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Hermite-Hadamard Type Integral Inequalities for Strongly p-convex Functions

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Abstract. In this paper we obtain the Hermite-Hadamard Inequality for strongly p-convex function. Using this strongly p-convex function we get the new theorem and corollary.

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1. Introduction

In recent years, several integral inequalities related to various classes of convex functions. Convex functions have played an important role in the development of various fields in pure and applied sciences. A significant class of convex functions is strongly convex functions. The strongly convex functions also play an important role in optimization theory and mathematical economics, see [7].

In this paper, we firstly list several definitions. Then, we discuss some properties of strongly p-convex functions.

Definition 1.1 ([7]). Let $I \subseteq \mathbb{R}$ be an interval and c be a positive number. A function $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ is called strongly convex with modulus $c > 0$, if

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y) - ct(1-t)\|y - x\|^2 \in I$$

for all $x, y \in I, t \in [0, 1]$.

Definition 1.2 ([10]). Let $I \subset (0, \infty)$ be a real interval and $p \in \mathbb{R} \setminus 0$. A function $f : I \to \mathbb{R}$ is said to be a p-convex function, if

$$f\left(\left[tx^p + (1-t)y^p\right]^{\frac{1}{p}}\right) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Proposition 1.3. *For $p = 1$ and $p = -1$, p-convexity reduces to ordinary convexity and harmonically convexity on $I \subset (0, \infty)$, respectively.*

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Theorem 1.4. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R} \setminus 0$ and $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then we have*

$$f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{b^p - a^p}\int_a^b f(x)x^{p-1}dx \leq \frac{f(a)+f(b)}{2}$$

This inequality known as Hermite-Hadamard inequality for p-convex function.

Some studies related to convex and strongly convex functions and several different types of them can be found in the literature, for example , in [1–6, 8, 9].

2. Main Results

In this section, we derive Hermite-Hadamard inequalities for strongly p-convex function.

Definition 2.1. Let $I \subset (0,\infty)$ be a interval, $f : I \to \mathbb{R}$ is said to be strongly p-convex function with modulus $c > 0$, if

$$f\left(\left[(1-t)x^p + ty^p\right]^{\frac{1}{p}}\right) \leq (1-t)f(x) + tf(y) - ct(1-t)\|y^p - x^p\|^2 \tag{2.1}$$

for all $x, y \in I$ and $t \in [0,1]$.

Lemma 2.2. *A function $f : I \subset (0,\infty) \to \mathbb{R}$ is strongly p-convex function with modulus $c > 0$, if and only if, the function $g(x) = f(x) - c\|x^p\|^2$ is p-convex function.*

Proof. Assume that f is strongly p-convex function with modulus $c > 0$.Using properties of the inner product, we have

$$\begin{aligned}
g\left(\left[tx^p + (1-t)y^p\right]^{\frac{1}{p}}\right) &= f\left(\left[tx^p + (1-t)y^p\right]^{\frac{1}{p}}\right) - c\|tx^p + (1-t)y^p\|^2 \\
&\leq tf(x) + (1-t)f(y) - ct(1-t)\|y^p - x^p\|^2 - c\left(t^2\|x^p\|^2 + 2t(1-t)x^py^p + (1-t)^2\|y^p\|^2\right) \\
&\leq tf(x) + (1-t)f(y) - c\left(t(1-t)\|y^p\|^2 - 2t(1-t)x^py^p + t(1-t)\|x^p\|^2 + t^2\|x^p\|^2 + 2t(1-t)x^py^p + (1-t)^2\|y^p\|^2\right) \\
&\leq tf(x) + (1-t)f(y) - c\left((1-t)\|y^p\|^2 + t\|x^p\|^2\right) \\
&\leq tf(x) - ct\|x^p\|^2 + (1-t)f(y) - c(1-t)\|y^p\|^2 \\
&= tg(x) + (1-t)g(y)
\end{aligned}$$

which gives that g is p-convex function.

