the structure of spectrum of diagonal operator matrices has been obtained in [16] and the compactness properties and belonging to Schatten-von Neumann classes of diagonal operator matrices in the direct sum of Hilbert spaces has been examined in [10].

In the present paper, we study the compactness properties of upper triangular one-band operator matrices in the infinite direct sum of Hilbert spaces. Then, belonging to Schatten-von Neumann classes of this type operators will be examined.

2. Boundedness and Compactness of Upper Triangular One-Band Block Operator Matrices

In this section, we will investigate the continuity and compactness properties of lower one-band diagonal block operator matrices which have the following form

$$A = \begin{pmatrix} 0 & A_1 & & & & & \\ & 0 & A_2 & & & & \\ & & 0 & A_3 & & 0 & \\ & & & \ddots & \ddots & & \\ & 0 & & & 0 & A_n & \\ & & & & & \ddots & \ddots \end{pmatrix}$$

in the direct sum $H = \bigoplus_{n=1}^{\infty} H_n$ of Hilbert Spaces H_n, $n \geq 1$ in case $A_n \in L(H_{n+1}, H_n)$, $n \geq 1$.

Firstly, let us prove the following:

Theorem 2.1. *In order for $A \in L(H)$ the sufficient condition is*

$$\sup_{n \geq 1} \|A_n\| < \infty.$$

Proof. Let $A \in L(H)$. In this case for every $x = (x_n) \in H$ we have

$$\begin{aligned} \|Ax\|_H^2 &= \sum_{n=1}^{\infty} \|A_n x_{n+1}\|_{H_n}^2 \\ &\leq \sum_{n=1}^{\infty} \|A_n\|^2 \|x_{n+1}\|_{H_{n+1}}^2 \\ &\leq \sum_{n=1}^{\infty} \left(\sup_{n\geq 1} \|A_n\|\right)^2 \|x_n\|_{H_n}^2 \\ &= \left(\sup_{n\geq 1} \|A_n\|\right)^2 \|x\|_H^2, \end{aligned}$$

that is, for each $x = (x_n) \in H$ we obtain that

$$\|Ax\|_H \leq \left(\sup_{n\geq 1} \|A_n\|\right) \|x\|.$$

From this it is clear that

$$A \in L(H).$$

□

Theorem 2.2. *If $A \in C_\infty(H)$, then for any $n \geq 1$, $A_n \in C_\infty(H_{n+1}, H_n)$. On the other hand, in case when $A_n \in C_\infty(H_{n+1}, H_n)$, $n \geq 1$ and*

$$\lim_{n\to\infty} \|A_n\| = 0,$$

then $A \in C_\infty(H)$.

Proof. The necessity of the theorem has been proved in [16]. To show the validity of second assertion of theorem consider the following sequence of operators in form

$$T_n := \begin{pmatrix} 0 & A_1 & & & & & & & \\ & 0 & A_2 & & & & & & \\ & & 0 & A_3 & & 0 & & & \\ & & & \ddots & \ddots & & & & \\ & 0 & & & 0 & A_n & & & \\ & & & & & 0 & 0 & & \\ & & & & & & 0 & 0 & \\ & & & & & & & \ddots & \ddots \end{pmatrix} : H \to H,\ n \geq 1.$$

In this case, for each $n \geq 1$, $T_n \in C_\infty(H)$. Moreover, for any $n \geq 1$ and $x = (x_n) \in H$ we have

$$\begin{aligned} \|(A - T_n)x\|_H^2 &= \sum_{m=n}^{\infty} \|A_{m+1} x_{m+2}\|_{H_{m+1}}^2 \\ &\leq \sum_{m=n}^{\infty} \|A_{m+1}\|^2 \|x_{m+2}\|_{H_{m+2}}^2 \\ &\leq \left(\sup_{m \geq n} \|A_n\|\right)^2 \|x\|_H^2. \end{aligned}$$

Hence

$$\|A - T_n\| \leq \sup_{m \geq n} \|A_n\|,\ n \geq 1.$$

From this and condition of theorem it is obtained that

$$\lim_{n \to \infty} T_n = A$$

in operator norm topology.

On the other hand, since $T_n \in C_\infty(H)$, $n \geq 1$ then by a well-known theorem of compact operators theory in [9], $A \in C_\infty(H)$. □

3. Characteristic Numbers of Upper Triangular One-Band Block Operator Matrices

We prove the following for singular numbers of operator A.

