http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 13.05.2018 *Year*: **2019**, *Number*: **26**, *Pages*: **1-12**
Published: 01.01.2019 *Original Article*

Generalized Forms of Upper and Lower Continuous Fuzzy Multifunctions

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Abstract − In this paper, we introduce the concepts of upper and lower $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunctions. It is in order to unify several characterizations and properties of some kinds of modifications of fuzzy upper and fuzzy lower semi-continuous fuzzy multifunctions, and to deduce a generalized form of these concepts, namely upper and lower $\eta\eta^*$-continuous fuzzy multifunctions.

Keywords − *General topology; fuzzy topology; multifunction; fuzzy multifunction.*

1 Introduction

Fuzzy multifunctions or multi-valued mappings have many applications in mathematical programming, probability, statistics, different inclusions, fixed point theorems and even in economics, and continuous fuzzy multifunctions have been generalized in manu ways. Many Mathematicians, see [1 - 6], devoted a great part of their research work on studying the generalized continuous fuzzy multifunctions, where their fuzzy fuzzy multifunction maps each point in a classical topological space into an arbitrary fuzzy set in a fuzzy topological space in the sense of Chang [7].

In this paper, we introduce the concepts of upper and lower $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunctions and prove that if α, β are operators on the topological space (X, T) and θ, θ^*, δ are fuzzy operators on the fuzzy topological space (Y, τ) in Šostak sense [8], and ℓ is a proper ideal on X, then a fuzzy multifunction $F : X \multimap Y$ is upper (resp. lower) $(\alpha, \beta, \theta \sqcap \theta^*, \delta, \ell)$-continuous fuzzy multifunction iff F is both of upper (resp. lower) $(\alpha, \beta, \theta, \delta, \ell)$-continuous and upper (resp. lower) $(\alpha, \beta, \theta^*, \delta, \ell)$-continuous fuzzy multifunction. Also, we introduce new generalized notions that

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cover many of the generalized forms of upper (resp. lower) semi-continuous fuzzy multifunctions.

2 Preliminaries

Throughout this paper, X refers to an initial universe, 2^X denotes the power set of X, I^X denotes the set of all fuzzy sets of X, $\lambda^c(x) = 1 - \lambda(x)\ \forall x \in X$ (where $I = [0,1], I_0 = (0,1]$).

As applications, $\alpha, \beta, id_X : 2^X \to 2^X$ are operators on X and $\theta, \delta, id_Y : I^Y \times I_0 \to I^Y$ are fuzzy operators on Y. Recall that an ideal ℓ on X [9], is a collection $\ell \subseteq 2^X$ that satisfies the following conditions:

(1) $A \in \ell$ and $B \subseteq A$ implies that $B \in \ell$,

(2) $A \in \ell$ and $B \in \ell$ implies that $A \cup B \in \ell$.

ℓ is proper if $X \notin \ell$. Let by (X, T) and (Y, τ) be meant the classical and the fuzzy topological spaces due to Šostak [8], respectively. The closure and the interior of any set A in (X, T) will be denoted by T-cl(A) and T-int(A) while the fuzzy closure and the fuzzy interior of any fuzzy set $\mu \in I^Y$ will be denoted by $\mathrm{cl}_\tau(\mu, r)$ and $\mathrm{int}_\tau(\mu, r)$. The notion of quasi-coincidence is given for two fuzzy sets $\lambda, \mu \in I^Y$, denoted by $\lambda\ q\ \mu$, iff there exists a $y \in Y$ such that $\lambda(y) + \mu(y) > 1$. If they are not quasi-coincidence, it will be denoted by $\lambda\ \hat{q}\ \mu$. Any fuzzy set $\mu \in I^Y$ is called r-fuzzy semi-closed [10] (resp. r-fuzzy preclosed [11]) iff $\mu \geq \mathrm{int}_\tau(\mathrm{cl}_\tau(\mu, r), r)$ (resp. $\mu \geq \mathrm{cl}_\tau(\mathrm{int}_\tau(\mu, r), r)$),
while

$$\mathrm{s\,cl}_\tau(\lambda, r) = \bigwedge\{\mu : \lambda \leq \mu \ \text{ and } \quad \mu \text{ is } \ r-\text{fuzzy semi-closed}\}$$

and

$$\mathrm{pre\,cl}_\tau(\lambda, r) = \bigwedge\{\mu : \lambda \leq \mu \ \text{ and } \quad \mu \text{ is } \ r-\text{fuzzy preclosed}\}.$$

Also, $A \subseteq X$ is strongly semi-open [12] (resp. semi-preopen [12]) if

$$A \subseteq T-\mathrm{int}(T-\mathrm{cl}(T-\mathrm{int}(A))) \ \ (\text{ resp. } A \subseteq T-\mathrm{cl}(T-\mathrm{int}(T-\mathrm{cl}(A)))),$$

while

$$T-\mathrm{ss\,int}(A) = \bigcup\{B : B \subseteq A \ \text{ and } \quad B \text{ is } \ \text{strongly semi-open}\}$$

and

$$T-\mathrm{spre\,int}(A) = \bigcup\{B : B \subseteq A \ \text{ and } \quad B \text{ is } \ \text{semi-preopen}\}.$$

A mapping $F : X \multimap Y$ is called a fuzzy multifunction [1] if for each $x \in X$, $F(x)$ is a fuzzy set in Y. The upper inverse $F^+(\lambda)$ and the lower inverse $F^-(\lambda)$ of $\lambda \in I^Y$ are defined as follows:

$F^+(\lambda) = \{x \in X : F(x) \leq \lambda\}$ and $F^-(\lambda) = \{x \in X : F(x)\ q\ \lambda\}$.

