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Original Article

Generalized Roughness of $(\in, \in \vee q)$-Fuzzy Ideals in Ordered Semigroups

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Abstaract — Ordered semigroups (OSGs) is a significant algebraic structure having partial ordered with associative binary operation. OSGs have broad applications in various fields such as coding theory, automata theory, fuzzy finite state machines and computer science etc. In this manuscript we investigate the notion of generalized roughness for fuzzy ideals in OSGs on the basis of isotone and monotone mappings. Then the notion of approximation is boosted to the approximation of fuzzy bi-ideals, approximations fuzzy interior ideals and approximations fuzzy quasi-ideals in OSGs and investigate their related properties. Furthermore $(\in, \in \vee q)$-fuzzy ideals are the generalization of fuzzy ideals. Also the generalized roughness for $(\in, \in \vee q)$-fuzzy ideals, fuzzy bi-ideals and fuzzy interior ideals have been studied in OSGs and discuss the basic properties on the basis of isotone and monotone mappings.

Keywords — *Fuzzy sets, Rough sets, Approximations of fuzzy ideals, Approximations of* $(\in, \in \vee q)$*-fuzzy ideals.*

1 Introduction

In real life, there exist some possible scenario in which the objects of a set are arrange through a specific order. For example the cost of certain commodities in a market can be debated by a terms such as very costly, costly, affordable, cheap and very cheap. We see that exist an order among these items and commodities. So it is clear that these commodities can be characterized through an order among their prices. This can be study in an algebraic structure called ordered semigroups (OSGs). OSGs is a set having partial ordered with associative binary operation. OSGs have broad

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applications in various fields such as coding theory, automata theory and computer science etc.

The paradigm of fuzzy set was originally initiated by Zadeh [36]. This theory has strong points of view to tackle with uncertainty. With the passage of time fuzzy set become the rich research area among the scholars. The model of fuzzy set has been generalized in several direction by different authors. The concept of fuzzy algebraic model was initiated by Rosenfeld [27] and presented the study of fuzzy subgroups. Kuroki [19] originated the theory of fuzzy semigroups. The theory of fuzzy ordered groupoids and OSGs was investigated by Kehayopulu and Tsingelis in [14, 15] and studied the concepts of fuzzy ideals and fuzzy filters in ordered groupoids. Bhakat and Das in [3, 4, 5] investigated the concepts of (α, β)-fuzzy subgroups in his pioneer work and the concept of $(\in, \in \vee q)$-fuzzy subgroups attracted more attention of the scholar towards the study of (α, β) structure. Concept of $(\in, \in \vee q)$-fuzzy subgroup is based on quasi-coincidence of fuzzy points. This notion is introduced in [24]. In algebraic structures the most significant topic fuzzy ideals (FIds) attract the attention of many scholars. In semigroups Kuroki [18] presented the ideas of FIds, fuzzy bi-ideals (FBIds) and study some of the fundamental properties of these ideals. Moreover in semigroups Kuroki [20] explored the notion of fuzzy quasi-ideals (FQIds) and fuzzy semiprime quasi-ideals and study some of the basic properties related to these ideals. Jun et al. [9] initiated the standpoints of $(\in, \in \vee q)$-FBIds of OSGs and given some characterizations Theorems. In semigroups the concepts of FQIds was studied by Ahsan [1]. The study of general form of fuzzy interior ideals (FIIds) and (α, β)-FIIds is initiated by Jun and Song [8] in semigroups. In semigroups the generalization of (α, β)-FIds of hemirings is presented by Jun et al. [10], and for more detail see [28, 29, 30].

Pawlak [23] is the pioneer who for the first time investigated the rudimentary concept of rough set. The fundamental concept of Pawlak rough set depend upon the equivalence relation. So due to confined knowledge about the objects of a certain set, it is too complicated to made the equivalence relation among the elements of a set. Here the authors are restricted by the properties of equivalence relation and many applications of Pawlak rough set have been reported. So different scholars studied the different structures for rough set with less constraint. The prototypes of fuzzy set and rough set are different but both of them have the ability to tackle with uncertainty. Both of these theories are combine very successfully by Dubois and Prade in [7]. The study of generalized rough sets was initiated by Davvaz [6]. In generalized rough set a set valued function play a vital role to define the approximations rather than equivalence relation of a set. Several authors presented the approximation of a set in different algebraic structure, such as in semigroups and fuzzy semigroups Kuroki [21] initiated the idea of roughness and in the same structure this idea is extended to the prime ideals in [31]. In OSGs rough approximations as proposed in [21] can be considered as a better idea. Rehman et al. initiated the concept of roughness in LA-semigroups. Qurashi and Shabir [25] presented the generalized roughness in quantales. The concepts of rough bipolar Γ-hyperideals was initiated by Yaqoob et al. [35] and for the detail study of roughness also see [33, 34, 37]. The rough study of ternary semigroups was presented by Yaqoob et al. [32]. As

OSGs is the relation of partial ordered and semigroups that is why to find the non-trivial equivalence relations for such a structure are difficult. Therefore in OSGs the study of generalized roughness was originated by Mahmood et al. [22] in fuzzy filters and fuzzy ideals with thresholds by defining the set valued homomorphisms. Furthermore they have studied the approximation of generalized structure of fuzzy filters and fuzzy ideals with thresholds in OSGs. In OSGs Ali et al. [2] initiated the rough study of $(\in, \in \vee qk)$-fuzzy filters and they also studied the approximation of generalization of fuzzy filters. Here in this manuscript we will originate the study of generalized roughness of fuzzy ideals in OSGs. Instead of equivalence relation the set valued maps will play a vital role to introduce this new concept of generalized roughness in fuzzy ideals of OSGs and these mapping will be in the form of isotone or monotone order. The order of the paper is as follows.

This paper is organized as, in Section 2, we will briefly recall some fundamental concepts related to OSGs, fuzzy sets, rough sets, FIds and their generalization which is the key for onward concepts. In Section 3, we will originate the approximations of FIds, FBIds, FIIds and FQIds of OSGs on the basis of isotone and monotone mapping. It is clear that these two mappings play a significant role for investigating the approximation of FIds in OSGs. Moreover in Section 4, the idea of approximation is generalized to $(\in, \in \vee q)$-FIds, FBIds, FIIds and FQIds. The final Section 5, consist of the conclusion of the proposed manuscript.

2 Preliminary

This section consist of brief and rudimentary standpoints about OSGs, fuzzy set, and rough set which will provide the key for onward concepts.

Let S be a nonempty set. OSGs $(S, \cdot, \leq)$ is the relation of partial ordered and semigroups in which S under multiplication is a semigroup and S under $\leq$ is a partially ordered set (po-set) and holds the following

$$(\forall z, z_1, z_2 \in S)(z_1 \leq z_2 \rightarrow z_1 z \leq z_2 z \text{ and } z z_1 \leq z z_2).$$

An ordered subsemigroup S_1 is a nonempty subset of S if it holds $S_1^2 \subseteq S_1$.

For $S_1 \subseteq S$, we denote $(S_1] := \{z_1 \in S / z_1 \leq z_2 \text{ for some } z_2 \in S_1\}$. If $S_1 = \{a\}$, then instead of $(\{a\}]$ we write $(a]$. For subsets $S_1 \neq \phi$ and $S_2 \neq \phi$ of S, we represent $S_1 S_2 = \{z_1 z_2 / z_1 \in S_1, z_2 \in S_2\}$.

In onward work the symbol S stands for an OSGs.

Definition 2.1. [13] Consider a nonempty subset I of S is known as a left (resp. right) ideal of S having the following conditions:

(I_1) $SI \subseteq I$(resp. $IS \subseteq I$)
(I_2) if $z_1 \in S$ and $z_2 \in I$ such that $z_1 \leq z_2$, then $z_1 \in I$.

So the set I is known to be an ideal of S if it is both a left and a right ideal.

Next we are going to define the generalized structure of ideals that is interior ideals, bi-ideals and quasi ideals in OSGs.

