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Betweenness centrality in convex amalgamation of graphs*

Research Article

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Abstract: Betweenness centrality measures the potential or power of a node to control the communication over the network under the assumption that information flows primarily over the shortest paths between pair of nodes. The removal of a node with highest betweenness from the network will most disrupt communications between other nodes because it lies on the largest number of paths. A large network can be thought of as inter-connection between smaller networks by means of different graph operations. Thus the structure of a composite graph can be studied by analysing its component graphs. In this paper we present the betweenness centrality of some classes of composite graphs constructed by the graph operation called amalgamation or merging.

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Keywords: Betweenness centrality, Central vertex, Extreme vertex, Convex subgraph, Vertex amalgamation, Edge amalgamation, Path amalgamation, Subgraph amalgamation.

1. Introduction

A large network can be thought of as inter-connection between smaller networks by means of different graph operations. Graph operations are important for constructing new classes of composite graphs and many of the structural properties of larger graphs can be derived from their component graphs. There are many operations on two graphs G_1 and G_2 which result in a larger graph G.

In this paper we define some betweenness centrality concepts, and derive the betweenness centrality for some classes of composite graphs constructed by the graph operation *subgraph-amalgamation*.

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2. Some betweenness centrality concepts

The concept of betweenness centrality of a vertex was first introduced by Bavelas in 1948 [1].

Definition 2.1. *[3]. Let G be a graph and $x \in V(G)$, then the betweenness centrality of x in G, denoted by $B_G(x)$ or simply $B(x)$ is defined as*

$$B_G(x) = \sum_{s,t \in V(G)\setminus\{x\}} \frac{\sigma_{st}(x)}{\sigma_{st}}$$

where $\sigma_{st}(x)$ denotes the number of shortest s-t paths in G passing through x and σ_{st}, the total number of shortest s-t paths in G. The ratio $\frac{\sigma_{st}(x)}{\sigma_{st}}$ is called pair dependency or partial betweenness of (s,t) on x, denoted by $\delta_G(s,t,x)$.

Betweenness centrality of some well known graphs has been studied in [7] and we use the following definitions [8].

2.1. Betweenness centrality of a vertex in a subgraph

Definition 2.2. *Let G be a graph and H a subgraph of G. Let $x \in V(H)$, then the betweenness centrality of x in H denoted by $B_H(x)$ is defined as*

$$B_H(x) = \sum_{s,t \in V(H)\setminus\{x\}} \frac{\sigma^H_{st}(x)}{\sigma^H_{st}}$$

where $\sigma^H_{st}(x)$ and σ^H_{st} denotes the number of shortest s-t paths passing through x and the total number of shortest s-t paths respectively, lying in H.

2.2. Betweenness centrality of a vertex induced by a subgraph

Definition 2.3. *Let G be a graph and H a subgraph of G. Let $x \in V(G)$, then the betweenness centrality of x induced by H denoted by $B(x,H)$ is defined as*

$$B(x,H) = \sum_{s,t(\neq x) \in V(H)} \frac{\sigma_{st}(x)}{\sigma_{st}}$$

where $\sigma_{st}(x)$ and σ_{st} denotes the number of shortest s-t paths passing through x and the total number of shortest s-t paths respectively in G.

The betweenness centrality of a vertex induced by a subset $S \subset V(G)$ is defined likewise.

2.3. Betweenness centrality of a vertex induced by a subset

Definition 2.4. *Let G be a graph and S a subset of $V(G)$. Let $x \in V(G)$, then the betweenness centrality of x induced by S denoted by $B(x,S)$ is defined as*

$$B(x,S) = \sum_{s,t(\neq x) \in S} \frac{\sigma_{st}(x)}{\sigma_{st}}$$

where $\sigma_{st}(x)$ and σ_{st} denotes the number of shortest s-t paths passing through x and the total number of shortest s-t paths respectively in G.

2.4. Betweenness centrality of a vertex induced by another vertex

Definition 2.5. *Let G be a graph and $s, x, t \in V(G)$, then the betweenness centrality of x induced by s in G, denoted by $B_G(x, s)$ or simply $B(x, s)$ is defined by*

$$B_G(x, s) = \sum_{t \in V(G) \setminus x} \frac{\sigma_{st}(x)}{\sigma_{st}}.$$

It can be easily seen that in any graph G, the betweenness centrality induced by a vertex on its extreme vertex or an end vertex is zero. Consider the following examples.

$B(x_i, x_j) = 0$ for $x_i, x_j \in K_n$. Let P_n be a path on n vertices $\{x_1, \ldots, x_n\}$, then

$$B(x_i, x_j) = \begin{cases} i - 1, & \text{if } i < j, \\ n - i, & \text{if } j < i. \end{cases}$$

If C_n is a cycle on n vertices $\{x_0, \ldots, x_{n-1}\}$, then
if n is even,

$$B(x_i, x_0) = \begin{cases} \frac{n-1-2i}{2}, & \text{if } 1 \leq d(x_i, x_0) < n/2, \\ 0, & \text{if } d(x_i, x_0) = n/2, \end{cases}$$

if n is odd,

$$B(x_i, x_0) = \frac{n - 1 - 2i}{2}, \quad \text{if } 1 \leq d(x_i, x_0) \leq \frac{n-1}{2}.$$

For a star S_n with central vertex x_0,

$$B(x_i, x_0) = 0,\ B(x_0, x_i) = n - 2 \text{ and} \tag{1}$$
$$B(x_i, x_j) = 0 \text{ for } i, j \neq 0. \tag{2}$$

For a wheel $W_n, n > 5$ with central vertex x_0,

$$B(x_i, x_0) = 0,\ \ B(x_0, x_i) = n - 5.$$

For $i, j \neq 0$,

$$B(x_i, x_j) = \begin{cases} 1/2, & \text{if } d(x_i, x_j) = 1, \\ 0, & \text{if } d(x_i, x_j) = 2. \end{cases}$$

It can be easily seen that for $x_i \in V(G)$, $B_G(x_i) = \frac{1}{2} \sum_{j \neq i} B_G(x_i, x_j)$.