For $A \subseteq X$, $F(A) = \bigvee\{F(x) : x \in A\}$. Also, $F^-(\lambda^c) = X - F^+(\lambda)$ for any $\lambda \in I^Y$.

3 Upper and Lower $(\alpha, \beta, \theta, \delta, \ell)$-continuous Fuzzy Multifunctions

Definition 3.1. *A mapping $F : (X, T) \multimap (Y, \tau)$ is said to be upper (resp. lower) $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction if for every $\mu \in I^Y$, $r \in I_0$, with $\tau(\mu) \geq r$,*

$$\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r))) \in \ell \quad (\text{resp. } \alpha(F^-(\delta(\mu, r))) - \beta(F^-(\theta(\mu, r))) \in \ell).$$

We can see that the above definition generalizes the concept of upper (resp. lower) semi-continuous fuzzy multifunction [13] when we choose α = identity operator, β = interior operator, δ = r-fuzzy identity operator, θ = r-fuzzy identity operator and $\ell = \{\emptyset\}$.

Let us give a historical justification of the definition:

(1) In 2015, Ramadan and Abd El-Latif [13], defined the concept of upper (resp. lower) almost continuous fuzzy multifunction as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then

$$F^+(\mu) \subseteq T\text{-int}(F^+(\text{int}_\tau(\text{cl}_\tau(\mu, r), r))) \quad (\text{resp. } F^-(\mu) \subseteq T\text{-int}(F^-(\text{int}_\tau(\text{cl}_\tau(\mu, r), r)))).$$

Here, α = identity operator, β = interior operator, δ = r-fuzzy identity operator, θ = r-fuzzy interior closure operator and $\ell = \{\emptyset\}$.

(2) In 2015, Ramadan and Abd El-Latif [13], defined the concept of upper (resp. lower) weakly continuous fuzzy multifunction as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then

$$F^+(\mu) \subseteq T - \text{int}(F^+(\text{cl}_\tau(\mu, r))) \quad (\text{resp. } F^-(\mu) \subseteq T - \text{int}(F^-(\text{cl}_\tau(\mu, r)))).$$

Here, α = identity operator, β = interior operator, δ = r-fuzzy identity operator, θ = r-fuzzy closure operator and $\ell = \{\emptyset\}$.

(3) The concept of upper (resp. lower) almost weakly continuous fuzzy multifunction is defined as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then

$$F^+(\mu) \subseteq T\text{-int}(T\text{-cl}(F^+(\text{cl}_\tau(\mu, r)))) \quad (\text{resp. } F^-(\mu) \subseteq T\text{-int}(T\text{-cl}(F^-(\text{cl}_\tau(\mu, r))))).$$

Here, α = identity operator, β = interior closure operator, δ = r-fuzzy identity operator, θ = r-fuzzy closure operator and $\ell = \{\emptyset\}$.

(4) The concept of upper (resp. lower) strongly semi-continuous fuzzy multifunction is defined as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then

$$F^+(\mu) \subseteq T\text{-int}(T\text{-cl}(T\text{-int}(F^+(\mu)))) \quad (\text{resp. } F^-(\mu) \subseteq T\text{-int}(T\text{-cl}(T\text{-int}(F^-(\mu))))).$$

Here, α = identity operator, β = interior closure interior operator, δ = r-fuzzy identity operator, θ = r-fuzzy identity operator and $\ell = \{\emptyset\}$.

(5) The concept of upper (resp. lower) almost strongly semi-continuous fuzzy multifunction is defined as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then
$F^+(\mu) \subseteq T\text{-ss}\,\text{int}(F^+(\text{s}\,\text{cl}_\tau(\mu, r)))$ (resp. $F^-(\mu) \subseteq T\text{-ss}\,\text{int}(F^-(\text{s}\,\text{cl}_\tau(\mu, r))))$.
Here, $\alpha =$ identity operator, $\beta =$ strongly semi-interior operator, $\delta = r$-fuzzy identity operator, $\theta = r$-fuzzy semi-closure operator and $\ell = \{\emptyset\}$.

(6) The concept of upper (resp. lower) weakly strongly semi-continuous fuzzy multifunction is defined as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then

$$F^+(\mu) \subseteq T - \text{int}(T - \text{cl}(T - \text{int}(F^+(\text{cl}_\tau(\mu, r)))))$$

$$(\text{resp. } F^-(\mu) \subseteq T - \text{int}(T - \text{cl}(T - \text{int}(F^-(\text{cl}_\tau(\mu, r)))))).$$

Here, $\alpha =$ identity operator, $\beta =$ interior closure interior operator, $\delta = r$-fuzzy identity operator, $\theta = r$-fuzzy closure operator and $\ell = \{\emptyset\}$.