Definition 2.2. [16] A subset $I \neq \phi$ of S is known to be a bi-ideal of S if it satisfies (I_2) and

(I_3) $ISI \subseteq I$
(I_4) $I^2 \subseteq I.$

Definition 2.3. [12] An interior ideal I is a nonempty subset of OSG S if it satisfies $(I_2), (I_4)$ and

(I_5) $SIS \subseteq I.$

Definition 2.4. [17] A quasi ideal $Q \neq \phi$ is a subset of S if it satisfy (I_2) and

(I_6) $(QS] \cap (SQ] \subseteq Q$

The paradigm of fuzzy set was originally initiated by Zadeh [36] and become the rich research area among the scholars. The model of fuzzy set has been generalized in several direction by different authors. Here in onward work we will present the combine study of fuzzy set with ideals that is FIds and their generalization.

Definition 2.5. [36] A fuzzy subset (FSS) μ is a mapping from S to $[0, 1]$.

Consider two FSSs μ_1 and μ_2 of S. Then $\mu_1 \subseteq \mu_2 \Longleftrightarrow \mu_1(z) \leq \mu_2(z)\ \forall\ z \in S$. Next $(\mu_1 \cap \mu_2)(z) = \min\{\mu_1(z), \mu_2(z)\}$ and $(\mu_1 \cup \mu_2)(z) = \max\{\mu_1(z), \mu_2(z)\}$.

Definition 2.6. A FSS μ of S of the form and for any $z_1 \in S$

$$\mu(z) = \begin{cases} t(t \neq 0) & \text{if} \quad z = z_1, \\ 0 & \text{if} \quad z \neq z_1. \end{cases}$$

then the fuzzy point is represented by $(z_1)_t$ with value t support by z_1. A fuzzy point $(z_1)_t$ 'belong to' FSS μ represented as $(z_1)_t \in \mu$, if $\mu(z_1) \geq t$, and a fuzzy point $(z_1)_t$ 'quasi-coincident' to FSS μ represented by $(z_1)_t\, q\mu$, if $\mu(z_1) + t > 1$.

Definition 2.7. [15] A FSS μ is called a fuzzy ordered subsemigroup of S if

(FI_1) $(\forall z_1, z_2 \in S)\,(\mu(z_1 z_2) \geq \min\{\mu(z_1), \mu(z_2)\})$.

Definition 2.8. [15] A FSS μ is known to be a fuzzy left (resp. right) ideals of S if it holds

(FI_2) $(\forall z_1, z_2 \in S)(z_1 \leq z_2$ this implies $\mu(z_1) \geq \mu(z_2))$
(FI_3) $(\forall z_1, z_2 \in S)(\mu(z_1 z_2) \geq \mu(z_2)($resp. $\mu(z_1 z_2) \geq \mu(z_1)))$.

A FSS μ of OSG S is said to be fuzzy ideal (FId), if μ is both sided ideal of S, that is a fuzzy left ideal (FLId) and as well as a fuzzy right ideal (FRId).

From this definition we can also conclude the following

Definition 2.9. A FSS μ is known to be FId of OSG S if it satisfy (FI_2) and

(FI_4) $(\forall z_1, z_2 \in S)(\mu(z_1 z_2) \geq \max\{\mu(z_1), \mu(z_2)\})$.

Proposition 2.10. Let μ_1 and μ_2 are the FLIds (resp. FRIds) of S. Then

i) $(\mu_1 \cap \mu_2)$ and

ii) $(\mu_1 \cup \mu_2)$ are FLIds (resp. FRIds) of S.

Proof. Proofs are straightforward. □

Definition 2.11. [12] A FSS μ is known as fuzzy interior ideal (FIId) of OSG S if it holds $(FI_1), (FI_2)$ and

(FI_5) $(\forall z_1, z_2, z_3 \in S)(\mu(z_1 z_3 z_2) \geq \mu(z_3))$.

Definition 2.12. [16] A FSS μ is known as fuzzy bi-ideal (FBId) of OSG S if it satisfies $(FI_1), (FI_2)$ and

(FI_6) $(\forall z_1, z_2, z_3 \in S)(\mu(z_1 z_2 z_3) \geq \min\{\mu(z_1), \mu(z_3)\})$.

Definition 2.13. Let $X \neq \phi$ be a subset of S, then we define a set X_{z_1} by

$$X_{z_1} = \{(z_2, z_3) \in S \times S / z_1 \leq z_2 z_3\}.$$

Let us consider the two fuzzy subsets μ_1 and μ_2 of S. Then we define $\mu_1 \circ \mu_2 : S \rightarrow [0,1]$, as

$$z_1 \rightarrow \mu_1 \circ \mu_2(z_1) = \begin{cases} V_{(z_2,z_3)\in X_{z_1}} \min\{\mu_1(z_2), \mu_2(z_3)\} & \text{if} \quad X_{z_1} \neq \phi \\ 0 & \text{if} \quad X_{z_1} = \phi. \end{cases} \tag{1}$$

$\mu_1 \leq \mu_2$ means $\mu_1(z) \leq \mu_2(z)$.

Pawlak [23] is the pioneer who for the first time investigated the rudimentary notion of rough set. The fundamental concept of Pawlak rough set depend upon the equivalence relation.

Consider the equivalence relation ξ on the initial universal set U. Then (U, ξ) is said to be the approximation space. Let $\phi \neq X \subseteq U$, so in this case the set X is called a definable subset of U if it is the collection of some equivalence classes of a universal set U else it is called not definable. Then the set X is approximated in the form of upper and lower approximations which are given as:

$$\overline{App}(X) = \left\{z_1 \in U : [z_1]_\xi \cap X \neq \phi\right\}$$
$$\underline{App}(X) = \left\{z_1 \in U : [z_1]_\xi \subseteq X\right\}$$

Then the rough set is a pair $\left(\overline{App}X, \underline{App}X\right)$, if $\overline{App}X \neq \underline{App}X$. The set X is a definable set if $\overline{App}X = \underline{App}X$.

In the following we will further generalized the concepts of upper and lower approximations to a FSS as well.

Definition 2.14. [11] Consider the approximation space (U, ξ), and for any $z_1 \in U$, the upper and lower approximations of a FSS μ is defined as

$$\overline{App}\left(\mu\right)\left(z_1\right) = \underset{z_2 \in [z_1]_\xi}{\vee} \mu\left(z_2\right) \text{ and } \underline{App}\left(\mu\right)\left(z_1\right) = \underset{z_2 \in [z_1]_\xi}{\wedge} \mu\left(z_2\right)$$

The pair $\left(\overline{App}\left(\mu\right), \underline{App}\left(\mu\right)\right)$ is said to be a rough fuzzy subset if $\overline{App}\left(\mu\right) \neq \underline{App}\left(\mu\right)$.

Definition 2.15. Consider the OSGs S_1 and S_2. Then the set-valued homomorphism $(SVH)$is a mapping $F : S_1 \longrightarrow P^*(S_2)$ if it satisfied:

(h_1) $F(z_1)F(z_2) = F(z_1z_2)$

Where $P^*(S_2) \neq \phi$ represents the collection of all subsets of S_2.

Definition 2.16. Let S_1 and S_2 be two OSGs. Then the set-valued monotone homomorphism $(SVMH)$ is a mapping $F : S_1 \longrightarrow P^*(S_2)$ if it satisfy the condition (h_1) of Definition 2.15, and

(h_2) if $z_1 \leq z_2$ this implies $F\left(z_1\right) \subseteq F\left(z_2\right)$ for each $z_1, z_2 \in S_1$.

Definition 2.17. Let S_1 and S_2 be two OSGs. Then the set-valued isotone homomorphism $(SVIH)$ is a mapping $F : S_1 \longrightarrow P^*(S_2)$ if it satisfy condition (h_1) of Definition 2.15, and

(h_3) $z_1 \leq z_2$ then $F\left(z_2\right) \subseteq F\left(z_1\right)$ for each $z_1, z_2 \in S_1$.

Definition 2.18. Consider that a $SVIH$ or $SVMH$ is a function $F : S \longrightarrow P^*(S)$. Then the generalized upper and lower approximations for any $z_1 \in S$, of a FSS μ with respect to the given mapping F is defined as

$$\overline{F}\left(\mu\right)\left(z_1\right) = \underset{z_2 \in F(z_1)}{\vee} \mu\left(z_2\right) \text{ and } \underline{F}\left(\mu\right)\left(z_1\right) = \underset{z_2 \in F(z_1)}{\wedge} \mu\left(z_2\right)$$

The rough fuzzy subset is a pair $\left(\overline{F}\left(\mu\right), \underline{F}\left(\mu\right)\right)$ if $\overline{F}\left(\mu\right) \neq \underline{F}\left(\mu\right)$.