2.5. Betweenness centrality donated by a vertex

Definition 2.6. *Let G be a graph and $x_0 \in V(G)$, then the betweenness centrality donated by x_0 in G, denoted by $DB_G(x_0)$ or simply $DB(x_0)$, is defined as the sum of betweenness values induced by x_0 on all other vertices in G, i.e.,*

$$DB_G(x_0) = \sum_{x \in V(G) \setminus x_0} B_G(x, x_0).$$

The betweenness centrality received by a vertex, $RB_G(x_0)$ is $B_G(x_0)$ by definition.

2.6. Betweenness centrality of a vertex induced by two disjoint subsets

Definition 2.7. *Let G be a graph and $x \in V(G)$. Let S, T be two disjoint subsets of $V(G)$, then the betweenness centrality of x induced by S and T denoted by $B(x, S, T)$ is defined as*

$$B(x, S, T) = B(x, S) + B(x, T).$$

2.7. Betweenness centrality of a vertex induced by two disjoint subsets, one against the other

Definition 2.8. *Let G be a graph and $x \in V(G)$. Let S, T be two disjoint subsets of $V(G)$ where $s(\neq x) \in S$ and $t(\neq x) \in T$, then the betweenness centrality of x induced by S against T, denoted by $B(x, S|T)$ is defined as*

$$B(x, S|T) = \sum_{s \in S,\, t \in T} \frac{\sigma_{st}(x)}{\sigma_{st}}$$

where $\sigma_{st}(x)$ and σ_{st} denotes the number of shortest s-t paths passing through x and the total number of shortest s-t paths respectively in G.

In metric graph theory, a convex subgraph of an undirected graph G is a subgraph that includes every shortest path in G between two of its vertices. A subgraph H of a graph G is an isometric subgraph, if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. Clearly, a convex subgraph is an isometric subgraph, but the converse need not be true.

3. Subgraph-amalgamation

One method of constructing composite graphs is *merging* or *pasting* two or more graphs together along a common subgraph. For any finite collection of graphs G_i, each with a fixed isomorphic subgraph H as common, the *subgraph-amalgamation* is the graph obtained by taking the union of all the G_i and identifying their fixed subgraphs H's. The simplest one is *vertex-amalgamation* or *vertex-merging*.

Theorem 3.1. *Let G be the graph obtained by merging the graphs $\{G_i\}_{i=1}^n$ along n copies of isomorphic induced common convex subgraph H where $H \subset G_i\ \forall i$. Let $S_i = V(G_i - H)\ \forall i$. Then, for $x \in H$ and $u \in G_k - H$,*

$$\begin{aligned} B_G(x) &= \sum_{i=1}^{n} B_{G_i}(x) - (n-1)B_H(x) + \sum_{i<j} B(x, S_i|S_j), \\ B_G(u) &= B_{G_k}(u) + \sum_{i \neq k} B(u, S_i|S_k). \end{aligned}$$

Proof. Consider the graphs $\{G_i\}_{i=1}^n$. Let G be the graph obtained by merging G_i along n copies of isomorphic induced common convex subgraph H where $H \subset G_i\ \forall i$. See Figure 1. Now the subgraphs $G_i - H$ and H form a partition of G. Any path joining a vertex of $G_i - H$ and a vertex of $G_j - H$ for $i \neq j$ passes through at least one vertex of H. Let $x \in H$, then its betweenness centrality $B_G(x)$ in G is due to the contribution of possible pairs of vertices from each G_i and from different pairs of $G_i - H$. Since H is convex, $B_H(x)$ is repeated n times in $\sum_{i=1}^n B_{G_i}(x)$ and hence we get $B_G(x) = \sum_{i=1}^n B_{G_i}(x) - (n-1)B_H(x) + \sum_{i<j} B(x, S_i|S_j)$. Let $u \in G_k - H$. To find $B_G(u)$, consider a pair of vertices $s, t \in G$ which contributes to $B_G(u)$, then $s \in S_k$, and t may be either in G_k or in its complement G'_k. Those $t \in G_K$ gives $B_{G_k}(u)$ and those $t \in G'_k$ gives $B(u, S_i|S_k)$ for $i \neq k$. Hence $B_G(u) = B_{G_k}(u) + \sum_{i \neq k} B(u, S_i|S_k)$. □

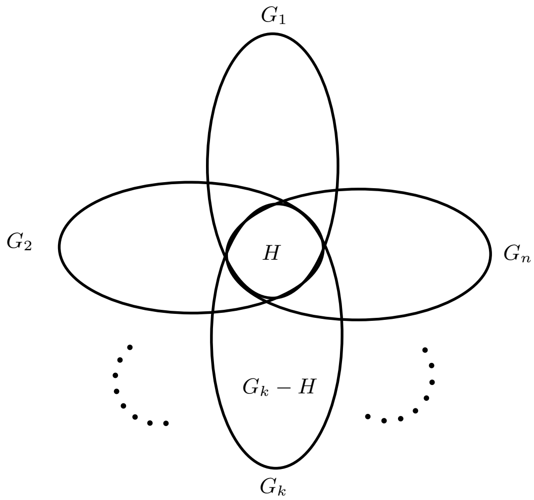

Figure 1. Subgraph-amalgamation

Algorithm 3.2. *Algorithm for Betweenness Computation of a Vertex $x \in V(H)$ in Subgraph Amalgamation*

Require: *Graphs $G_1, G_2, \dots, G_n$, convex common subgraph H, vertex $x \in H$*
Ensure: *Betweenness of vertex x, $B_G(x)$*
1: $S_i = V(G_i - H)$ *for* $i = 1, \dots, n$.
2: *Compute betweenness of x in G_i, $B_{G_i}(x)$ for $i = 1, \dots, n$. Add all these betweenness values to get $B_{ComponentSum}(x)$*
3: *Compute betweenness of x in H, $B_H(x)$.*
4: *Compute betweenness of x in S_i, $B_{S_i}(x)$ for $i = 1, \dots, n$.*
5: *Find $B(x, S_i|S_j) = \sum_{s \in S_i, t \in S_j} \frac{\sigma_{st}(x)}{\sigma_{st}}$ for all $i < j, i = 1, \dots, n$. Add all these betweenness values to get $B_{SubsetSum}(x)$*
6: *Determine $B_G(x) = B_{ComponentSum}(x) - (n-1) * B_H(x) + B_{SubsetSum}(x)$*

Theorem 3.3. *Let G be the graph obtained by merging the graphs $\{G_i\}_{i=1}^n$ along n copies of isomorphic induced common convex subgraph H where $H \subset G_i \forall i$. Then Algorithm 3.2 correctly computes the betweenness value of vertex $x \in H$ in $O(n'm')$ time where n' is the order of the largest component graph G_i and m' the number of edges in G_i.*

Proof. Proof of *Algorithm 3.2* follows from Theorem 3.1.