(7) The concept of upper (resp. lower) semi-precontinuous fuzzy multifunction is defined as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then
$F^+(\mu) \subseteq T\text{-cl}(T\text{-int}(T\text{-cl}(F^+(\mu))))$ (resp. $F^-(\mu) \subseteq T\text{-cl}(T\text{-int}(T\text{-cl}(F^-(\mu)))))$.
Here, $\alpha =$ identity operator, $\beta =$ closure interior closure operator, $\delta = r$-fuzzy identity operator, $\theta = r$-fuzzy identity operator and $\ell = \{\emptyset\}$.

(8) The concept of upper (resp. lower) almost semi-precontinuous fuzzy multifunction is defined as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then
$F^+(\mu) \subseteq T\text{-spre}\,\text{int}(F^+(\text{s}\,\text{cl}_\tau(\mu, r)))$ (resp. $F^-(\mu) \subseteq T\text{-spr}\,\text{int}(F^-(\text{s}\,\text{cl}_\tau(\mu, r))))$.
Here, $\alpha =$ identity operator, $\beta =$ semi-preinterior operator, $\delta = r$-fuzzy identity operator, $\theta = r$-fuzzy semi-closure operator and $\ell = \{\emptyset\}$.

(9) The concept of upper (resp. lower) weakly semi-precontinuous fuzzy multifunction is defined as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then

$$F^+(\mu) \subseteq T - \text{cl}(T - \text{int}(T - \text{cl}(F^+(\text{cl}_\tau(\mu, r)))))$$

$$(\text{resp. } F^-(\mu) \subseteq T - \text{cl}(T - \text{int}(T - \text{cl}(F^-(\text{cl}_\tau(\mu, r)))))).$$

Here, $\alpha =$ identity operator, $\beta =$ closure interior closure operator, $\delta = r$-fuzzy identity operator, $\theta = r$-fuzzy closure operator and $\ell = \{\emptyset\}$.

(10) The concept of upper (resp. lower) precontinuous fuzzy multifunction is defined as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then

$$F^+(\mu) \subseteq T - \text{int}(T - \text{cl}(F^+(\mu))) \quad (\text{resp. } F^-(\mu) \subseteq T - \text{int}(T - \text{cl}(F^-(\mu)))).$$

Here, $\alpha =$ identity operator, $\beta =$ interior closure operator, $\delta = r$-fuzzy identity operator, $\theta = r$-fuzzy identity operator and $\ell = \{\emptyset\}$.

(11) The concept of upper (resp. lower) strongly precontinuous fuzzy multifunction as: For every $\mu \in I^Y, r \in I_0$ with $\tau(\mu) \geq r$, then

$$F^+(\mu) \subseteq T\text{−int}(T\text{−pre cl}(F^+(\mu))) \text{ (resp. } F^-(\mu) \subseteq T\text{−int}(T\text{−pre cl}(F^-(\mu)))).$$

Here, α = identity operator, β = interior preclosure operator, δ = r-fuzzy identity operator, $\theta = r$-fuzzy identity operator and $\ell = \{\emptyset\}$.

Definition 3.2. *A mapping $F : (X, T) \multimap (Y, \tau)$ is called upper (resp. lower) P-continuous fuzzy multifunction* iff *$F^+(\mu) \in T$ (resp. $F^-(\mu) \in T$) for every $\mu \in I^Y$, $r \in I_0$, with $\tau(\mu) \geq r$, such that μ satisfies the property P.*

Let $\theta_P : I^Y \times I_0 \to I^Y$ be a fuzzy operator defined as:

$$\theta_P(\mu, r) = \begin{cases} \mu & \text{if } \mu \in I^Y, r \in I_0 \text{ with } \tau(\mu) \geq r \text{ and } \mu \text{ satisfies the property } P, \\ \overline{1} & \text{otherwise} \end{cases}$$

Theorem 3.3. A map $F : (X, T) \multimap (Y, \tau)$ is upper (resp. lower) P-continuous fuzzy multifunction iff it is upper (resp. lower) $(id_X, T\text{-int}, \theta_P, id_Y, \{\emptyset\})$-continuous fuzzy multifunction.

Proof. Suppose that F is an upper P-continuous fuzzy multifunction and let $\mu \in I^Y$, $r \in I_0$ with $\tau(\mu) \geq r$.

Case 1. If μ satisfies the property P, $\theta_P(\mu, r) = \mu$, and then by hypothesis $F^+(\mu) \in T$ and $F^+(\mu) \subseteq T\text{-int}(F^+(\mu)) = T\text{-int}(F^+(\theta_P(\mu, r)))$.

Case 2. μ does not satisfy the property P, then $\theta_P(\mu, r) = \overline{1}$, and thus $F^+(\mu) \subseteq X = T\text{-int}(F^+(\theta_P(\mu, r)))$. That is, F is upper $(id_X, T\text{-int}, \theta_P, id_Y, \{\emptyset\})$-continuous fuzzy multifunction.