3 Approximations of FIds in OSGs

In this section study of roughness of FIds in OSGs is being presented on the bases of $SVIH$ or $SVMH$. Thus we will start from the following.

Theorem 3.1. Suppose that $F : S \rightarrow P^*\left(S\right)$ be a $SVIH$ or $SVMH$ and a FSS μ be a fuzzy ordered subsemigroup of S. Then the upper approximation $\overline{F}\left(\mu\right)$ is a fuzzy ordered subsemigroup of S.

Proof. For any $z_1, z_2 \in S$. Consider

$$\begin{aligned}
\overline{F}(\mu)(z_1 z_2) &= \bigvee_{z_1' \in F(z_1 z_2)} \mu\left(z_1'\right) \\
&= \bigvee_{z_1' \in F(z_1)F(z_2)} \mu\left(z_1'\right) \\
&= \bigvee_{z_2' z_3' \in F(z_1)F(z_2)} \mu\left(z_2' z_3'\right) \quad \left(\begin{array}{c} \text{as } z_1' = z_2' z_3' \text{ such that } z_2' \in F(z_1) \\ \text{and } z_3' \in F(z_2) \end{array} \right) \\
&= \bigvee_{\substack{z_2' \in F(z_1) \\ z_3' \in F(z_2)}} \mu\left(z_2' z_3'\right) \\
&\geq \bigvee_{\substack{z_2' \in F(z_1) \\ z_3' \in F(z_2)}} \min\left\{\mu\left(z_2'\right), \mu\left(z_3'\right)\right\} \\
&= \min\left\{ \bigvee_{z_2' \in F(z_1)} \mu\left(z_2'\right), \bigvee_{z_3' \in F(z_2)} \mu\left(z_3'\right) \right\}
\end{aligned}$$

implies

$$\overline{F}(\mu)(z_1 z_2) \geq \min\left\{\overline{F}(\mu)(z_1), \overline{F}(\mu)(z_2)\right\}$$

Therefore $\overline{F}(\mu)$ is a fuzzy ordered subsemigroup of S. □

Theorem 3.2. Suppose that a FSS μ be a fuzzy ordered subsemigroup of S and $F : S \to P^*(S)$ be a $SVIH$ or $SVMH$. Then $\underline{F}(\mu)$ is a fuzzy ordered subsemigroup of OSG S.

Proof. Similarly as above Theorem 3.1. □

In onward discussion the study of roughness of FIds in OSGs is being presented.

Theorem 3.3. Consider the $SVMH$ $F : S \to P^*(S)$ and a FSS μ be a FLId (resp. FRId) of OSG S. Then $\underline{F}(\mu)$ is a FLId (resp. FRId) of S.

Proof. For each $z_1, z_2 \in S$ with $z_1 \leq z_2$, then $F(z_1) \subseteq F(z_2)$. Now we may consider the following

$$\begin{aligned}
\underline{F}(\mu)(z_1) &= \bigwedge_{z_1' \in F(z_1)} \mu\left(z_1'\right) \\
&\geq \bigwedge_{z_2' \in F(z_2)} \mu\left(z_2'\right)
\end{aligned}$$

implies

$$\underline{F}(\mu)(z_1) \geq \underline{F}(\mu)(z_2)$$

Next

$$\begin{aligned}
\underline{F}(\mu)(z_1z_2) &= \bigwedge_{z_1^{'}\in F(z_1z_2)} \mu\left(z_1^{'}\right) \\
&= \bigwedge_{z_1^{'}\in F(z_1)F(z_2)} \mu\left(z_1^{'}\right) \\
&= \bigwedge_{z_2^{'}z_3^{'}\in F(z_1)F(z_2)} \mu\left(z_2^{'}z_3^{'}\right) \quad \left(\begin{array}{c}\text{as } z_1^{'}=z_2^{'}z_3^{'} \text{ such that } z_2^{'}\in F(z_1)\\ \text{and } z_3^{'}\in F(z_2)\end{array}\right) \\
&= \bigwedge_{\substack{z_2^{'}\in F(z_1)\\ z_3^{'}\in F(z_2)}} \mu\left(z_2^{'}z_3^{'}\right) \\
&\geq \bigwedge_{z_3^{'}\in F(z_2)} \mu\left(z_3^{'}\right)
\end{aligned}$$

implies

$$\underline{F}(\mu)(z_1z_2) \ \geq \ \underline{F}(\mu)(z_2)$$

Hence $\underline{F}(\mu)$ is a FLId of S. Analogously, we can prove that $\underline{F}(\mu)$ is a FRId of S. □

Here by counter example it is shown that upper approximation $\overline{F}(\mu)$ does not hold in general for a FId μ, when F is a $SVMH$.

Example 3.4. Let us suppose a set $S = \{\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5, \tilde{a}_6\}$ with the following multiplication table and order relation "$\leq$".

Multiplication table for S

$\cdot$	$\tilde{a}_1$	$\tilde{a}_2$	$\tilde{a}_3$	$\tilde{a}_4$	$\tilde{a}_5$	$\tilde{a}_6$
$\tilde{a}_1$	$\tilde{a}_1$	$\tilde{a}_1$	$\tilde{a}_1$	$\tilde{a}_1$	$\tilde{a}_1$	$\tilde{a}_1$
$\tilde{a}_2$	$\tilde{a}_1$	$\tilde{a}_2$	$\tilde{a}_2$	$\tilde{a}_4$	$\tilde{a}_2$	$\tilde{a}_2$
$\tilde{a}_3$	$\tilde{a}_1$	$\tilde{a}_2$	$\tilde{a}_3$	$\tilde{a}_4$	$\tilde{a}_5$	$\tilde{a}_5$
$\tilde{a}_4$	$\tilde{a}_1$	$\tilde{a}_1$	$\tilde{a}_4$	$\tilde{a}_4$	$\tilde{a}_4$	$\tilde{a}_4$
$\tilde{a}_5$	$\tilde{a}_1$	$\tilde{a}_2$	$\tilde{a}_3$	$\tilde{a}_4$	$\tilde{a}_5$	$\tilde{a}_5$
$\tilde{a}_6$	$\tilde{a}_1$	$\tilde{a}_2$	$\tilde{a}_3$	$\tilde{a}_4$	$\tilde{a}_5$	$\tilde{a}_6$

Table 1

and $\leq:=\{(\tilde{a}_1,\tilde{a}_1),(\tilde{a}_2,\tilde{a}_2),(\tilde{a}_3,\tilde{a}_3),(\tilde{a}_4,\tilde{a}_4),(\tilde{a}_5,\tilde{a}_5),(\tilde{a}_6,\tilde{a}_6),(\tilde{a}_1,\tilde{a}_4),(\tilde{a}_1,\tilde{a}_5),(\tilde{a}_4,\tilde{a}_5),(\tilde{a}_2,\tilde{a}_6),(\tilde{a}_3,\tilde{a}_5),(\tilde{a}_3,\tilde{a}_6),(\tilde{a}_2,\tilde{a}_5),(\tilde{a}_6,\tilde{a}_5)\}$. Then $(S,\cdot,\leq)$ is an OSG. Right ideals of S are $\{\tilde{a}_1,\tilde{a}_4\}, \{\tilde{a}_1,\tilde{a}_2,\tilde{a}_4\}$ and S. Left ideals of S are $\{\tilde{a}_1\}, \{\tilde{a}_1,\tilde{a}_2\}, \{\tilde{a}_1,\tilde{a}_4\}$, $\{\tilde{a}_1,\tilde{a}_2,\tilde{a}_4\}$, $\{\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,\tilde{a}_4\}$, $\{\tilde{a}_1,\tilde{a}_2,\tilde{a}_4,\tilde{a}_5,\tilde{a}_6\}$ and S. Define a FSS $\mu : S \to [0,1]$ by $\mu(\tilde{a}_1) = 0.8$, $\mu(\tilde{a}_2) = 0.5$, $\mu(\tilde{a}_4) = 0.6$ and $\mu(\tilde{a}_3) = \mu(\tilde{a}_5) = \mu(\tilde{a}_6) = 0.4$. Then FSS μ is a FId of S.