Conversely, if g is p-convex function, then we have

$$\begin{aligned}
f\left(\left[tx^p + (1-t)y^p\right]^{\frac{1}{p}}\right) &= g\left(\left[tx^p + (1-t)y^p\right]^{\frac{1}{p}}\right) + c\|tx^p + (1-t)y^p\|^2 \\
&\leq tg(x) + (1-t)g(y) + c\left(t^2\|x^p\|^2 + 2t(1-t)x^py^p + (1-t)^2\|y^p\|^2\right) \\
&\leq tg(x) + (1-t)g(y) + c(1-t)\|y^p\|^2 - ct(1-t)\|y^p\|^2 + 2ct(1-t)x^py^p + ct\|x^p\|^2 - ct(1-t)\|x^p\|^2 \\
&= tg(x) + ct\|x^p\|^2 + (1-t)g(y) + c(1-t)\|y^p\|^2 - ct(1-t)\|y^p\|^2 + 2ct(1-t)x^py^p - ct(1-t)\|x^p\|^2 \\
&= tf(x) + (1-t)f(y) - ct(1-t)\|y^p - x^p\|^2
\end{aligned}$$

which shows that f is strongly p-convex with modulus $c > 0$. □

Theorem 2.3. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a strongly p-convex function with modulus $c > 0$ and all $x, y \in I$, $t \in [0,1]$. If $f \in L[a,b]$, then*

$$f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) + \frac{c}{12}\|b^p - a^p\|^2 \leq \frac{p}{b^p - a^p}\int_a^b f(x)x^{p-1}dx \leq \frac{f(a)+f(b)}{2} - \frac{c}{6}\|b^p - a^p\|^2 \tag{2.2}$$

Proof. Since $f : I \subset (0,\infty) \to \mathbb{R}$ be a strongly p-convex function, we have, $\forall x, y \in I$, (with $t = \frac{1}{2}$ in the inequality (2.1)).

$$f\Big(\Big[\frac{x^p}{2} + \frac{y^p}{2}\Big]^{\frac{1}{p}}\Big) \le \frac{f(x) + f(y)}{2} - \frac{c}{4}\|y^p - x^p\|^2$$

Choosing $x = [(1-t)a^p + tb^p]^{\frac{1}{p}}, y = [ta^p + (1-t)b^p]^{\frac{1}{p}}$ we get

$$f\Big(\Big[\frac{a^p + b^p}{2}\Big]^{\frac{1}{p}}\Big) \le \frac{f\Big([(1-t)a^p + tb^p]^{\frac{1}{p}} + f\Big([ta^p + (1-t)b^p]^{\frac{1}{p}}\Big)}{2} - \frac{c}{4}\|(1-t)a^p + tb^p - ta^p - (1-t)b^p\|^2$$

By integrating for $t \in [0,1]$, we have

$$f\Big(\Big[\frac{a^p + b^p}{2}\Big]^{\frac{1}{p}}\Big) \le \frac{1}{2}\Big[\int_0^1 f\Big([(1-t)a^p + tb^p]^{\frac{1}{p}} + \int_0^1 f\Big([ta^p + (1-t)b^p]^{\frac{1}{p}}\Big)dt\Big] - \frac{c}{4}\|b^p - a^p\|^2 \int_0^1 (2t-1)^2 dt$$

$$f\Big(\Big[\frac{a^p + b^p}{2}\Big]^{\frac{1}{p}}\Big) + \frac{c}{12}\|b^p - a^p\|^2 \le \frac{p}{b^p - a^p}\int_a^b f(x)x^{p-1}dx$$

We get the left hand side of the inequality (2.2). Furthermore, we observe that for all $t \in [0,1]$

$$f\Big([(1-t)a^p + tb^p]^{\frac{1}{p}} \le (1-t)f(a) + tf(b) - ct(1-t)\|b^p - a^p\|^2$$

By integrating this inequality with respect to t over $[0,1]$, we have the right-hand side of the inequality (2.2).

$$\le \int_0^1 \Big((1-t)f(a) + tf(b)\Big)dt - c\|b^p - a^p\|^2 \int_0^1 t(1-t)dt = \frac{f(a) + f(b)}{2} - \frac{c}{6}\|b^p - a^p\|^2$$

□

Theorem 2.4. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a strongly p-convex function with modulus $c > 0$, and $\forall x, y \in I, t \in [0,1]$. Then*

$$f\Big(\Big[\frac{a^p + b^p}{2}\Big]^{\frac{1}{p}}\Big) + \frac{c}{12}\|b^p - a^p\|^2 \le \phi(x)$$

$$\le \frac{p}{b^p - a^p}\int_a^b f(x)x^{p-1}dx$$

$$\psi(x) \le \frac{f(a) + f(b)}{2} - \frac{c}{6}\|b^p - a^p\|^2$$

where

$$\phi(x) = \frac{1}{2}\Big[f\Big(\Big[\frac{3a^p + b^p}{4}\Big]^{\frac{1}{p}}\Big) + f\Big(\Big[\frac{a^p + 3b^p}{4}\Big]^{\frac{1}{p}}\Big)\Big] + \frac{c}{48}\|b^p - a^p\|^2$$

$$\psi(x) = \frac{1}{2}\Big[f\Big(\Big[\frac{a^p + b^p}{2}\Big]^{\frac{1}{p}}\Big) + \frac{f(a) + f(b)}{2}\Big] - \frac{c}{24}\|b^p - a^p\|^2$$