Conversely, suppose that $F^+(\mu) \subseteq T\text{-int}(F^+(\theta_P(\mu, r)))$ for each $\mu \in I^Y$, $r \in I_0$ with $\tau(\mu) \geq r$. Take μ satisfying the property P, then $\theta_P(\mu, r) = \mu$, and thus $F^+(\mu) \subseteq T\text{-int}(F^+(\theta_P(\mu, r))) = T\text{-int}(F^+(\mu))$. We conclude that $F^+(\mu) \in T$ and thus F is an upper P-continuous fuzzy multifunction.

For lower P-continuous fuzzy multifunction, the proof is similar.

Definition 3.4. *If γ and γ^* are fuzzy operators on X, then the operator $\gamma \sqcap \gamma^*$ is defined as follows:*

$$(\gamma \sqcap \gamma^*)(\lambda, r) = \gamma(\lambda, r) \wedge \gamma^*(\lambda, r) \quad \forall \lambda \in I^X, \ r \in I_0.$$

The fuzzy operators γ and γ^* are said to be mutually dual if $\gamma \sqcap \gamma^*$ is the identity operator.

Theorem 3.5. Let (X, T) be a topological space, (Y, τ) a fuzzy topological space and ℓ a proper ideal on X. Let α, β, β^* be operators on (X, T) and δ, θ, θ^* be fuzzy operators on (Y, τ). Then $F : X \multimap Y$ is:

(1) upper (resp. lower) $(\alpha, \beta, \theta \sqcap \theta^*, \delta, \ell)$-continuous fuzzy multifunction iff it is both upper (resp. lower) $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction and upper (resp. lower) $(\alpha, \beta, \theta^*, \delta, \ell)$-continuous fuzzy multifunction provided that for all $A, B \subseteq X$, we have $\beta(A \cap B) = \beta(A) \cap \beta(B)$.

(2) upper (resp. lower) $(\alpha, \beta \sqcap \beta^*, \theta, \delta, \ell)$-continuous fuzzy multifunction iff it is both upper (resp. lower) $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction and upper (resp. lower) $(\alpha, \beta^*, \theta, \delta, \ell)$-continuous fuzzy multifunction.

Proof. (1) If F is both upper $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction and upper $(\alpha, \beta, \theta^*, \delta, \ell)$-continuous fuzzy multifunction, then, for every $\mu \in I^Y$, $r \in I_0$, with $\tau(\mu) \geq r$, we have

$$\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r))) \in \ell \ \text{ and } \ \alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta^*(\mu, r))) \in \ell$$

and then

$$(\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r)))) \ \cup \ (\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta^*(\mu, r)))) \ \in \ \ell.$$

But

$$\begin{aligned} &(\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r)))) \ \cup \ (\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta^*(\mu, r)))) \\ &\left\{ \begin{array}{ll} = & \alpha(F^+(\delta(\mu, r))) - (\beta(F^+(\theta(\mu, r))) \ \cap \ \beta(F^+(\theta^*(\mu, r)))) \\ = & \alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r) \wedge \theta^*(\mu, r))) \\ = & \alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta \sqcap \theta^*(\mu, r))). \end{array} \right. \end{aligned}$$

That is, F is upper $(\alpha, \beta, \theta \sqcap \theta^*, \delta, \ell)$-continuous fuzzy multifunction.

Conversely; if F is upper $(\alpha, \beta, \theta \sqcap \theta^*, \delta, \ell)$-continuous fuzzy multifunction, then

$$\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta \sqcap \theta^*(\mu, r))) \ \in \ \ell$$

Now, by the above equalities, we get that

$$(\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r)))) \ \cup \ (\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta^*(\mu, r)))) \ \in \ \ell,$$

which implies that

$$\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r))) \in \ell \ \text{ and } \ \alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta^*(\mu, r))) \in \ell$$

which means that F is both upper $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction and upper $(\alpha, \beta, \theta^*, \delta, \ell)$-continuous fuzzy multifunction.

(2) Similar to the proof in (1).

The proof for lower continuity is typical.

Let Φ be the set of all operators on the topological space (X, T). Then a partial order could be defined by the relation:

$$\alpha \sqsubseteq \beta \quad \text{iff} \quad \alpha(A) \subseteq \beta(A) \text{ for all } A \in 2^X \ [14].$$

Theorem 3.6. Let (X, T) be a topological space, (Y, τ) a fuzzy topological space and ℓ a proper ideal on X. Let $\alpha, \alpha^*, \beta, \beta^* : 2^X \to 2^X$ be operators on (X, T) and $\delta, \theta, \theta^* : I^Y \times I_0 \to I^Y$ are fuzzy operators on (Y, τ) and $F : X \multimap Y$ is a fuzzy multifunction. Then,

(1) If β is a monotone, $\theta \sqsubseteq \theta^*$ and F is upper (resp. lower) $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction, then F is upper (resp. lower) $(\alpha, \beta, \theta^*, \delta, \ell)$-continuous fuzzy multifunction,

(2) If $\alpha^* \sqsubseteq \alpha$ and F is upper (resp. lower) $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction, then F is upper (resp. lower) $(\alpha^*, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction,

(3) If $\beta \sqsubseteq \beta^*$ and F is upper (resp. lower) $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction, then F is upper (resp. lower) $(\alpha, \beta^*, \theta^*, \delta, \ell)$-continuous fuzzy multifunction.