Next suppose that a $SVMH$ $F : S \to P^*(S)$ i.e.

(i) $F(\tilde{a}_1)F(\tilde{a}_2) = F(\tilde{a}_1\tilde{a}_2)$
(ii) if $\tilde{a}_1 \leq \tilde{a}_2 \to F(\tilde{a}_1) \subseteq F(\tilde{a}_2)$.

Where $P^*(S)$ consist of all non-empty subset of S. Now if $F(\tilde{a}_5) = \{\tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5, \tilde{a}_6\}$ and $F(\tilde{a}_6) = \{\tilde{a}_3, \tilde{a}_6\}$, as $\tilde{a}_6 \leq \tilde{a}_5 \to F(\tilde{a}_6) \subseteq F(\tilde{a}_5)$ but $\overline{F}(\mu)(\tilde{a}_6) \ngeq \overline{F}(\mu)(\tilde{a}_5)$. Hence in $SVMH$ it is prove that $\overline{F}(\mu)$ is not a FId of S.

Theorem 3.5. Suppose that a FSS μ be a FLId (resp. FRId) of S and $F : S \to P^*(S)$ be a $SVIH$. Then $\overline{F}(\mu)$ is a FLId (resp. FRId) of OSG S.

Proof. For each $z_1, z_2 \in S$ such that $z_1 \leq z_2$, then $F(z_2) \subseteq F(z_1)$. Now consider the following

$$\begin{aligned} \overline{F}(\mu)(z_1) &= \bigvee_{z_1' \in F(z_1)} \mu\left(z_1'\right) \\ &\geq \bigvee_{z_2' \in F(z_2)} \mu\left(z_2'\right) \end{aligned}$$

implies

$$\overline{F}(\mu)(z_1) \geq \overline{F}(\mu)(z_2)$$

Next

$$\begin{aligned} \overline{F}(\mu)(z_1 z_2) &= \bigvee_{z_1' \in F(z_1 z_2)} \mu\left(z_1'\right) \\ &= \bigvee_{z_1' \in F(z_1)F(z_2)} \mu\left(z_1'\right) \\ &= \bigvee_{z_2' z_3' \in F(z_1)F(z_2)} \mu\left(z_2' z_3'\right) \quad \left(\begin{array}{c}\text{as } z_1' = z_2' z_3' \text{ such that } z_2' \in F(z_1) \\ \text{and } z_3' \in F(z_2)\end{array}\right) \\ &= \bigvee_{\substack{z_2' \in F(z_1) \\ z_3' \in F(z_2)}} \mu\left(z_2' z_3'\right) \\ &\geq \bigvee_{z_3' \in F(z_2)} \mu\left(z_3'\right) \end{aligned}$$

implies

$$\overline{F}(\mu)(z_1 z_2) \geq \overline{F}(\mu)(z_2)$$

Hence this prove that $\overline{F}(\mu)$ is a FLId (resp. FRId) of S. □

Here by counter example it is shown that upper approximation $\underline{F}(\mu)$ does not hold in general for a FId μ, when F is a $SVIH$.

Example 3.6. Suppose a FId μ of OSG S as shown in example 3.4. Now consider a $SVIH$ $F : S \to P^*(S)$ i.e.

(i) $F(\tilde{a}_1)F(\tilde{a}_2) = F(\tilde{a}_1\tilde{a}_2)$
(ii) if $\tilde{a}_1 \leq \tilde{a}_2 \Rightarrow F(\tilde{a}_2) \subseteq F(\tilde{a}_1)$.

Now if $F(\tilde{a}_6) = \{\tilde{a}_1, \tilde{a}_2, \tilde{a}_4\}$ and $F(\tilde{a}_5) = \{\tilde{a}_1, \tilde{a}_4\}$, as $\tilde{a}_6 \leq \tilde{a}_5 \Rightarrow F(\tilde{a}_5) \subseteq F(\tilde{a}_6)$ but $\underline{F}(\mu)(\tilde{a}_6) \ngeq \underline{F}(\mu)(\tilde{a}_5)$. Hence in $SVIH$ it is prove that $\underline{F}(\mu)$ is not a FId of S.

Theorem 3.7. Suppose that $F : S \to P^*(S)$ be $SVMH$ and a FSS μ be a FIId of OSG S. Then $\underline{F}(\mu)$ is a FIId of S.

Proof. From Theorem 3.3, we have $z_1 \leq z_2$, implies $F(z_1) \subseteq F(z_2)$, for each $z_1, z_2 \in S$, then $\underline{F}(\mu)(z_1) \geq \underline{F}(\mu)(z_2)$. Next consider

$$\begin{aligned}
\underline{F}(\mu)(z_1 z_2) &= \bigwedge_{z_1' \in F(z_1 z_2)} \mu\left(z_1'\right) \\
&= \bigwedge_{z_1' \in F(z_1)F(z_2)} \mu\left(z_1'\right) \\
&= \bigwedge_{z_2' z_3' \in F(z_1)F(z_2)} \mu\left(z_2' z_3'\right) \quad \left(\begin{array}{c} \text{as } z_1' = z_2' z_3', \text{ where } z_2' \in F(z_1) \\ \text{and } z_3' \in F(z_2) \end{array}\right) \\
&= \bigwedge_{\substack{z_2' \in F(z_1) \\ z_3' \in F(z_2)}} \mu\left(z_2' z_3'\right) \\
&\geq \bigwedge_{\substack{z_2' \in F(z_1) \\ z_3' \in F(z_2)}} \min\left\{\mu\left(z_2'\right), \mu\left(z_3'\right)\right\} \\
&= \min\left\{\bigwedge_{z_2' \in F(z_1)} \mu\left(z_2'\right), \bigwedge_{z_3' \in F(z_2)} \mu\left(z_3'\right)\right\}
\end{aligned}$$

implies

$$\underline{F}(\mu)(z_1 z_2) \geq \min\left\{\underline{F}(\mu)(z_1), \underline{F}(\mu)(z_2)\right\}$$

Consider

$$\begin{aligned}
\underline{F}(\mu)(z_1 z_3 z_2) &= \bigwedge_{z_1' \in F(z_1 z_3 z_2)} \mu\left(z_1'\right) \\
&= \bigwedge_{z_1' \in F(z_1)F(z_3)F(z_2)} \mu\left(z_1'\right) \\
&= \bigwedge_{tuv \in F(z_1)F(z_3)F(z_2)} \mu(tuv) \quad \left(\begin{array}{c} \text{as } z_1' = tuv \text{ where } t \in F(z_1), \\ u \in F(z_3) \text{ and } v \in F(z_2) \end{array}\right) \\
&= \bigwedge_{\substack{t \in F(z_1) \\ u \in F(z_3) \\ v \in F(z_2)}} \mu(tuv) \\
&\geq \bigwedge_{u \in F(z_3)} \mu(u)
\end{aligned}$$

implies

$$\underline{F}(\mu)(z_1 z_3 z_2) \geq \underline{F}(\mu)(z_3)$$

Therefore, $\underline{F}(\mu)$ satisfies all the conditions of FIId, so $\underline{F}(\mu)$ is a FIId of OSG S. □

Theorem 3.8. Consider a FSS μ be a FIId of OSG S and $F : S \to P^*(S)$ be a $SVIH$. Then $\overline{F}(\mu)$ is a FIId of OSG S.