Theorem 3.1. *The characteristic numbers of the operator $A \in C_\infty(H)$*

$$\{s_m(A) : m \geq 1\} = \bigcup_{n=1}^{\infty} \{s_k(A_n) : k \geq 1\} \cup \{0\}.$$

Proof. For the operator A

$$A^* A = \bigoplus_{n=1}^{\infty} A_n^* A_n.$$

and

$$\sqrt{A^* A} = \bigoplus_{n=1}^{\infty} \sqrt{A_n^* A_n}.$$

holds. From this and Theorem 2. 3 [16] we obtaine that

$$\sigma_p\left(\sqrt{A^* A}\right) = \bigcup_{n=1}^{\infty} \sigma_p\left(\sqrt{A_n^* A_n}\right) \cup \{0\}.$$

Hence, we have

$$\{s_m(A) : m \geq 1\} = \bigcup_{n=1}^{\infty} \{s_k(A_n) : k \geq 1\} \cup \{0\}.$$ □

With the use of this theorem, we can present the following results.

Corollary 3.2. *If $A \in C_p(H),\ 1 \le p \le \infty$, then for every $n \ge 1$ $A_n \in C_p(H_{n+1}, H_n)$.*

Proof. For $p = \infty$ this proposition has been proved in Theorem 2.2. Let us $1 \le p < \infty$. Since $A \in C_p(H)$, then the series $\sum\limits_{m=1}^{\infty} s_m^p(A)$ is convergent. With the use of following inequality

$$\sum_{k=1}^{\infty} s_k^p(A_n) \le \sum_{m=1}^{\infty} s_m^p(A),\ n \ge 1$$

one can see that the series $\sum\limits_{k=1}^{\infty} s_k^p(A_n)$ is also convergent. This means that for every $n \ge 1,\ A_n \in C_p(H_{n+1}, H_n)$. □

Theorem 3.3. *$A \in C_p(H),\ 1 \le p < \infty$ if and only if the series $\sum\limits_{n=1}^{\infty}\sum\limits_{k=1}^{\infty} s_k^p(A_n)$ is convergent.*

Proof. If $A \in C_p(H)$, then the series $\sum\limits_{m=1}^{\infty} s_m^p(A)$ is convergent. In this case by the Theorem 3.1 and important theorem on the convergence of rearrangement series it is obtained that the series $\sum\limits_{n=1}^{\infty}\sum\limits_{k=1}^{\infty} s_k^p(A_n)$ is convergent.

On the contrary, if the series $\sum\limits_{n=1}^{\infty}\sum\limits_{k=1}^{\infty} s_k^p(A_n)$ is convergent, the series $\sum\limits_{m=1}^{\infty} s_m^p(A)$ being a rearrangement of the above series is also convergent. So $A \in C_p(H)$. □

Corollary 3.4. *For $n \ge 1,\ A_n \in C_{p_n}(H_n, H_{n+1})$ and $p = \sup\limits_{n\ge1} p_n < \infty$. Then $A \in C_p(H)$ if and only if the series $\sum\limits_{n=1}^{\infty}\sum\limits_{k=1}^{\infty} s_k^p(A_n)$ converges.*

Indeed, in this case for each $n \ge 1,\ A_n \in C_{p_n}(H_{n+1}, H_n)$. Then from Theorem 3.3 it is obtained that the validity of proposition.

Corollary 3.5. *For each $n \ge 1,\ A_n \in C_{p_n}(H_{n+1}, H_n)$ and $p = \sup\limits_{n\ge1} p_n < \infty$. If the series $\sum\limits_{n=1}^{\infty}\sum\limits_{k=1}^{\infty} s_k^{p_n}(A_n)$ convergent, then $A \in C_p(H)$.*

Remark 3.6. The similar problems can be studied for the block operator-matrices in form

$$A = \begin{pmatrix} 0 & \dots & \dots & A_1 & & & & & \\ & 0 & & & A_2 & & & & \\ & & 0 & & & A_3 & & 0 & \\ & & & \ddots & & & \ddots & & \\ & & 0 & & 0 & & & A_n & \\ & & & & & \ddots & & & \ddots \end{pmatrix},\ A : H \to H.$$

Example: Let

$$H_n = \left(\mathbb{C}^2, |\cdot|\right),\ n \ge 1,$$
$$A_n = \begin{pmatrix} 0 & \alpha^n \\ \alpha^n & 0 \end{pmatrix},\ n \ge 1,\ \alpha \in \mathbb{C},\ |\alpha| < 1.$$

Then, we have $\|A_n\| = |\alpha|^n$ and $\lim\limits_{n\to\infty} \|A_n\| = 0$.