Proof. (1) Since F is upper $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction, then for every $\mu \in I^Y$, $r \in I_0$ with $\tau(\mu) \geq r$, it happens that

$$\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r))) \in \ell.$$

We know that $\theta \sqsubseteq \theta^*$, and then, for every $\mu \in I^Y$, $r \in I_0$, $\theta(\mu, r) \leq \theta^*(\mu, r)$, and thus $F^+(\theta(\mu, r)) \subseteq F^+(\theta^*(\mu, r))$ and $\beta(F^+(\theta(\mu, r))) \subseteq \beta(F^+(\theta^*(\mu, r)))$. Therefore,

$$\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta^*(\mu, r))) \subseteq \alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r))) \in \ell,$$

which means that F is upper $(\alpha, \beta, \theta^*, \delta, \ell)$-continuous fuzzy multifunction.

(2) and (3) are similar.

The case of lower continuity is similar.

Definition 3.7. *A fuzzy operator γ on a fuzzy topological space (X, τ) induces another fuzzy operator $(\mathrm{int}_\tau \gamma)$ defined as follows: $(\mathrm{int}_\tau \gamma)(\mu, r) = \mathrm{int}_\tau(\gamma(\mu, r), r)$. Note that: $\mathrm{int}_\tau \gamma \sqsubseteq \gamma$.*

Theorem 3.8. Let $\alpha, \beta : 2^X \to 2^X$ be operators on (X, T) and $\delta, \theta : I^Y \times I_0 \to I^Y$ are fuzzy operators on (Y, τ) and ℓ a proper ideal on X. If $F : X \multimap Y$ is an upper (resp. lower) $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunction and

$$\beta(F^+(\mu)) \subseteq \beta(F^+(\mathrm{int}_\tau(\mu, r))) \ \ (\text{ resp. } \ \beta(F^-(\mu)) \subseteq \beta(F^-(\mathrm{int}_\tau(\mu, r)))),$$

for every $\mu \in I^Y$, $r \in I_0$. Then F is upper (resp. lower) $(\alpha, \beta, \mathrm{int}_\tau \theta, \delta, \ell)$-continuous fuzzy multifunction.

Proof. Let $\mu \in I^Y$, $r \in I_0$ with $\tau(\mu) \geq r$, we have that $\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r))) \in \ell$. Since $\beta(F^+(\mu)) \subseteq \beta(F^+(\mathrm{int}_\tau(\mu, r)))$, then $\beta(F^+(\theta(\mu, r))) \subseteq \beta(F^+(\mathrm{int}_\tau \theta(\mu, r)))$. Thus, $\alpha(F^+(\delta(\mu, r))) - \beta(F^+(\mathrm{int}_\tau \theta(\mu, r))) \subseteq \alpha(F^+(\delta(\mu, r))) - \beta(F^+(\theta(\mu, r))) \in \ell$, and it follows that F is upper $(\alpha, \beta, \mathrm{int}_\tau \theta, \delta, \ell)$-continuous fuzzy multifunction.

Definition 3.9. *Let (X, τ) be a fuzzy topological space, θ is a fuzzy operator on X and $\mu \in I^X$, $r \in I_0$. Then μ is called fuzzy θ-compact if for each family $\{\lambda_j \in I^X : \tau(\lambda_j) \geq r,\ j \in J\}$ with $\mu \leq \bigvee_{j \in J} (\lambda_j)$, there exists a finite subset $J_0 \subseteq J$ such that $\mu \leq \bigvee_{j \in J_0} (\theta(\lambda_j, r))$.*

An ordinary subset $A \in 2^X$ is called fuzzy θ-compact if for each family $\{\lambda_j \in I^X : \tau(\lambda_j) \geq r,\ j \in J\}$ with $\chi_A \leq \bigvee_{j \in J} (\lambda_j)$, there exists a finite subset $J_0 \subseteq J$ such that $\chi_A \leq \bigvee_{j \in J_0} (\theta(\lambda_j, r))$.

In crisp case (X, T); a fuzzy set $K \in 2^X$ is called θ-compact if for each family $\{B_j \in 2^X : B_j \in T\}$ with $K \subseteq \bigcup_{j \in J} (B_j)$, there exists a finite subset $J_0 \subseteq J$ such that $K \subseteq \bigcup_{j \in J_0} (\theta(B_j))$.

Theorem 3.10. Let (X,T) be a topological space, (Y,τ) a fuzzy topological space, $\alpha : 2^X \to 2^X$ an operator on (X,T) with $A \subseteq \alpha(A)\ \forall A \in 2^X$ and $\delta, \theta : I^Y \times I_0 \to I^Y$ with $\delta(\lambda, r) \geq \lambda\ \ \forall \lambda \in I^Y,\ r \in I_0$ are fuzzy operators on (Y,τ). If $F : X \multimap Y$ is upper (resp. lower) $(\alpha, T\text{-int}, \theta, \delta, \{\emptyset\})$-continuous fuzzy multifunction and K is a compact subset of X, then, $F(K)$ is fuzzy θ-compact in I^Y.