Proof. By applying (2.2) on each of the interval $\left[a,\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right]$ ve $\left[\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}},b\right]$, we have

$$f\left(\left[\frac{a^p+\frac{a^p+b^p}{2}}{2}\right]^{\frac{1}{p}}\right)+\frac{c}{12}\left\|\left(\frac{a^p+b^p}{2}\right)-a^p\right\|^2$$

$$\leq \frac{p}{\frac{a^p+b^p}{2}-a^p}\int_a^{\frac{a^p+b^p}{2}} f(x)x^{p-1}dx$$

$$\leq \frac{f(a)+f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)}{2}-\frac{c}{6}\left\|\frac{a^p+b^p}{2}-a^p\right\|^2$$

$$f\left(\left[\frac{3a^p+b^p}{4}\right]^{\frac{1}{p}}\right)+\frac{c}{48}\|b^p-a^p\|^2$$

$$\leq \frac{2p}{b^p-a^p}\int_a^{\frac{a^p+b^p}{2}} f(x)x^{p-1}dx \leq \frac{1}{2}\left[f(a)+f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right]-\frac{c}{24}\|b^p-a^p\|^2$$

and similarly,

$$f\left(\left[\frac{a^p+3b^p}{4}\right]^{\frac{1}{p}}\right)+\frac{c}{48}\|b^p-a^p\|^2$$

$$\leq \frac{2p}{b^p-a^p}\int_{\frac{a^p+b^p}{2}}^b f(x)x^{p-1}dx \leq \frac{1}{2}\left[f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+f(b)\right]-\frac{c}{24}\|b^p-a^p\|^2$$

Summing up side by side, we obtain

$$\phi(x)=\frac{1}{2}\left[f\left(\left[\frac{3a^p+b^p}{4}\right]^{\frac{1}{p}}\right)+f\left(\left[\frac{a^p+3b^p}{4}\right]^{\frac{1}{p}}\right)\right]+\frac{c}{48}\|b^p-a^p\|^2$$

$$\leq \frac{p}{b^p-a^p}\int_a^b f(x)x^{p-1}dx \leq \frac{1}{2}\left[f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+\frac{f(a)+f(b)}{2}\right)\right]-\frac{c}{24}\|b^p-a^p\|^2$$

$$\leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+\frac{f(a)+f(b)}{2}-\frac{c}{4}\|b^p-a^p\|^2\right]-\frac{c}{24}\|b^p-a^p\|^2$$

$$\leq \frac{f(a)+f(b)}{2}-\frac{c}{6}\|b^p-a^p\|^2$$

□

Theorem 2.5. *Let $f, g : I \subset (0, \infty) \to \mathbb{R}$ be a strongly p-convex function with modulus $c > 0$. If $f, g \in L[a, b]$, then*

$$\frac{p}{b^p - a^p} \int_a^b f(x) g\Big([a^p + b^p - x^p]^{\frac{1}{p}}\Big) x^{p-1} dx$$

$$\leq \frac{1}{6} M(a, b) + \frac{1}{3} N(a, b) - \frac{c}{12} \|b^p - a^p\|^2 S(a, b) - \frac{c^2}{30} \|b^p - a^p\|^4$$

where

$$M(a, b) = f(a)g(a) + f(b)g(b) \tag{2.3}$$

$$N(a, b) = f(a)g(b) + f(b)g(a) \tag{2.4}$$

$$S(a, b) = f(a) + f(b) + g(a) + g(b) \tag{2.5}$$

Proof. Let f, g be strongly p-convex functions with modulus $c > 0$. Then

$$\frac{p}{b^p - a^p} \int_a^b f(x) g\Big([a^p + b^p - x^p]^{\frac{1}{p}}\Big) x^{p-1} dx$$

$$= \int_0^1 f\Big([(1-t)a^p + tb^p]^{\frac{1}{p}}\Big) g\Big([ta^p + (1-t)b^p]^{\frac{1}{p}}\Big) dt$$

$$\leq \int_0^1 \Big[(1-t)f(a) + tf(b) - ct(1-t)\|b^p - a^p\|^2\Big]\Big[tg(a) + (1-t)g(b) - ct(1-t)\|b^p - a^p\|^2\Big] dt$$

$$= f(a)g(b) \int_0^1 (1-t)^2 dt + f(b)g(a) \int_0^1 t^2 dt + [f(a)g(a) + f(b)g(b)] \int_0^1 t(1-t) dt$$