In the direct sum $H = \bigoplus\limits_{n=1}^{\infty} H_n$ consider the following upper triangular one-band block operator matrix in form

$$A = \begin{pmatrix} \begin{matrix} 0 & 0 \\ & 0 \end{matrix} & \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} & & \\ & \begin{matrix} 0 & 0 \\ & 0 \end{matrix} & \begin{pmatrix} 0 & \alpha^2 \\ \alpha^2 & 0 \end{pmatrix} & \\ & & \ddots & \end{pmatrix}.$$

Then $A \in C_\infty(H)$ and characteristic numbers of A are in form

$$s_n(A) = |\alpha|^n,\ n \geq 1.$$

Moreover, for any $p,\ 1 \leq p < \infty,\ A \in \sigma_p(H)$.

References

[1] Akhmedov, A. M., El-Shabrawy, S. R., *On the Spectrum of the Generalized Lower Triangle Double-Band Matrices* , Lviv, (2010), 17-21. 1

[2] Akhmedov, A. M., El-Shabrawy, S. R., *Notes on the Spectrum of Lower Triangular Double-Band Matrices* , Thai Journal of Mathematics, **10**(2012), 415-421. 1

[3] Baliarsingh, P., Dutta, S., *On a Spectral Classification of the operator Δ_ν^r Over the Sequence Space c_0* , Proc. Math. Acad. Sci. India, Sect. A Phys., 84, **4**(2014), 555-561. 1

[4] Başar, F., Karaisa, A., *Spectrum and Fine Spectrum of the Generalized Difference Operator Defined by Double Sequential Upper Band Matrix Over the Sequence Spaces l_p, $(1 < p < \infty)$*, Hacet. J. Math., 44, **6**(2015), 1315-1332. 1

[5] Böttcher, A., Silbermann, B., Analysis of Teoplitz Operators, Berlin, Springer-Verlag, 1990. 1

[6] Böttcher, A., Grudsky, S., Toeplitz Matrices, Asymptotic Linear Algebra anf Functional Analysis, Berlin, Springer-Verlag, 1991. 1

[7] Dunford, N., Schwartz, J. T., Linear Operators I, II, Second ed., Interscience, New York, 1958; 1963. 1

[8] El-Shabrawy, S. R., *Spectra and Fine Spectra of Certain Lower Triangular Double-Band Matrices as Operator on c_0* , Journal of Inequalities and Applications, 241, **1**(2014), 2-9. 1

[9] Gohberg, I., Goldberg, S., Kaashock, M. A., Basic Classes of Linear Operators, Springer, 1990. 2

[10] Ismailov, Z. I., Otkun Çevik, E., Unluyol, E., *Compact Inverses of Multipoint Normal Differential Operators for First Order* , Electronic Journal of Differential Equations, **89**(2011), 1-11. 1

[11] Jeribi, A., Spectral Theory and Applications of Linear Operators and Block Operator Matrices, Springer, 2015. 1

[12] Karakaya, V., Dzh. Manafov, M., Şimşek, N., *On the Fine Spectrum of the Second Order Difference Operator Over the Sequence Spaces l_p and bv_p, $(1 < p < \infty)$* , Mathematical and Computer Modelling, **55**(2012), 426-431. 1

[13] Kochubei, A. N., *Symmetric Operators and Nonclassical Spectral Problems* , Mat. Zametki, 25, **3**(1979), 425-434. 1

[14] Langer, M., Strauss, M., *Spectral Properties of Unbounded $j-$Self Adjoint Block Operator Matrices* , arXiv:1410.1213 [math.SP], 1-37. 1

[15] Nagel, R. , *The Spectrum of Unbounded Operator Matrices with Non-Diagonal Domain, Journal of Functional Analysis* , **89**(1990), 291-302. 1

[16] Otkun Çevik, E., Ismailov, Z. I., *Spectrum of the Direct Sum of Operators, Electronic Journal of Differential Equations* , **210**(2012), 1-8. 1, 2, 3

[17] Tretter, Ch., Spectral Theory of Block Operator Matrices and Applications, Londan: Imperial CollegePress, 264p, 2008. 1

[18] Tripathy, B. C., Das, R., *Spectrum and Fine Spectrum of the Lower Triangular Matrix $B(r, o, s)$ Over the Sequences Spaces* , Appl. Math. Inf. Sci, 9, **4**(2015), 2139-2145. 1

[19] Zettl, A., Sturm-Liouville Theory, First ed., Amer. Math. Survey and Monographs vol. 121, USA, 2005. 1