The efficient algorithm by Brandes [2] compute betweenness centrality of all vertices in an unweighted graph in $O(nm)$ time and $O(m+n)$ space where m is the number of edges in the graph and n is the number of vertices. Here we compute the betweenness centrality of a vertex using partial betweenness of different components of the amalgamation. Therefore, the complexity gets reduced to $O(n'm')$ where n' is the order of the largest component graph G_i and m' the number of edges in G_i. □

Algorithm 3.4. *Algorithm for Betweenness Computation of Vertex u in $G_k - H$ in a Subgraph Amalgamation*

Require: *Graphs $G_1, G_2, \dots, G_n$, convex common subgraph H, vertex $u \in V(G_k) - V(H)$*
Ensure: *Betweenness of vertex x, $B_G(x)$*
1: $S_i = V(G_i - H)$ *for* $i = 1, \dots, n$
2: *Compute betweenness of u in G_k, $B_{G_k}(u)$.*
3: *Compute betweenness of u in S_i, $B_{S_i}(u)$ for $i = 1, \dots, n$.*

4: Find $B(u, S_i|S_k) = \sum_{s\in S_i,\, t\in S_k} \frac{\sigma_{st}(u)}{\sigma_{st}}$ *for all* $i \neq k$. *Add all these betweenness values to get* $B_{SubsetSum}(u)$

5: Determine $B_G(u) = B_{G_k}(u) + B_{SubsetSum}(x)$

Theorem 3.5. *Let G be the graph obtained by merging the graphs $\{G_i\}_{i=1}^{n}$ along n copies of isomorphic induced common convex subgraph H where $H \subset G_i \forall i$. Then Algorithm 3.4 correctly computes the betweenness value of vertex u in $G_k - H$ in $O(n_k m_k)$ time where n_k is the order of G_k and m_k the number of edges in G_k.*

Proof. Proof of *Algorithm 3.4* follows from Theorem 3.1. □

3.1. Path-amalgamation of graphs

Two cycles connected by merging along a common subgraph

Proposition 3.6. *Let G be the graph obtained by merging two cycles C_m and C_n along a common path $P_p = \{x_1, \ldots, x_p\}$ as common subgraph where $p < min\{\lceil \frac{m}{2} \rceil, \lceil \frac{n}{2} \rceil\}$ and $U = \{u_1, \ldots, u_{m-p}\}$ and $V = \{v_1, \ldots, v_{n-p}\}$, be the vertex sets of $C_m - P_p$ and $C_n - P_p$ respectively, then the betweenness centrality of C_m in G is given by*

$$B_G(x_r) = B_{C_m}(x_r) + B_{C_n}(x_r) - (r-1)(p-r) + B(x_r, U|V), \ \textit{for } x_r \in P_p,$$
$$B_G(u_r) = B_{C_m}(u_r) + B(u_r, U|V), \ \textit{for } u_r \in U,$$

where $B(x_r, U|V)$ and $B(u_r, U|V)$ are given by

Case 1: *When C_m and C_n are even*

$$B(x_r, U|V) = \begin{cases} mn/2 - (m+n)(p-1/2) + 2(p^2 - p + 1/3), & \textit{for } 1 < r < p, \\ 3mn/4 - (m+n)(p-1/4) + (9p^2 - 6p + 2)/6, & \textit{for } r = 1, p. \end{cases}$$

$$B(u_r, U|V) = \begin{cases} (n-p)(m-p-2r)/2, & \textit{for } 1 \le r \le \frac{m}{2} - p, \\ n(2p-3)/4 - (3p^2 - 3p - 2)/6, & \textit{for } r = \frac{m}{2} - p + 1, \\ k(k-p+1) + (n-p)(p-2)/2 + 1/6, & \textit{for } r = \frac{m}{2} - p + 1 + k, \ 1 \le k \le p-2. \end{cases}$$

Case 2: *When C_m and C_n are odd*

$$B(x_r, U|V) = \begin{cases} 2[(m+1)/2 - p][(n+1)/2 - p], & \textit{for } 1 < r < p, \\ 3mn/4 - (m+n)(p-1/4) + (6p^2 - 4p + 1)/4, & \textit{for } r = 1, p. \end{cases}$$

$$B(u_r, U|V) = \begin{cases} (n-p)(m-p-2r)/2, & \textit{for } 1 \le r \le \frac{m+1}{2} - p, \\ (p-2)(n-p-1)/2, & \textit{for } r = \frac{m+1}{2} - p + 1, \\ k(k-p+2) + (p-2)(n-p-1)/2, & \textit{for } r = \frac{m+1}{2} - p + 1 + k, \ 1 \le k \le p-2. \end{cases}$$