Proof. From Theorems 3.1 and 3.5, if $z_1 \leq z_2$ implies $F(z_2) \subseteq F(z_1)$ for each $z_1, z_2 \in S$, then $\overline{F}(\mu)(z_1) \geq \overline{F}(\mu)(z_2)$ and $\overline{F}(\mu)(z_1z_2) \geq \min\left\{\overline{F}(\mu)(z_1), \overline{F}(\mu)(z_2)\right\}$. Next we may consider the following

$$\begin{aligned}
\overline{F}(\mu)(z_1z_3z_2) &= \bigvee_{z_1' \in F(z_1z_3z_2)} \mu\left(z_1'\right) \\
&= \bigvee_{z_1' \in F(z_1)F(z_3)F(z_2)} \mu\left(z_1'\right) \\
&= \bigvee_{tuv \in F(z_1)F(z_3)F(z_2)} \mu(tuv) \quad \left(\begin{array}{c} \text{as } z_1' = tuv \text{ where } t \in F(z_1), \\ u \in F(z_3) \text{ and } v \in F(z_2) \end{array}\right) \\
&= \bigvee_{\substack{t \in F(z_1) \\ u \in F(z_3) \\ v \in F(z_2)}} \mu(tuv) \\
&\geq \bigvee_{u \in F(z_3)} \mu(u)
\end{aligned}$$

implies

$$\overline{F}(\mu)(z_1z_3z_2) \geq \overline{F}(\mu)(z_3)$$

Therefore, it is prove that $\overline{F}(\mu)$ is a FIId of S. □

Theorem 3.9. Suppose that $F: S \to P^*(S)$ be a $SVMH$ and a FSS μ be a FBId of OSG S. Then $\underline{F}(\mu)$ is a FBId of S.

Proof. From Theorem 3.7, we have for each $z_1, z_2 \in S$, such that $z_1 \leq z_2$, implies $F(z_1) \subseteq F(z_2)$, then $\underline{F}(\mu)(z_1) \geq \underline{F}(\mu)(z_2)$, and also

$$\underline{F}(\mu)(z_1z_2) \geq \min\{\underline{F}(\mu)(z_1), \underline{F}(\mu)(z_2)\}$$

Next for each $z_1, z_2, z_3 \in S$, consider the following

$$\begin{aligned}
\underline{F}(\mu)(z_1z_2z_3) &= \bigwedge_{z_1' \in F(z_1z_2z_3)} \mu\left(z_1'\right) \\
&= \bigwedge_{z_1' \in F(z_1)F(z_2)F(z_3)} \mu\left(z_1'\right) \\
&= \bigwedge_{tuv \in F(z_1)F(z_2)F(z_3)} \mu(tuv) \quad \left(\begin{array}{c} \text{as } z_1' = tuv \text{ where } t \in F(z_1), \\ u \in F(z_2) \text{ and } v \in F(z_3) \end{array}\right) \\
&= \bigwedge_{\substack{t \in F(z_1) \\ u \in F(z_2) \\ v \in F(z_3)}} \mu(tuv) \\
&\geq \bigwedge_{\substack{t \in F(z_1) \\ v \in F(z_3)}} \min\{\mu(t), \mu(v)\} \\
&= \min\left\{\bigwedge_{t \in F(z_1)} \mu(t), \bigwedge_{v \in F(z_3)} \mu(v)\right\}
\end{aligned}$$

implies

$$\underline{F}(\mu)(z_1z_2z_3) \geq \min\{\underline{F}(\mu)(z_1), \underline{F}(\mu)(z_3)\}$$

Hence $\underline{F}(\mu)$ satisfies all the conditions of a FBId of S, so $\underline{F}(\mu)$ is a FBId of S. □

Theorem 3.10. Suppose that a FSS μ be a FBId of S and $F : S \to P^*(S)$ be a $SVIH$. Then we have to prove that $\overline{F}(\mu)$ is a FBId of OSG S.

Proof. From Theorem 3.8, for each $z_1, z_2 \in S$ such that $z_1 \le z_2$, implies $F(z_2) \subseteq F(z_1)$, then $\overline{F}(\mu)(z_1) \ge \overline{F}(\mu)(z_2)$ and also $\overline{F}(\mu)(z_1 z_2) \ge \min\left\{\overline{F}(\mu)(z_1), \overline{F}(\mu)(z_2)\right\}$. Next for each $z_1, z_2, z_3 \in S$, we consider

$$\begin{aligned}
\overline{F}(\mu)(z_1 z_2 z_3) &= \bigvee_{z_1' \in F(z_1 z_2 z_3)} \mu\left(z_1'\right) \\
&= \bigvee_{z_1' \in F(z_1) F(z_2) F(z_3)} \mu\left(z_1'\right) \\
&= \bigvee_{tuv \in F(z_1) F(z_2) F(z_3)} \mu(tuv) \quad \left(\begin{array}{c} \text{as } z_1' = tuv \text{ where } t \in F(z_1), \\ u \in F(z_2) \text{ and } v \in F(z_3) \end{array}\right) \\
&= \bigvee_{\substack{t \in F(z_1) \\ u \in F(z_2) \\ v \in F(z_3)}} \mu(tuv) \\
&\ge \bigvee_{\substack{t \in F(z_1) \\ v \in F(z_3)}} \min\{\mu(t), \mu(v)\} \\
&= \min\left\{\bigvee_{t \in F(z_1)} \mu(t), \bigvee_{v \in F(z_3)} \mu(v)\right\}
\end{aligned}$$

implies

$$\overline{F}(\mu)(z_1 z_2 z_3) \ \ge \ \min\left\{\overline{F}(\mu)(z_1), \overline{F}(\mu)(z_3)\right\}$$

Hence $\overline{F}(\mu)$ satisfies all the conditions of a FBId, so $\overline{F}(\mu)$ is a FBId of S. □

Theorem 3.11. Let us suppose that $F : S \to P^*(S)$ be a $SVMH$ and a FSS μ be a FQId of S and . Then we have to prove that $\underline{F}(\mu)$ is a FQId of S.

Proof. As $\underline{F}(\mu)$ is a FLId (resp. FRId) of OSG S, therefore by Theorem 3.3, for each $z_1, z_2 \in S$ such that $z_1 \le z_2$, implies $F(z_1) \subseteq F(z_2)$, then $\underline{F}(\mu)(z_1) \ge \underline{F}(\mu)(z_2)$. Next consider

$$\begin{aligned}
\underline{F}(\mu)(z_1) &= \bigwedge_{\bar{a} \in F(z_1)} \mu(\bar{a}) \\
&\ge \bigwedge_{\bar{a} \in F(z_1)} ((\mu \circ 1) \wedge (1 \circ \mu))(\bar{a}) \\
&= \underline{F}((\mu \circ 1) \wedge (1 \circ \mu))(z_1)
\end{aligned}$$

implies

$$\underline{F}(\mu)(z_1) \ \ge \ \underline{F}((\mu \circ 1) \wedge (1 \circ \mu))(z_1)$$

Hence from the proof it is clear that $\underline{F}(\mu)$ is a FQId of OSG S. □

Theorem 3.12. Suppose that a $SVIH$ $F : S \to P^*(S)$ and a FSS μ be a FQId of OSG S. Then $\overline{F}(\mu)$ is a FQId of OSG S.

Proof. As we know from Theorem 3.5 that $\overline{F}(\mu)$ is a FLId (resp. FRId) of S, therefore for each $z_1, z_2 \in S$ such that $z_1 \leq z_2$ implies $F(z_2) \subseteq F(z_1)$, then $\overline{F}(\mu)(z_1) \geq \overline{F}(\mu)(z_2)$. Next consider

$$\begin{aligned}\overline{F}(\mu)(z_1) &= \bigvee_{\bar{a}\in F(z_1)} \mu(\bar{a}) \\ &\geq \bigvee_{\bar{a}\in F(z_1)} ((\mu\circ 1)\wedge(1\circ\mu))(\bar{a}) \\ &= \overline{F}((\mu\circ 1)\wedge(1\circ\mu))(z_1)\end{aligned}$$

implies

$$\overline{F}(\mu)(z_1) \geq \overline{F}((\mu\circ 1)\wedge(1\circ\mu))(z_1)$$

Hence from the proof it is clear that $\overline{F}(\mu)$ is a FQId of S. $\square$

4 Approximations of $(\in,\in\vee q)$-FIds in OSGs

In this section, roughness of $(\in,\in\vee q)$-FIds is being studied on the bases of $SVIH$ and $SIMH$.