Proof. Suppose that each family $\{\mu_j : j \in J, r \in I_0$ with $\tau(\mu_j) \geq r\}$ satisfies that $F(K) \leq \bigvee_{j\in J} \mu_j$. By F is upper $(\alpha, T\text{-int}, \theta, \delta, \{\emptyset\})$-continuous fuzzy multifunction, then for each $j \in J$, we have $\alpha(F^+(\delta(\mu_j, r))) \subseteq T\text{-int}(F^+(\theta(\mu_j, r))) \subseteq F^+(\theta(\mu_j, r))$. Then there exists $G_j \in T$ such that $\alpha(F^+(\delta(\mu_j, r))) \subseteq G_j \subseteq F^+(\theta(\mu_j, r))$. Also, since $F^+(\delta(\mu_j, r)) \subseteq \alpha(F^+(\delta(\mu_j, r)))$ and $\mu_j \leq \delta(\mu_j, r)$, then

$$K \subseteq F^+(F(K)) \subseteq \bigcup_{j\in J} F^+(\mu_j) \subseteq \bigcup_{j\in J} G_j.$$

From the compactness of K, there exists a finite subset J_0 of J such that $K \subseteq \bigcup_{j\in J_0} G_j$. Then

$$F(K) \leq \bigvee_{j\in J_0} F(G_j) \leq \bigvee_{j\in J_0} F(F^+(\theta(\mu_j, r))) \leq \bigvee_{j\in J_0} \theta(\mu_j, r).$$

which means that $F(K)$ is fuzzy θ-compact.

Corollary 3.11. *Let (X,T) be a topological space and (Y,τ) a fuzzy topological space. Let $F : X \multimap Y$ be an upper (resp. lower) weakly continuous fuzzy multifunction and K a compact subset of X, then $F(K)$ is a fuzzy almost compact set in I^Y.*

Proof. Take $\alpha =$ identity operator on X, $\beta = T$-int, $\delta = r$-fuzzy identity operator, $\theta = r$-fuzzy closure operator on Y and $\ell = \{\emptyset\}$. Then the result is fulfilled directly from Theorem 2.5.

Corollary 3.12. *Let (X,T) be a topological space and (Y,τ) a fuzzy topological space. Let $F : X \multimap Y$ be an upper (resp. lower) almost continuous fuzzy multifunction and K a compact subset of X, then $F(K)$ is a fuzzy nearly compact set in I^Y.*

Proof. Take $\alpha =$ identity operator on X, $\beta = T$-int, $\delta = r$-fuzzy identity operator, $\theta = r$-fuzzy closure operator on Y and $\ell = \{\emptyset\}$. Then the result follows from Theorem 2.5.

4 Upper and Lower $\eta\eta^*$-continuous Fuzzy Multifunctions

Let X and Y be nonempty sets and $\eta \subseteq 2^X$ be any collection of subsets of X and $\eta^* : I^Y \to I$ any function.

Definition 4.1. *A function $F : X \multimap Y$ is said to be upper (resp. lower) $\eta\eta^*$-continuous fuzzy multifunction if $F^+(\mu) \in \eta$ (resp. $F^-(\mu) \in \eta$) whenever $\mu \in I^Y, r \in I_0$ with $\eta^*(\mu) \geq r$.*

Remark 4.2. *A generalized topology on a set X ([15]) is a collection η of subsets of X such that $\emptyset \in \eta$ and η is closed under arbitrary unions. Also, a generalized fuzzy topology on a set Y ([15]) is a function $\eta^* : I^Y \to I$ such that $\eta^*(\overline{0}) = 1$ and $\eta^*(\bigvee_{j\in J} \mu_j) \geq \bigwedge_{j\in J}(\eta^*(\mu_j))\ \forall \mu_j \in I^Y$. Observe that if Definition 3.1, η and η^* are generalized topology and generalized fuzzy topology on X and Y respectively, then we just obtain the notion of upper (resp. lower) η, η^*-continuous fuzzy multifunctions. In [16], Maki et al., introduced the notion of minimal structure on a set X, as the collection m_X of subsets of X such that $\emptyset \in m_X$ and $X \in m_X$. Also, in [17], Yoo et al., introduced the notion of fuzzy minimal structure on a set Y, as $m_Y : I^Y \to I$ such that $m_Y(\overline{0}) = m_Y(\overline{1}) = 1$. Now, if in Definition 3.1, $\eta = m_X$ and $\eta^* = m_Y$, we obtain the notion of upper (resp. lower) m_X, m_Y-continuous fuzzy multifunctions.*

Any collection η of subsets of a set X and any function $\eta^* : I^Y \to I$ determine in a natural form an operator $\theta_\eta : 2^X \to 2^X$ and a fuzzy operator $\theta_{\eta^*} : I^Y \times I_0 \to I^Y$ respectively, so that

$$\theta_\eta(A) \;=\; \begin{cases} A & \text{if } A \in \eta \\ X & \text{otherwise} \end{cases}$$

and

$$\theta_{\eta^*}(\mu, r) \;=\; \begin{cases} \mu & \text{if } \mu \in I^Y, r \in I_0 \text{ with } \eta^*(\mu) \geq r \\ \overline{1} & \text{otherwise} \end{cases}$$