Case 3: *When C_m is even and C_n is odd*

$$B(x_r, U|V) = \begin{cases} (m-2p+1)(n-2p+1)/2, & \textit{for } 1 < r < p, \\ 3mn/4 - (m+n)(p-1/4) + (6p^2 - 4p + 1)/4, & \textit{for } r = 1, p. \end{cases}$$

$$B(u_r, U|V) = \begin{cases} (n-p)(m-p-2r)/2, & \textit{for } 1 \le r \le \frac{m}{2} - p, \\ 1/2[(n-p)(p-2) + (n+1)/2 - p], & \textit{for } r = \frac{m}{2} - p + 1, \\ k(k-p+1) + (n-p)(p-2)/2, & \textit{for } r = \frac{m}{2} - p + 1 + k, \ 1 \le k \le p-2. \end{cases}$$

$$B(v_r, U|V) = \begin{cases} (m-p)(n-p-2r)/2, & \textit{for } 1 \le r \le \frac{n+1}{2} - p, \\ 1/2[m(p-2) - p^2 + p + 2], & \textit{for } r = \frac{n+1}{2} - p + 1, \\ k(k-p+2) + (p/2)(m-p+1) - m + 1, & \textit{for } r = \frac{n+1}{2} - p + 1 + k, \ 1 \le k \le p-2. \end{cases}$$

Proof. Let C_m and C_n be two cycles merged along a common path P_p induced by the common vertices $X = \{x_1, \ldots, x_p\}$ where $p < \min\{\lceil \frac{m}{2} \rceil, \lceil \frac{n}{2} \rceil\}$. Let $U = \{u_1, \ldots, u_{m-p}\}$ and $V = \{v_1, \ldots, v_{n-p}\}$ be the remaining vertices in C_m and C_n respectively. Consider the three cycles C_m, C_n and C', where C' is the cycle induced by $U, V, \{x_1\}$ and $\{x_p\}$. By symmetry, the vertices in each of the sets X, U, V from either ends have the same betweenness centrality in G.

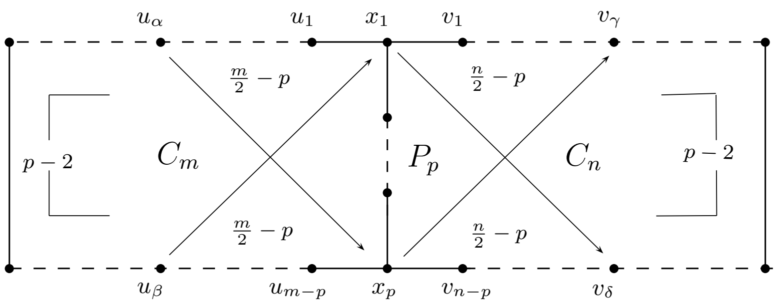

Figure 2. Two even cycles merged along a common subgraph

Case 1: Both C_m and C_n are even.

Let u_α, u_β be the eccentric vertices of x_p and x_1 respectively in C_m and v_δ, v_γ be their eccentric vertices in C' where $\alpha = \frac{m}{2} - p + 1$, $\beta = \frac{m}{2}$, $\gamma = \frac{n}{2} - p + 1$ and $\delta = \frac{n}{2}$. See Figure 2. Let $x_r \in X$. By Theorem 3.1, $B_G(x_r) = B_{C_m}(x_r) + B_{C_n}(x_r) - B_P(x_r) + B(x_r, U|V)$. Let us find $B(x_r, U|V)$ as the others are known. Consider $u \in U$ and $v \in V$, then for each (u, v) pair there are at most three shortest $u - v$ paths and in the case of (u_α, v_δ) and (u_β, v_γ) there are three shortest $u - v$ paths P_1, P_2 and P_3 passing through the end vertices x_1, x_p and the whole path P.

(a) Consider the vertex x_r for $1 < r < p$, then those pairs (u_i, v_j) for $1 \leq i \leq \alpha$, $\delta \leq j \leq n - p$, or (u_i, v_j) for $\beta \leq i \leq m - p$, $1 \leq j \leq \gamma$ contribute betweenness centrality $\left(\frac{m}{2} - p\right)\left(\frac{n}{2} - p\right) + \frac{1}{2}\left(\frac{m}{2} - p + \frac{n}{2} - p\right) + \frac{1}{3}$ to x_r. Thus by Theorem 3.1 we get,

$$B_G(x_r) = \frac{1}{8}\left[(m-2)^2 + (n-2)^2\right] - (r-1)(p-r) + 2\left[\left(\frac{m}{2} - p\right)\left(\frac{n}{2} - p\right) + \frac{1}{2}\left(\frac{m}{2} - p + \frac{n}{2} - p\right) + \frac{1}{3}\right].$$

(b) Consider the vertex x_1, then the pairs (u_i, v_j) for $1 \leq i < \alpha$, $1 \leq j \leq n - p$ contributes $(\frac{m}{2} - p)(n - p)$; Now for $i = \alpha$, the pairs for $1 \leq j < \beta$ contributes $\frac{n}{2} - 1$, $j = \beta$ contributes $2/3$, $\beta < j \leq n - p$ contributes $\frac{1}{2}(\frac{n}{2} - p)$ giving the sum $\frac{1}{12}(9n - 6p - 4)$ to x_1. The vertices lying between u_α and u_β contribute $\frac{1}{2}(n-p)(p-2)$. Consider u_β, it makes pairs with v_j, $1 \leq j \leq \gamma$ and gives $\frac{n}{2} - p + \frac{2}{3}$. Again the vertices on the right of u_β with the same set give $(\frac{m}{2} - p)\left[(\frac{n}{2} - p) + \frac{1}{2}\right]$. Summing all these contributions, we get $B(x_1, U|V)$.

(c) Consider the vertex u_r for $1 \leq r < \alpha$, now the vertices lying between u_r and u_α make pairs with all vertices of V giving the contribution $(\frac{m}{2} - p - r)(n - p)$. Again u_α and the vertices lying between u_α and u_β contribute the sum as earlier. But u_β makes pairs with v_j for $1 \leq j \leq \gamma$ and gives $\frac{1}{2}(\frac{n}{2} - p) + \frac{1}{3}$ and the right of u_β has no contribution. Summing all the above values gives $B(u_r, U|V)$ for $1 \leq r < \alpha$.

(d) Consider the vertex u_α, now the vertices u_i for $\alpha < i \leq \beta$ give the sum $\frac{1}{2}(n-p)(p-2) + \frac{1}{2}(\frac{n}{2} - p) + \frac{1}{3}$.