Definition 4.1. A FSS μ of OSG S is known as an $(\in,\in\vee q)$-fuzzy ordered sub-semigroup of OSG S if:

$$(FI_8) \quad (\text{for each } z_1, z_2 \in S)(\text{for all } t_1, t_2 \in (0,1])\begin{pmatrix}(z_1)_{t_1}, (z_2)_{t_2} \in \mu \text{ implies} \\ (z_1z_2)_{\min\{t_1,t_2\}} \in \vee q\mu\end{pmatrix}$$

Definition 4.2. A FSS μ is known as $(\in,\in\vee q)$-FLId (resp. FRId) of S if the following conditions are holds:

$$(FI_9) \quad (\text{for all } z_1, z_2 \in S)(\text{for } all\ t_1 \in (0,1])\begin{pmatrix} z_1 \leq z_2, \text{ then } (z_2)_{t_1} \in \mu \text{ implies} \\ (z_1)_{t_1} \in \vee q\mu\end{pmatrix}$$

$$(FI_{10}) \quad (\text{for all } z_1, z_2 \in S)(\text{for } all\ t_1 \in (0,1])\begin{pmatrix}(z_2)_{t_1} \in \mu \text{ implies } (z_1z_2)_{t_1} \in \vee q\mu \\ (\text{resp. } (z_2z_1)_{t_1} \in \vee q\mu)\end{pmatrix}$$

A FSS μ is known as $(\in,\in\vee q)$-FId of S, if it is both $(\in,\in\vee q)$-FLId and $(\in,\in\vee q)$-FRId of S.

Definition 4.3. [12] A FSS μ is known to be an $(\in,\in\vee q)$-FIId of OSG S if it holds $(FI_8), (FI_9)$ and

$$(FI_{11}) \quad (\text{for all } z_1, z_2, z_3 \in S)(\text{for all } t_1 \in (0,1])\left((z_3)_{t_1} \in \mu \text{ implies } (z_1z_3z_2)_{t_1} \in \vee q\mu\right)$$

Definition 4.4. A FSS μ is said to be an $(\in,\in\vee q)$-FBId of OSG S if holds $(FI_8), (FI_9)$ and

$$(FI_{12}) \quad (\text{for all } z_1, z_2, z_3 \in S)(\text{for all } t_1, t_2 \in (0,1])\begin{pmatrix}(z_1)_{t_1}, (z_3)_{t_2} \in \mu \text{ implies} \\ (z_1z_2z_3)_{\min\{t_1,t_2\}} \in \vee q\mu\end{pmatrix}$$

Definition 4.5. A FSS μ is known as $(\in,\in\vee q)$-FQId of S if it holds (FI_9) and

$$(FI_{13}) \quad (\text{for all } z_1 \in S)(\text{for all } t_1 \in (0,1])\left((z_1)_{t_1} \in (\mu\circ 1)\wedge(1\circ\mu) \text{ implies } (z_1)_{t_1} \in \vee q\mu\right).$$

Lemma 4.6. [12] A FSS μ is known as $(\in, \in \vee q)$-FLId (resp. FRId) of OSG $S \Leftrightarrow$ it holds

(FI_{14}) (for all $z_1, z_2 \in S$) $(z_1 \leq z_2, \mu(z_1) \geq \min\{\mu(z_2), 0.5\})$,
(FI_{15}) (for all $z_1, z_2 \in S$) $(\mu(z_1 z_2) \geq \min\{\mu(z_2), 0.5\})$
(resp. $\mu(z_1 z_2) \geq \min\{\mu(z_1), 0.5\})$.

Lemma 4.7. [12] A FSS μ is said to be an $(\in, \in \vee q)$-FBId of OSG $S \Leftrightarrow$ it holds (FI_{14}) of lemma 4.6 and

(FI_{16}) (for all $z_1, z_2 \in S$) $(\mu(z_1 z_2) \geq \min\{\mu(z_1), \mu(z_2), 0.5\})$
(FI_{17}) (for all $z_1, z_2, z_3 \in S$) $(\mu(z_1 z_2 z_3) \geq \min\{\mu(z_1), \mu(z_3), 0.5\})$

Lemma 4.8. [12] μ is known as $(\in, \in \vee q)$-FIId of $S \Leftrightarrow$ it holds $(FI_{14}), (FI_{16})$ of lemmas 4.6 and 4.7 and

(FI_{18}) (for all $z_1, z_2 \in S$) $(\mu(z_1 z_3 z_2) \geq \min\{\mu(z_3), 0.5\})$

Lemma 4.9. A FSS μ is said to be an $(\in, \in \vee q)$-FQId of OSG $S \Leftrightarrow$ it holds (FI_{14}) of lemma 4.6 and

(FI_{19}) (for all $z_1, z_2 \in S$) $(\mu(z_1) \geq \min\{((\mu \circ 1) \wedge (1 \circ \mu))(z_1), 0.5\})$

Theorem 4.10. Suppose that FSS μ be an $(\in, \in \vee q)$-fuzzy ordered subsemigroup of S and $F : S \to P^*(S)$ be a $SVMH$ or $SVIH$. Then $\underline{F}(\mu)$ is an $(\in, \in \vee q)$-fuzzy ordered subsemigroup of S.

Proof. To prove this theorem we have to see that, $\underline{F}(\mu)$ satisfies (FI_{16}). If for each $z_1, z_2 \in S$, now consider

$$\begin{aligned}
\underline{F}(\mu)(z_1 z_2) &= \bigwedge_{z_1' \in F(z_1 z_2)} \mu\left(z_1'\right) \\
&= \bigwedge_{z_1' \in F(z_1) F(z_2)} \mu\left(z_1'\right) \\
&= \bigwedge_{ab \in F(z_1) F(z_2)} \mu(ab) \quad \left(\begin{array}{c} \text{as } z_1' = ab \text{ such that } a \in F(z_1), \\ \text{and } b \in F(z_2) \end{array}\right) \\
&= \bigwedge_{\substack{a \in F(z_1) \\ b \in F(z_2)}} \mu(ab) \\
&\geq \bigwedge_{\substack{a \in F(z_1) \\ b \in F(z_2)}} \min\{\mu(a), \mu(b), 0.5\} \\
&= \min\left\{\bigwedge_{a \in F(z_1)} \mu(a), \bigwedge_{b \in F(z_2)} \mu(b), 0.5\right\}
\end{aligned}$$

implies

$$\underline{F}(\mu)(z_1 z_2) \geq \min\{\underline{F}(\mu)(z_1), \underline{F}(\mu)(z_2), 0.5\}$$

Hence $\underline{F}(\mu)$ is an $(\in, \in \vee q)$-fuzzy ordered subsemigroup of OSG S. □

Theorem 4.11. Consider that a FSS μ be $(\in, \in \vee q)$-fuzzy ordered subsemigroup of S and $F : S \to P^*(S)$ be a $SVMH$ or $SVIH$ and. Then $\overline{F}(\mu)$ is an $(\in, \in \vee q)$-fuzzy ordered subsemigroup of S.

Proof. Straightforward as Theorem 4.10. □

Theorem 4.12. Consider that a FSS μ be $(\in, \in \vee q)$-FLId (resp. FRId) of S and $F : S \to P^*(S)$ be a $SVMH$. Then $\underline{F}(\mu)$ is $(\in, \in \vee q)$-FLId (resp. FRId) of S.

Proof. To prove this theorem, we have to see that $\underline{F}(\mu)$ satisfies (FI_{14}) and (FI_{15}). If for each $z_1, z_2 \in S$ with $z_1 \leq z_2$, implies $F(z_1) \subseteq F(z_2)$. Now consider

$$\begin{aligned}\min\{\underline{F}(\mu)(z_2), 0.5\} &= \min\left\{\bigwedge_{z_2^{'}\in F(z_2)} \mu\left(z_2^{'}\right), 0.5\right\}\\ &= \bigwedge_{z_2^{'}\in F(z_2)} \min\left\{\mu\left(z_2^{'}\right), 0.5\right\}\\ &\leq \bigwedge_{z_1^{'}\in F(z_1)} \mu\left(z_1^{'}\right)\end{aligned}$$

implies

$$\min\{\underline{F}(\mu)(z_2), 0.5\} \leq \underline{F}(\mu)(z_1)$$

Next consider

$$\begin{aligned}\underline{F}(\mu)(z_1z_2) &= \bigwedge_{z_1^{'}\in F(z_1z_2)} \mu\left(z_1^{'}\right)\\ &= \bigwedge_{z_1^{'}\in F(z_1)F(z_2)} \mu\left(z_1^{'}\right)\\ &= \bigwedge_{ab\in F(z_1)F(z_2)} \mu(ab) \quad \left(\begin{array}{c}\text{as } z_1^{'} = ab \text{ such that } a \in F(z_1),\\ b \in F(z_2)\end{array}\right)\\ &= \bigwedge_{\substack{a\in F(z_1)\\ b\in F(z_2)}} \mu(ab)\\ &\geq \bigwedge_{b\in F(z_2)} \min\{\mu(b), 0.5\}\\ &= \min\left\{\bigwedge_{b\in F(z_2)} \mu(b), 0.5\right\}\end{aligned}$$

implies

$$\underline{F}(\mu)(z_1z_2) \geq \min\{\underline{F}(\mu)(z_2), 0.5\}$$

Therefore it is clear that $\underline{F}(\mu)$ is an $(\in, \in \vee q)$-FLId (resp. FRId) ideal of S. □

Theorem 4.13. Consider that $F : S \to P^*(S)$ be a $SVIH$ and a FSS μ be $(\in, \in \vee q)$-FLId (resp. FRId) of S. Then $\overline{F}(\mu)$ is $(\in, \in \vee q)$-FLId (resp. FRId) of S.