In the case that η is a generalized topology on X and η^* is a generalized fuzzy topology on Y, we obtain other operations (see [15]) that are important for its applications:

$$\eta - \mathrm{int}(A) = \bigcup\{B : B \subseteq A \ \text{ and } \ B \in \eta\},$$

$$\eta - \mathrm{cl}(A) = \bigcap\{B : A \subseteq B \ \text{ and } \ X - B \in \eta\},$$

$$\mathrm{int}_{\eta^*}(\lambda, r) = \bigvee\{\mu : \mu \leq \lambda \ \text{ and } \ \eta^*(\mu) \geq r\},$$

$$\mathrm{cl}_{\eta^*}(\lambda, r) = \bigwedge\{\mu : \lambda \leq \mu \ \text{ and } \ \eta^*(\overline{1} - \mu) \geq r\}.$$

Note that: η-int $\subseteq id_X \subseteq \theta_\eta$ and $\mathrm{int}_{\eta^*} \sqsubseteq id_Y \sqsubseteq \theta_{\eta^*}$. Similarly, in the case of a minimal structure m_X (see [18]) and a fuzzy minimal structure m_Y (see [17]), we have

$$m_X - \mathrm{int}(A) = \bigcup\{B : B \subseteq A \ \text{ and } \ B \in m_X\},$$

$$m_X - \mathrm{cl}(A) = \bigcap\{B : A \subseteq B \ \text{ and } \ X - B \in m_X\},$$

$$\mathrm{int}_{m_Y}(\mu, r) = \bigvee\{\nu : \nu \leq \mu \ \text{ and } \ m_Y(\nu) \geq r\},$$

$$\mathrm{cl}_{m_Y}(\mu, r) = \bigwedge\{\nu : \mu \leq \nu \ \text{ and } \ m_Y(\overline{1} - \nu) \geq r\}.$$

Note that: m_X-int $\subseteq id_X \subseteq \theta_\eta$ and $\mathrm{int}_{m_Y} \sqsubseteq id_Y \sqsubseteq \theta_{\eta^*}$. Also, m_X-int$(A) = A$ if $A \in m_X$ while m_X-int$(A) \in m_X$ whenever m_X is a minimal structure with the

Maki property [16]. $\text{int}_{m_Y}(\lambda, r) = \lambda$ if $m_Y(\lambda) \geq r$ while $m_Y(m_Y\text{-int}(\lambda, r)) \geq r$ whenever m_Y is a fuzzy minimal structure with the Yoo property [17].

The following results give the relationship between upper (resp. lower) $\eta\eta^*$-continuous fuzzy multifunctions and upper (resp. lower) $(\alpha, \beta, \theta, \delta, \ell)$-continuous fuzzy multifunctions. We obtain some interesting properties of upper (resp. lower) $\eta\eta^*$-continuous fuzzy multifunctions.

Theorem 4.3. Let X and Y be nonempty sets, $\eta \subseteq 2^X$, $\eta^* : I^Y \to I$. If $X \in \eta$, then $F : X \multimap Y$ is upper (resp. lower) $\eta\eta^*$-continuous fuzzy multifunction iff $F : X \multimap Y$ is upper (resp. lower) $(\theta_\eta, id_X, \theta_{\eta^*}, id_Y, \{\emptyset\})$-continuous fuzzy multifunction.

Proof. Suppose that $F : X \multimap Y$ is upper $\eta\eta^*$-continuous fuzzy multifunction. Let $\mu \in I^Y$, $r \in I_0$, we have two cases:
Case 1. If $\eta^*(\mu) \geq r$, then $\theta_{\eta^*}(\mu, r) = \mu$ and $\theta_\eta(F^+(\mu)) = F^+(\mu)$. This follows that $\theta_\eta(F^+(id_Y(\mu, r))) = F^+(\mu) = id_X(F^+(\theta_{\eta^*}(\mu, r)))$, and consequently

$$\theta_\eta(F^+(id_Y(\mu, r))) \subseteq id_X(F^+(\theta_{\eta^*}(\mu, r))).$$

Case 2. If $\eta^*(\mu) = 0$, $\theta_{\eta^*}(\mu, r) = \overline{1}$, then $\theta_\eta(F^+(id_Y(\mu, r))) \subseteq X = F^+(\overline{1}) = id_X(F^+(\theta_{\eta^*}(\mu, r)))$. Hence,

$$\theta_\eta(F^+(id_Y(\mu, r))) - id_X(F^+(\theta_{\eta^*}(\mu, r))) = \emptyset$$

for all $\mu \in I^Y, r \in I_0$. Thus, F is an upper $(\theta_\eta, id_X, \theta_{\eta^*}, id_Y, \{\emptyset\})$-continuous fuzzy multifunction.

Necessity; suppose that F is upper $(\theta_\eta, id_X, \theta_{\eta^*}, id_Y, \{\emptyset\})$-continuous fuzzy multifunction, then $\theta_\eta(F^+(id_Y(\mu, r))) - id_X(F^+(\theta_{\eta^*}(\mu, r))) = \emptyset$ for all $\mu \in I^Y, r \in I_0$ with $\eta^*(\mu) \geq r$. This implies that $\theta_\eta(F^+(\mu)) \subseteq F^+(\theta_{\eta^*}(\mu, r)))$. Assume that there is $\nu \in I^Y, r \in I_0$ such that $\eta^*(\nu) \geq r$ and $F^+(\nu)$ does not belong to η. Then we obtain $X \subseteq F^+(\nu)$. So, $F^+(\nu) = X$. Now, our hypothesis $X \in \eta$ implies that $F^+(\nu) \in \eta$, and a contradiction. Therefore, $F^+(\mu) \in \eta$ whenever $\mu \in I^Y, r \in I_0$ with $\eta^*(\mu) \geq r$, and thus $F : X \multimap Y$ is an upper $\eta\eta^*$-continuous fuzzy multifunction.