(e) Consider u_r for $\alpha < r \leq \lceil \frac{m-p}{2} \rceil$. Let $r = \alpha + k$ where $1 \leq k \leq \lceil \frac{p}{2} \rceil - 1$. Now u_α and u_β gives $u_{\alpha+k}$ the same contribution as $\frac{1}{2}(\frac{n}{2} - p) + \frac{1}{3}$. The vertices lying between u_α and $u_{\alpha+k}$ gives the sum $(k-1)(\frac{n}{2} - p + \frac{1}{2}) + \frac{k(k-1)}{2}$ and between $u_{\alpha+k}$ and u_β gives $\frac{1}{2}(n - k - p)(p - k - 2)$. The total of these contribution gives the expression for $B(u_r, U|V)$ for $r > \alpha$.

Case 2: Both C_m and C_n are odd.

Consider u_α and u_β, a pair of extreme vertices of the end vertices x_p and x_1 of P where $\alpha = \frac{m+1}{2} - p + 1$ and $\beta = \frac{m-1}{2}$ and v_δ and v_γ be their eccentric vertices in C' where $\delta = \frac{n-1}{2}$, $\gamma = \frac{n+1}{2} - p + 1$. If $u_i \in U$ and $v_j \in V$, then for $x_r \in P$ where $1 < r < p$, $\{(u_i, v_j) : 1 \leq i < \alpha, \delta < j \leq n-p\}$ or $\{(u_i, v_j) : \beta < i \leq m-p, 1 \leq j < \gamma\}$ contributes the sum $\left(\frac{m+1}{2} - p\right)\left(\frac{n+1}{2} - p\right)$. Since there exists no more pair, $B(x_r, U|V) = 2\left(\frac{m+1}{2} - p\right)\left(\frac{n+1}{2} - p\right)$. Consider the vertex x_1. Now the pairs $\{(u_i, v_j) : 1 \leq i < \alpha, 1 \leq j \leq n-p\}$ contributes the sum $\left(\frac{m+1}{2} - p\right)(n-p)$. The vertices $\{u_i : \alpha \leq i \leq \beta\}$ contributes the sum $\frac{1}{2}(n-p)(p-1)$ and the pairs $\{(u_i, v_j) : \beta < i \leq m-p, 1 \leq j \leq \frac{n+1}{2} - p\}$ contributes the sum $\left(\frac{m+1}{2} - p\right)\left(\frac{n+1}{2} - p\right)$. Hence $B(x_1, U|V) = \left(\frac{m+1}{2} - p\right)\left(\frac{n+1}{2} - p\right) + (n-p)\left(\frac{m+p}{2} - p\right)$. Consider the vertex $u_r, 1 \leq r < \alpha$, the vertices lying between u_r and u_α contributes the sum $(\alpha - r - 1)(n-p)$, vertices from u_α to u_β as given above and no vertex from the right. Hence $B(u_r, U|V) = \left(\frac{m+1}{2} - p - r\right)(n-p) + \frac{1}{2}(n-p)(p-1)$ for $1 \leq r < \alpha$. For u_α, the vertices from $u_{\alpha+1}$ to u_β contributes $\frac{1}{2}(p-2)(n-p-1)$. Hence $B(u_\alpha, U|V) = \frac{1}{2}(p-2)(n-p-1)$. Consider a vertex on the left of u_α, say $u_{\alpha+k}$, then no vertex on the right of u_α is considered. The vertices from u_α to $u_{\alpha+k-1}$ contributes $\frac{1}{2}k(n+k-2p+1)$ and the vertices from $u_{\alpha+k+1}$ to u_β contribute $\frac{1}{2}(p-k-2)(n-p-k-1)$ and no more vertex from the right. Hence, $B(u_{\alpha+k}, U|V) = \frac{1}{2}k(n+k-2p+1) + \frac{1}{2}(p-k-2)(n-p-k-1)$.

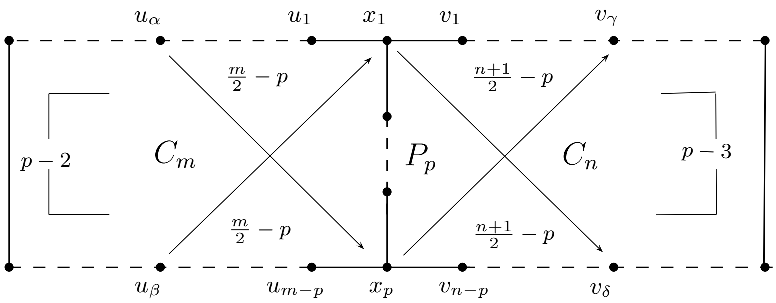

Figure 3. Even and odd cycles merged along a common subgraph

Case 3: C_m is even and C_n is odd.

Let u_α and u_β where $\alpha = \frac{m}{2} - p + 1$, $\beta = \frac{m}{2}$ be the eccentric vertices in C_m for the end vertices x_p, x_1 of P and v_δ and v_γ be their eccentric vertices in C' where $\delta = \frac{n-1}{2}$, $\gamma = \frac{n+1}{2} - p + 1$. See Figure 3.

Consider $x_r \in P$ such that $1 < r < p$. If $u_i \in U$ and $v_j \in V$, then for each pair $\{(ui, v_j) : 1 \leq i < \alpha, \delta < j \leq n-p$ or $\beta < i \leq m-p, 1 \leq j < \gamma\}$ there exists a geodesic passing through x_r and contributes the sum $(\frac{m}{2} - p)\left(\frac{n+1}{2} - p\right)$ and $\frac{1}{2}\left(\frac{n+1}{2} - p\right)$ for $i = \alpha$. Since there exists no such other pair, $B(x_r, U|V) = \frac{1}{2}(m-2p+1)(n-2p+1)$.