Proof. To prove this theorem, we have to see that $\overline{F}(\mu)$ satisfies (FI_{14}) and (FI_{15}). If for each $z_1, z_2 \in S$ with $z_1 \leq z_2$, implies $F(z_2) \subseteq F(z_1)$. Next suppose the following

$$\begin{aligned} \min\left\{\overline{F}(\mu)(z_2), 0.5\right\} &= \min\left\{\bigvee_{z_2' \in F(z_2)} \mu\left(z_2'\right), 0.5\right\} \\ &= \bigvee_{z_2' \in F(z_2)} \min\left\{\mu\left(z_2'\right), 0.5\right\} \\ &\leq \bigvee_{z_1' \in F(z_1)} \mu\left(z_1'\right) \end{aligned}$$

implies

$$\min\left\{\overline{F}(\mu)(z_2), 0.5\right\} \leq \overline{F}(\mu)(z_1)$$

Next consider

$$\begin{aligned} \overline{F}(\mu)(z_1 z_2) &= \bigvee_{z_1' \in F(z_1 z_2)} \mu\left(z_1'\right) \\ &= \bigvee_{z_1' \in F(z_1) F(z_2)} \mu\left(z_1'\right) \\ &= \bigvee_{ab \in F(z_1) F(z_2)} \mu(ab) \quad \left(\begin{array}{c} \text{as } z_1' = ab \text{ such that } a \in F(z_1), \\ b \in F(z_2) \end{array}\right) \\ &= \bigvee_{\substack{a \in F(z_1) \\ b \in F(z_2)}} \mu(ab) \\ &\geq \bigvee_{b \in F(z_2)} \min\{\mu(b), 0.5\} \\ &= \min\left\{\bigvee_{b \in F(z_2)} \mu(b), 0.5\right\} \end{aligned}$$

implies

$$\overline{F}(\mu)(z_1 z_2) \geq \min\left\{\overline{F}(\mu)(z_2), 0.5\right\}$$

Therefore $\overline{F}(\mu)$ is an $(\in, \in \vee q)$-FLId of S. Similarly, we can prove that $\overline{F}(\mu)$ is an $(\in, \in \vee q)$-FRId of S. □

Theorem 4.14. Suppose that a FSS μ be $(\in, \in \vee q)$-FIId of S and $F : S \to P^*(S)$ be a $SVMH$. Then $\underline{F}(\mu)$ is $(\in, \in \vee q)$-FIId of S.

Proof. From Theorems 4.10 and 4.12,we see that $\underline{F}(\mu)$ satisfies (FI_{14}) and (FI_{16}).

Next we consider the following for each $z_1, z_2, z_3 \in S$.

$$\begin{aligned}
\underline{F}(\mu)(z_1 z_3 z_2) &= \bigwedge_{z_1' \in F(z_1 z_3 z_2)} \mu\left(z_1'\right) \\
&= \bigwedge_{z_1' \in F(z_1)F(z_3)F(z_2)} \mu\left(z_1'\right) \\
&= \bigwedge_{acb \in F(z_1)F(z_3)F(z_2)} \mu(acb) \quad \left(\begin{array}{c} \text{as } z_1' = acb \text{ such that } a \in F(z_1), \\ c \in F(z_3) \text{ and } b \in F(z_2) \end{array}\right) \\
&= \bigwedge_{\substack{a \in F(z_1) \\ c \in F(z_3) \\ b \in F(z_2)}} \mu(acb) \\
&\geq \bigwedge_{c \in F(z_3)} \min\{\mu(c), 0.5\} \\
&= \min\left\{\bigwedge_{z_3' \in F(z_3)} \mu\left(z_3'\right), 0.5\right\}
\end{aligned}$$

implies

$$\underline{F}(\mu)(z_1 z_3 z_2) \geq \min\{\underline{F}(\mu)(z_3), 0.5\}$$

Hence it is proved that $\underline{F}(\mu)$ is $(\in, \in \vee q)$-FIId of S. $\square$

Theorem 4.15. Let us consider that $F : S \rightarrow P^*(S)$ be a $SVIH$ and a FSS μ be $(\in, \in \vee q)$-FIId of S. Then $\overline{F}(\mu)$ is $(\in, \in \vee q)$-FIId of S.

Proof. From Theorem 4.13, we have for each $z_1, z_2 \in S$, if $z_1 \leq z_2$ implies $F(z_2) \subseteq F(z_1)$. Then $\min\left\{\overline{F}(\mu)(z_2), 0.5\right\} \leq \overline{F}(\mu)(z_1)$, Next let for each $z_1, z_2 \in S$,

$$\begin{aligned}
\overline{F}(\mu)(z_1 z_2) &= \bigvee_{z_1' \in F(z_1 z_2)} \mu\left(z_1'\right) \\
&= \bigvee_{z_1' \in F(z_1)F(z_2)} \mu\left(z_1'\right) \\
&= \bigvee_{ab \in F(z_1)F(z_2)} \mu(ab) \quad \left(\begin{array}{c} \text{as } z_1' = ab \text{ where } a \in F(z_1) \\ \text{and } b \in F(z_2) \end{array}\right) \\
&= \bigvee_{\substack{a \in F(z_1) \\ b \in F(z_2)}} \mu(ab) \\
&\geq \bigvee_{\substack{z_1' \in F(z_1) \\ z_2' \in F(z_2)}} \min\left\{\mu\left(z_1'\right), \mu\left(z_2'\right), 0.5\right\} \\
&= \min\left\{\bigvee_{z_1' \in F(z_1)} \mu\left(z_1'\right), \bigvee_{z_2' \in F(z_2)} \mu\left(z_2'\right), 0.5\right\}
\end{aligned}$$

implies

$$\overline{F}(\mu)(z_1 z_2) \geq \min\left\{\overline{F}(\mu)(z_1), \overline{F}(\mu)(z_2), 0.5\right\}$$

Next consider

$$\begin{aligned}
\overline{F}(\mu)(z_1z_3z_2) &= \bigvee_{z_1'\in F(z_1z_3z_2)} \mu\left(z_1'\right) \\
&= \bigvee_{z_1'\in F(z_1)F(z_3)F(z_2)} \mu\left(z_1'\right) \\
&= \bigvee_{acb\in F(z_1)F(z_3)F(z_2)} \mu(acb) \quad \left(\begin{array}{c}\text{as } z_1' = acb \text{ where } a\in F(z_1), \\ c\in F(z_3) \text{ and } b\in F(z_2)\end{array}\right) \\
&= \bigvee_{\substack{a\in F(z_1)\\ c\in F(z_3)\\ b\in F(z_2)}} \mu(acb) \\
&\geq \bigvee_{c\in F(z_3)} \min\{\mu(c), 0.5\} \\
&= \min\left\{\bigvee_{c\in F(z_3)} \mu(c), 0.5\right\}
\end{aligned}$$

implies

$$\overline{F}(\mu)(z_1z_3z_2) \geq \min\left\{\overline{F}(\mu)(z_3), 0.5\right\}$$

Therefore, it is prove that $\overline{F}(\mu)$ is $(\in,\in\vee q)$-FIId of S. □

Theorem 4.16. Suppose that a FSS μ be $(\in,\in\vee q)$-FBId of S and $F : S \to P^*(S)$ be a $SVMH$. Then $\underline{F}(\mu)$ is $(\in,\in\vee q)$-FBId of S.