In the case that η is a generalized topology, then the following result is obtained.

Theorem 4.4. If η is a generalized topology such that $X \in \eta$ and $\eta^* : I^Y \to I$ is a function. Then $F : X \multimap Y$ is upper (resp. lower) $\eta\eta^*$-continuous fuzzy multifunction iff $F : X \multimap Y$ is upper (resp. lower) $(id_X, \eta\text{-int}, \theta_{\eta^*}, id_Y, \{\emptyset\})$-continuous fuzzy multifunction.

Proof. Suppose that $F : X \multimap Y$ is upper $\eta\eta^*$-continuous fuzzy multifunction. Let $\mu \in I^Y$, $r \in I_0$. Then consider two cases:
Case 1. If $\eta^*(\mu) \geq r$, then $\theta_{\eta^*}(\mu, r) = \mu$ and $id_X(F^+(\mu)) = F^+(\mu) = \eta\text{-int}(F^+(\mu))$. This follows that $id_X(F^+(id_Y(\mu, r))) = F^+(\mu) = \eta\text{-int}(F^+(\theta_{\eta^*}(\mu, r)))$, and consequently

$$id_X(F^+(id_Y(\mu, r))) \subseteq \eta - \text{int}(F^+(\theta_{\eta^*}(\mu, r))).$$

Case 2. If $\eta^*(\mu) = 0$, $\theta_{\eta^*}(\mu, r) = \overline{1}$, since $X \in \eta$, then

$$id_X(F^+(id_Y(\mu, r))) \subseteq X = F^+(\overline{1}) = \eta - \text{int}(F^+(\theta_{\eta^*}(\mu, r))).$$

So,

$$id_X(F^+(id_Y(\mu,r))) \ - \ \eta - \mathrm{int}(F^+(\theta_{\eta^*}(\mu,r))) \ = \ \emptyset$$

for every $\mu \in I^Y, r \in I_0$. Thus, F is an upper $(id_X, \eta\text{-int}, \theta_{\eta^*}, id_Y, \{\emptyset\})$-continuous fuzzy multifunction.

Necessity; suppose that F is upper $(id_X, \eta\text{-int}, \theta_{\eta^*}, id_Y, \{\emptyset\})$-continuous fuzzy multifunction. Then

$$id_X(F^+(id_Y(\mu,r))) \ - \ \eta - \mathrm{int}(F^+(\theta_{\eta^*}(\mu,r))) \ = \ \emptyset$$

for every $\mu \in I^Y, r \in I_0$ with $\eta^*(\mu) \geq r$. This implies that

$$F^+(\mu) \ \subseteq \ \eta - \mathrm{int}(F^+(\theta_{\eta^*}(\mu,r))) = \eta - \mathrm{int}(F^+(\mu)).$$

Assume that there is $\nu \in I^Y, r \in I_0$ such that $\eta^*(\nu) \geq r$ and $F^+(\nu)$ does not belong to η. Then we obtain $F^+(\nu) \ \subseteq \ \eta\text{-int}(F^+(\nu))$, and thus $F^+(\nu) \ = \ \eta\text{-int}(F^+(\nu))$, and $F^+(\nu) \in \eta$, and a contradiction. Therefore, $F^+(\mu) \in \eta$ whenever $\mu \in I^Y, r \in I_0$ with $\eta^*(\mu) \geq r$, that is, $F : X \multimap Y$ is an upper $\eta\eta^*$-continuous fuzzy multifunction.

The following corollaries are direct results.

Corollary 4.5. *Let $F : X \multimap Y$ be a fuzzy fuzzy multifunction. If F is upper (resp. lower) $m_X m_Y$-continuous fuzzy multifunction, then F is upper (resp. lower) $(id_X, m_X\text{-int}, \theta_{m_Y}, id_Y, \ell)$-continuous fuzzy multifunction whenever m_X has the Maki property.*

Corollary 4.6. *Let η be a generalized topology on X and η^* a generalized fuzzy topology on Y such that $X \in \eta$. Then, $F : X \multimap Y$ is upper (resp. lower) $\eta\eta^*$-continuous fuzzy multifunction* iff *F is upper (resp. lower) $(id_X, \eta\text{-int}, \mathrm{int}_{\eta^*}, id_Y, \{\emptyset\})$-continuous fuzzy multifunction.*

5 Conclusions

In this article, we have introduced the notions of upper and lower continuous multifunctions from an ordinary topological space into a fuzzy topological space in Šostak sense.. We have investigated some of its properties. There are many other properties of the introduced notions, those could be investigated and applied for investigations in other branches of technology.

Acknowledgments

The authors are partially supported by Deanship of Scientific Research, Jazan University, Saudi Arabia.

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