Consider the vertex x_1. Now $\{(u_i, v_j) : 1 \leq i < \alpha, 1 \leq j \leq n-p\}$ contributes the sum $(\frac{m}{2} - p)(n-p)$. $\{(u_\alpha, v_j) : 1 \leq j \leq \delta\}$ and $\{(u_\alpha, v_j) : \delta < j \leq n-p\}$ contributes the sum $\frac{n-1}{2} + \frac{1}{2}\left(\frac{n+1}{2} - p\right)$. $\{(u_\beta, v_j) : 1 \leq j \leq \gamma - 1\}$ contributes the sum $\frac{n+1}{2} - p$. The vertices lying between u_α and u_β contributes the sum $\frac{1}{2}(n-p)(p-2)$ and each vertex on the right of u_β gives the sum $\frac{n+1}{2} - p$. The total of these sums gives $B(x_1, U|V)$ which is $\frac{1}{4}\left[3mn - (4p-1)(m+n) + 6p^2 - 4p + 1\right]$.

Consider the vertex u_r, for $1 \leq r < \alpha$. Now the vertices lying between u_r and u_α contributes the sum $(\alpha - r - 1)(n-p)$. Again u_α and the vertices lying between u_α and u_β have the same contribution as mentioned above, u_β gives $\frac{1}{2}\left(\frac{n+1}{2} - p\right)$. For the vertex u_α, vertices from $u_{\alpha+1}$ to $u_{\beta-1}$ and then u_β contributes to it as earlier. Consider a vertex on the right of u_α, say $u_{\alpha+k}$. Again u_α and u_β contributes the same as $\frac{1}{2}\left(\frac{n+1}{2} - p\right)$. The vertices lying between u_α and $u_{\alpha+k}$ contribute $(k-1)\left(\frac{n+1}{2} - p\right) + \frac{1}{2}k(k-1)$. The vertices lying between $u_{\alpha+k}$ and u_β contribute $\frac{1}{2}(n-p-k)(p-k-2)$. The sum of these gives $B(u_{\alpha+k}, U|V)$. Consider the vertex v_r in C_n for

$1 \le r < \gamma$. The vertices lying between v_r and v_γ offers $(\frac{n+1}{2}-p-r)(m-p)$ and the vertices from v_γ to v_δ offers $(p-1)(m-p)/2$ giving $B(v_r, U|V) = (m-p)(n-p-2r)/2$. For v_γ, the vertices from $v_{\gamma+1}$ to v_δ gives $B(v_\gamma, U|V) = 1/2[m(p-2)-p^2+p+2]$. For $v_{\gamma+k}$, $1 \le k \le p-2$, the vertices from $v_{\gamma+k+1}$ to v_δ offers $\frac{1}{2}(p-k-2)(m-p-k-1)$ and vertices from v_γ to $v_{\gamma+k-1}$ offers $\frac{k}{2}(m+k-2+1)$ so that vertices symmetric from v_γ and v_δ offers the same giving $B(v_{\gamma+k}, U|V) = k(k-p+2)+(p/2)(m-p+1)-m+1$.

□

Corollary 3.7. *Let G be the graph obtained by merging m copies of cycle C_n along a common path $P_p = \{x_1, \ldots, x_p\}$ as common subgraph where $p < \lceil \frac{n}{2} \rceil$ and $U = \{u_1, \ldots, u_{n-p}\}$ be the vertex sets of $C_n - P_p$. Then the betweenness centrality of C_n in G is given by*

$$\begin{aligned} B_G(x_r) &= mB_{C_n}(x_r) - (m-1)(r-1)(p-r) + \binom{m}{2} B(x_r, U|V), \text{ for } x_r \in P_p, \\ B_G(u_r) &= B_{C_n}(u_r) + (m-1)B(u_r, U|V), \text{ for } u_r \in U. \end{aligned}$$

where $B(x_r, U|V)$ and $B(u_r, U|V)$ are given by

Case 1: *If C_n is even, then*

$$\begin{aligned} B(x_r, U|V) &= \begin{cases} n^2/2 - 2n(p-1/2) + 2(p^2-p+1/3), & \text{for } 1 < r < p, \\ 3n^2/4 - 2n(p-1/4) + (9p^2-6p+2)/6, & \text{for } r = 1,\, p. \end{cases} \\ B(u_r, U|V) &= \begin{cases} (n-p)(n-p-2r)/2, & \text{for } 1 \le r \le \frac{n}{2} - p, \\ n(2p-3)/4 - (3p^2-3p-2)/6, & \text{for } r = \frac{n}{2} - p + 1, \\ k(k-p+1) + (n-p)(p-2)/2 + 1/6, & \text{for } r = \frac{n}{2} - p + 1 + k,\ 1 \le k \le p-2. \end{cases} \end{aligned}$$

Case 2: *If C_n is odd, then*

$$B(x_r, U|V) = \begin{cases} 2\big[(n+1)/2 - p\big]^2, & \text{for } 1 < r < p, \\ 3n^2/4 - 2n(p-1/4) + (6p^2-4p+1)/4, & \text{for } r = 1,\, p. \end{cases}$$

$$B(u_r, U|V) = \begin{cases} (n-p)(n-p-2r)/2, & \text{for } 1 \le r \le \frac{n+1}{2} - p, \\ (p-2)(n-p-1)/2, & \text{for } r = \frac{n+1}{2} - p + 1, \\ k(k-p+2) + (p-2)(n-p-1)/2, & \text{for } r = \frac{n+1}{2} - p + 1 + k,\ 1 \le k \le p-2. \end{cases}$$

Proposition 3.8. *Let G be the graph obtained by merging both ends of m copies of paths P_n together where $V(P_n) = \{1, 2, \ldots, n\}$. Then the betweenness centrality of G is given by*

$$B(r) = \begin{cases} \frac{1}{4}m(m-1)(n-2)^2, & \text{for } r = 1, n, \\ \frac{1}{2}(n-1)(n-3) + \frac{1}{m} + \frac{(m-2)}{2}\Big[(n-1-r)^2 + (r-2)^2\Big], & \text{for } 1 < r < n. \end{cases}$$