$$-c\|b^p - a^p\|^2 [f(a) + g(b)] \int_0^1 t(1-t)^2 dt - c\|b^p - a^p\|^2 [f(b) + g(a)] \int_0^1 t^2(1-t) dt - c^2 \|b^p - a^p\|^4 \int_0^1 t^2(1-t)^2 dt$$

$$= \frac{f(a)g(a) + f(b)g(b)}{6} + \frac{f(a)g(b) + f(b)g(a)}{3} - \frac{c}{12} \|b^p - a^p\|^2 [f(a) + f(b) + g(a) + g(b)] - \frac{c^2}{30} \|b^p - a^p\|^4$$

$$= \frac{1}{6} M(a, b) + \frac{1}{3} N(a, b) - \frac{c}{12} \|b^p - a^p\|^2 S(a, b) - \frac{c^2}{30} \|b^p - a^p\|^4$$

□

If $f = g$ in Theorem 2.5, then it reduces to the following result.

Corollary 2.6. *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a strongly p-convex function with modulus $c > 0$. If $f \in L[a, b]$, then*

$$\frac{p}{b^p - a^p} \int_a^b f(x) f\Big([a^p + b^p - x^p]^{\frac{1}{p}}\Big) x^{p-1} dx$$

$$\leq \frac{f^2(a) + f^2(b)}{6} + \frac{2[f(a)f(b)]}{3} - \frac{c}{6} \|b^p - a^p\|^2 [f(a) + f(b)] - \frac{c^2}{30} \|b^p - a^p\|^4$$

Theorem 2.7. *Let $f, g : I \subset (0, \infty) \to \mathbb{R}$ be a strongly p-convex function with modulus $c > 0$.If $fg \in L[a, b]$, then*

$$\frac{p}{b^p - a^p} \int_a^b f(x) g(x) x^{p-1} dx \leq \frac{1}{3} M(a, b) + \frac{1}{6} N(a, b) - \frac{c}{12} \|b^p - a^p\|^2 S(a, b) - \frac{c^2}{30} \|b^p - a^p\|^4$$

where $M(a, b), N(a, b), S(a, b)$ are given by (2.3), (2.4) and (2.5) respectively.

Proof. Let f, g be strongly p-convex functions with modulus $c > 0$. Then

$$\frac{p}{b^p - a^p}\int_a^b f(x)g(x)x^{p-1}dx = \int_0^1 f\Big(\big[(1-t)a^p + tb^p\big]^{\frac{1}{p}}\Big)g\Big(\big[(1-t)a^p + tb^p\big]^{\frac{1}{p}}\Big)dt$$

$$\leq \int_0^1 \Big[(1-t)f(a) + tf(b) - ct(1-t)\|b^p - a^p\|^2\Big]\Big[(1-t)g(a) + tg(b) - ct(1-t)\|b^p - a^p\|^2\Big]dt$$

$$= f(a)g(a)\int_0^1 (1-t)^2dt + f(b)g(b)\int_0^1 t^2dt + [f(a)g(b) + f(b)g(a)]\int_0^1 t(1-t)dt$$

$$-c\|b^p - a^p\|^2[f(a) + g(a)]\int_0^1 t(1-t)^2dt - c\|b^p - a^p\|^2[f(b) + g(b)]\int_0^1 t^2(1-t)dt - c^2\|b^p - a^p\|^4\int_0^1 t^2(1-t)^2dt$$

$$\frac{f(a)g(a) + f(b)g(b)}{3} + \frac{f(a)g(b) + f(b)g(a)}{6} - \frac{c}{12}\|b^p - a^p\|^2[f(a) + f(b) + g(a) + g(b)] - \frac{c^2}{30}\|b^p - a^p\|^4$$

$$\frac{1}{3}M(a,b) + \frac{1}{6}N(a,b) - \frac{c}{12}\|b^p - a^p\|^2 S(a,b) - \frac{c^2}{30}\|b^p - a^p\|^4$$

□

If $f = g$ in Theorem 2.7, then it reduces to the following result.

Corollary 2.8. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a strongly p-convex function with modulus $c > 0$. If $f \in L[a,b]$, then*

$$\frac{p}{b^p - a^p}\int_a^b f^2(x)x^{p-1}dx$$

$$\leq \frac{[f(a)f(b)]}{3} + \frac{f^2(a) + f^2(b)}{3} - \frac{c}{6}\|b^p - a^p\|^2[f(a) + f(b)] - \frac{c^2}{30}\|b^p - a^p\|^4$$

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