Proof. From Theorem 4.14, we have for each $z_1, z_2 \in S$, if $z_1 \leq z_2$ implies $F(z_1) \subseteq F(z_2)$. Then $\min\{\underline{F}(\mu)(z_2), 0.5\} \leq \underline{F}(\mu)(z_1)$, and also $\underline{F}(\mu)(z_1z_2) \geq \min\{\underline{F}(\mu)(z_1), \underline{F}(\mu)(z_2), 0.5\}$. Next let for each $z_1, z_2, z_3 \in S$,

$$\begin{aligned}
\underline{F}(\mu)(z_1z_2z_3) &= \bigwedge_{z_1'\in F(z_1z_2z_3)} \mu\left(z_1'\right) \\
&= \bigwedge_{z_1'\in F(z_1)F(z_2)F(z_3)} \mu\left(z_1'\right) \\
&= \bigwedge_{abc\in F(z_1)F(z_3)F(z_2)} \mu(abc) \quad \left(\begin{array}{c}\text{as } z_1' = abc \text{ where } a\in F(z_1), \\ b\in F(z_2) \text{ and } c\in F(z_3)\end{array}\right) \\
&= \bigwedge_{\substack{a\in F(z_1)\\ c\in F(z_3)\\ b\in F(z_2)}} \mu(acb) \\
&\geq \bigwedge_{\substack{a\in F(z_1)\\ c\in F(z_3)}} \min\{\mu(a), \mu(c), 0.5\} \\
&= \min\left\{\bigwedge_{a\in F(z_1)} \mu(a), \bigwedge_{c\in F(z_3)} \mu(c), 0.5\right\}
\end{aligned}$$

implies

$$\underline{F}(\mu)(z_1z_2z_3) \geq \min\{\underline{F}(\mu)(z_1), \underline{F}(\mu)(z_3), 0.5\}$$

Hence $\underline{F}(\mu)$ satisfies all the conditions of $(\in,\in\vee q)$-FBId of S. Therefore $\underline{F}(\mu)$ is an $(\in,\in\vee q)$-FBId of S. □

Theorem 4.17. Let us consider that $F : S \to P^*(S)$ be a $SVIH$ and a FSS μ be $(\in, \in \vee q)$-FBId of OSG S. Then $\overline{F}(\mu)$ is $(\in, \in \vee q)$-FBId of S.

Proof. From Theorem 4.15, we have for each $z_1, z_2 \in S$, if $z_1 \leq z_2$ implies $F(z_2) \subseteq F(z_1)$. Then $\min\{\overline{F}(\mu)(z_2), 0.5\} \leq \overline{F}(\mu)(z_1)$, and also $\overline{F}(\mu)(z_1 z_2) \geq \min\{\overline{F}(\mu)(z_1), \overline{F}(\mu)(z_2), 0.5\}$. Next we consider the following for each $z_1, z_2, z_3 \in S$,

$$\begin{aligned}
\overline{F}(\mu)(z_1 z_2 z_3) &= \bigvee_{z_1' \in F(z_1 z_2 z_3)} \mu\left(z_1'\right) \\
&= \bigvee_{z_1' \in F(z_1)F(z_2)F(z_3)} \mu\left(z_1'\right) \\
&= \bigvee_{abc \in F(z_1)F(z_3)F(z_2)} \mu(abc) \quad \left(\begin{array}{c} \text{as } z_1' = abc \text{ where } a \in F(z_1), \\ b \in F(z_2) \text{ and } c \in F(z_3) \end{array}\right) \\
&= \bigvee_{\substack{a \in F(z_1) \\ c \in F(z_3) \\ b \in F(z_2)}} \mu(acb) \\
&\geq \bigvee_{\substack{a \in F(z_1) \\ c \in F(z_3)}} \min\{\mu(a), \mu(c), 0.5\} \\
&= \min\left\{\bigvee_{a \in F(z_1)} \mu(a), \bigvee_{c \in F(z_3)} \mu(c), 0.5\right\}
\end{aligned}$$

implies

$$\overline{F}(\mu)(z_1 z_2 z_3) \geq \min\left\{\overline{F}(\mu)(z_1), \overline{F}(\mu)(z_3), 0.5\right\}$$

Hence $\overline{F}(\mu)$ satisfies all the conditions of $(\in, \in \vee q)$-FBId of S. Therefore $\overline{F}(\mu)$ is $(\in, \in \vee q)$-FBId of S. □

Theorem 4.18. Let us suppose that a FSS μ be $(\in, \in \vee q)$-FQId of S and $F : S \to P^*(S)$ be a $SVMH$. Then we have to prove that $\underline{F}(\mu)$ is $(\in, \in \vee q)$-FQId of S.

Proof. From Theorem 4.12, we have for each $z_1, z_2 \in S$, if $z_1 \leq z_2$ implies $F(z_1) \subseteq F(z_2)$. Then $\min\{\underline{F}(\mu)(z_2), 0.5\} \leq \underline{F}(\mu)(z_1)$, Next let for each $z_1 \in S$,

$$\begin{aligned}
\min\{\underline{F}((\mu \circ 1) \wedge (1 \circ \mu))(z_1), 0.5\} &= \min\left\{\bigwedge_{z_1' \in F(z_1)} ((\mu \circ 1) \wedge (1 \circ \mu))\left(z_1'\right), 0.5\right\} \\
&= \bigwedge_{z_1' \in F(z_1)} \min\left(((\mu \circ 1) \wedge (1 \circ \mu))\left(z_1'\right), 0.5\right) \\
&\leq \bigwedge_{z_1' \in F(z_1)} \mu\left(z_1'\right)
\end{aligned}$$

implies

$$\min\{\underline{F}((\mu \circ 1) \wedge (1 \circ \mu))(z_1), 0.5\} \leq \underline{F}(\mu)(z_1)$$

Hence $\underline{F}(\mu)$ is an $(\in, \in \vee q)$-FQId of S. □

Theorem 4.19. Let a FSS μ be $(\in, \in \vee q)$-FQId of S and consider that $F : S \to P^*(S)$ be a $SVIH$. Then we have to prove that $\overline{F}(\mu)$ is $(\in, \in \vee q)$-FQId of S.

Proof. From Theorem 4.13, we have for each $z_1, z_2 \in S$, if $z_1 \leq z_2$ implies $F(z_2) \subseteq F(z_1)$. Then $\min\left\{\overline{F}(\mu)(z_2), 0.5\right\} \leq \overline{F}(\mu)(z_1)$, Next let for each $z_1 \in S$,

$$\begin{aligned} \min\left\{\overline{F}((\mu \circ 1) \wedge (1 \circ \mu))(z_1), 0.5\right\} &= \min\left\{\bigvee_{z_1' \in F(z_1)} ((\mu \circ 1) \wedge (1 \circ \mu))\left(z_1'\right), 0.5\right\} \\ &= \bigvee_{z_1' \in F(z_1)} \min\left\{((\mu \circ 1) \wedge (1 \circ \mu))\left(z_1'\right), 0.5\right\} \\ &\leq \bigvee_{z_1' \in F(z_1)} \mu\left(z_1'\right) \end{aligned}$$

implies

$$\min\left\{\overline{F}((\mu \circ 1) \wedge (1 \circ \mu))(z_1), 0.5\right\} \leq \overline{F}(\mu)(z_1)$$

Hence it is proved that $\overline{F}(\mu)$ is an $(\in, \in \vee q)$-FQId of S. □

5 Conclusion

OSGs is a significant algebraic structure having partial ordered with associative binary operations. OSGs have broad applications in various fields such as coding theory, automata theory and computer science etc. In this manuscript we have originated the approximations of FIds, FBIds, FIIds and FQIds of OSGs on the basis of isotone and monotone mapping. It is clear that these two mappings play a significant role for investigating the approximation of FIds in OSGs. Moreover in the idea of approximation is generalized to $(\in, \in \vee q)$-FIds, FBIds, FIIds and FQIds.

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