Proof. Since the ends of m copies of path P_n are separately merged, any two copies of P_n form an even cycle C_{2n-2} in G. Since there are $\binom{m}{2}$ such cycles, $B(r) = \binom{m}{2} B_{C_{2n-2}}(r) = \frac{1}{4}m(m-1)(n-2)^2$ for $r = 1, n$. Consider an internal vertex of any path P_n say $P_n^{(i)}$. Now $P_n^{(i)}$ and $P_n^{(j)}$ for $i \ne j$ form a cycle of $2n-2$ vertices in G, where the pair of end vertices gives the centrality $\frac{1}{m}$ instead of $\frac{1}{2}$. Consider the path $P_n^{(k)}$ where $k \ne j \ne i$. Let U_i and U_k denotes the vertex sets of $P_{n-2}^{(i)}$ and $P_{n-2}^{(k)}$ respectively where the merged end vertices 1 and n are deleted. Therefore, for any internal vertex r of P_n,

$$\begin{aligned} B_G(r) &= B_{C_{2n-2}}(r) - \frac{1}{2} + \frac{1}{m} + \sum_{k \ne i \ne j} B(r, U_i|U_k) \\ &= \frac{1}{2}(n-1)(n-3) + \frac{1}{m} + \frac{(m-2)}{2}\Big[(n-1-r)^2 + (r-2)^2\Big] \end{aligned}$$

since,

$$\begin{aligned} B(r, U_i, |U_k) &= \frac{\big[n-(r+1)\big]^2}{2} + \frac{1}{2}[1+3+\ldots(r-2) \text{ terms}] \\ &= \frac{1}{2}\big[(n-1-r)^2 + (r-2)^2\big]. \end{aligned}$$

□

3.2. Edge-amalgamation of graphs

The edge amalgamation of $\{G_i\}_{i=1}^{n}$ is the graph obtained by taking the union of all the G_i and identifying their fixed edges. An amalgamation of two edges $e_1 = u_1v_1$ of a graph G_1 and $e_2 = u_2v_2$ of a graph G_2 is a graph created by identifying u_1 with u_2 and v_1 with v_2 and then deleting one of the two edges corresponding to e_1 or e_2. The other edge will be called the *amalgamated edge*.

The edge amalgamation of cycles are called *generalized books* [5].

Proposition 3.9. *Let two cycles C_m and C_n are connected by merging a pair of edges. Let u_i, v_i be any vertices on C_m and C_n respectively at a distance i from the merged edge. Then betweenness centrality of the resulting graph is given by*

Case 1: *If both C_m and C_n are even, then*

$$B(u_i) = \begin{cases} \frac{(m-2)^2}{8} + \frac{(n-2)^2}{8} + \frac{(m-3)(n-3)}{4} + \frac{(m-2)(n-2)}{2} + \frac{1}{12}, & \text{for } i = 0, \\ \frac{(m-2)^2}{8} + \frac{(n-2)(m-2-2i)}{2}, & \text{for } 1 \le i < \frac{m}{2} - 1, \\ \frac{(m-2)^2}{8} + \frac{n}{4} - \frac{2}{3}, & \text{for } i = \frac{m}{2} - 1. \end{cases}$$

$B(v_i)$ are obtained on interchanging m and n.

Case 2: *If both C_m and C_n are odd (See Figure 4), then*

$$B(u_i) = \begin{cases} \frac{(m-1)(m-3)}{8} + \frac{(n-1)(n-3)}{8} + \frac{(m-3)(n-3)}{4} + \frac{(m-2)(n-2)}{2}, & \text{for } i = 0, \\ \frac{(m-1)(m-3)}{8} + \frac{(n-2)(m-2-2i)}{2}, & \text{for } 1 \le i < \frac{m-1}{2}, \\ \frac{(m-1)(m-3)}{8}, & \text{for } i = \frac{m-1}{2}. \end{cases}$$

$B(v_i)$ are obtained on interchanging m and n.

Case 3: *If C_m is even and C_n is odd, then*

$$B(u_i) = \begin{cases} \frac{(m-2)^2}{8} + \frac{(n-1)(n-3)}{8} + \frac{(m-3)(n-3)}{4} + \frac{(m-2)(n-2)}{2}, & \text{for } i = 0, \\ \frac{(m-2)^2}{8} + \frac{(n-2)(m-2-2i)}{2}, & \text{for } 1 \le i < \frac{m}{2} - 1, \\ \frac{(m-2)^2}{8} + \frac{n}{4} - \frac{2}{3}, & \text{for } i = \frac{m}{2} - 1. \end{cases}$$

$$B(v_i) = \begin{cases} \frac{(n-1)(n-3)}{8} + (m-2)(n-2-2i)/2, & \text{for } 1 \le i < \frac{n-1}{2}, \\ \frac{(n-1)(n-3)}{8}, & \text{for } i = \frac{n-1}{2}. \end{cases}$$

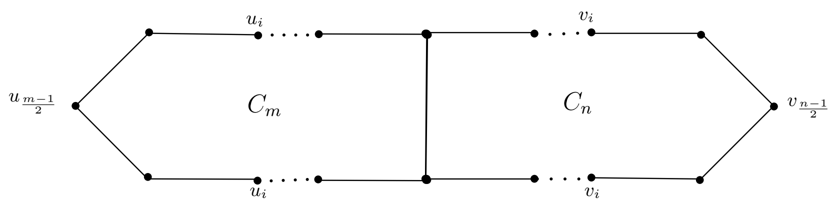

Figure 4. Two odd cycles C_m and C_n identifying a pair of edges

Proof. Consider a merged vertex. Since the merged vertex lies on both cycles C_m and C_n, each cycle induces a value for its betweenness centrality. C_m against C_n also induces a value. The betweenness centrality of merged vertex is their sum. For any other vertex u, it lies on one of the cycle and betweenness centrality is the sum of the betweenness centrality induced by that cycle and the other one on it. □

Corollary 3.10. *Let the odd cycles $C_{n_1}, \dots, C_{n_m}$ share a common edge. Then, for u_0, an end vertex of the common edge and u_i, a vertex in C_{n_k} at a distance i from the common edge, we have*

$$B(u_0) = \frac{1}{8}\sum_i (n_i-1)(n_i-3) + \frac{1}{4}\sum_{i<j}(n_i-3)(n_j-3) + \frac{1}{2}\sum_{i<j}(n_i-2)(n_j-2),$$

$$B(u_i) = \begin{cases} \frac{(n_k-1)(n_k-3)}{8} + (\frac{n_k}{2}-1-i)\sum_{i\neq k}(n_i-2), & \text{for } 1 \le i < \frac{n_k-1}{2}, \\ \frac{(n_k-1)(n_k-3)}{8}, & \text{for } i = \frac{n_k-1}{2}. \end{cases}$$

Corollary 3.11. *If m copies of n-cycles, each are merged at an edge, then for u_0, an end vertex of the common edge and u_i, a vertex at a distance i from the common edge in any n-cycle (See Figure 5), then we have*

Case 1: *If n is even*

$$B(u_i) = \begin{cases} \frac{m(n-2)^2}{8} + \binom{m}{2}\left[\frac{(n-2)^2}{2} + \frac{(n-3)^2}{4} + \frac{1}{12}\right], & \text{when } i = 0, \\ \frac{(n-2)^2}{8} + (m-1)(n-2)(\frac{n}{2}-1-i), & \text{when } 1 \le i < \frac{n}{2}-1, \\ \frac{(n-2)^2}{8} + (m-1)(\frac{n}{4}-\frac{2}{3}), & \text{when } i = \frac{n}{2}-1. \end{cases}$$

Case 2: *If n is odd*

$$B(u_i) = \begin{cases} \frac{m(n-1)(n-3)}{8} + \binom{m}{2}\left[\frac{(n-2)^2}{2} + \frac{(n-3)^2}{4}\right], & \text{when } i = 0, \\ \frac{(n-1)(n-3)}{8} + (m-1)(n-2)(\frac{n}{2}-1-i), & \text{when } 1 \le i < \frac{n-1}{2}, \\ \frac{(n-1)(n-3)}{8}, & \text{when } i = \frac{n-1}{2}. \end{cases}$$

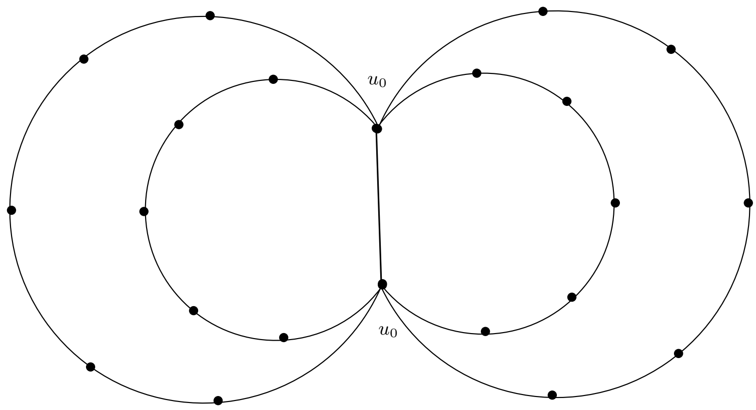

Figure 5. **Four 7-cycles sharing a common edge**

3.3. Vertex-amalgamation of graphs

Definition 3.12. *[6] A graph G in which a vertex is distinguished from other vertices is called a rooted graph and the vertex is called a root (or terminal) of G.*

Identifying the terminal vertices of two graphs is known as vertex-amalgamation and the new graph obtained by vertex-amalgamation of G_1 and G_2 is denoted by $G_1 \cdot G_2$ [4].

Proposition 3.13. *Let $G_1, \ldots, G_k$ be vertex disjoint graphs of order $n_1, \ldots, n_k$ respectively and G, the graph obtained by identifying the vertices $v_i \in G_i$, $\forall i$; if v_0 denotes the merged vertex and $v(\neq v_i) \in G_i$, then*

$$B_G(v_0) = \sum_{i=1}^{k} B_{G_i}(v_i) + \sum_{i<j}(n_i - 1)(n_j - 1) \text{ and}$$

$$B_G(v) = B_{G_i}(v) + B_{G_i}(v, v_0) \sum_{j \neq i}(n_j - 1).$$

Proof. Consider the graph G obtained by identifying the vertices $v_i \in G_i$. If v_0 denotes the merged vertex, i.e, $v_i = v_0$ for all i, v_0 is a cut vertex (See Figure 6) and the removal of v_0 disconnects the graph G into k components. The betweenness centrality of v_0 in G is calculated over all pairs of vertices and each pair belongs to the same component or different ones. The pairs of vertices lying in the same component give the sum $\sum_{i=1}^{k} B_{G_i}(v_i)$ and the pairs of vertices lying in different components give $\sum_{i<j}(n_i-1)(n_j-1)$. Now their sum gives the result. For $v \in G_i$, $v \neq v_i$, a pair of vertices in G provides a contribution to the centrality of v, if one of which belongs to G_i. If $B_{G_i}(v, v_0)$ denotes the betweenness centrality of v induced by v_0 in G_i, then clearly $B_G(v) = B_{G_i}(v_i) + B_{G_i}(v, v_0) \sum_{j \neq i}(n_j - 1)$. □

Algorithm 3.14. *Algorithm for Betweenness Computation of Merged Vertex v_0 in a Vertex Amalgamation*

Require: *Vertex disjoint graphs $G_1, G_2, \ldots, G_k$ of order $n_1, n_2, \ldots, n_k$, and v_0 the merged vertex*
Ensure: *Betweenness of vertex v_0, $B_G(v_0)$*

1. *Compute betweenness of v_i in G_i, $B_{G_i}(v_i)$ for $i = 1, \ldots, k$. Add all these betweenness values to get $B_{ComponentSum}$*
2. *Find $B_{Contribution} = \sum_{i=1,\ldots,k-1}(n_i - 1) * (n_j - 1)$ for $i < j$.*
3. *Determine $B_G(v_0) = B_{ComponentSum} + B_{Contribution}$*

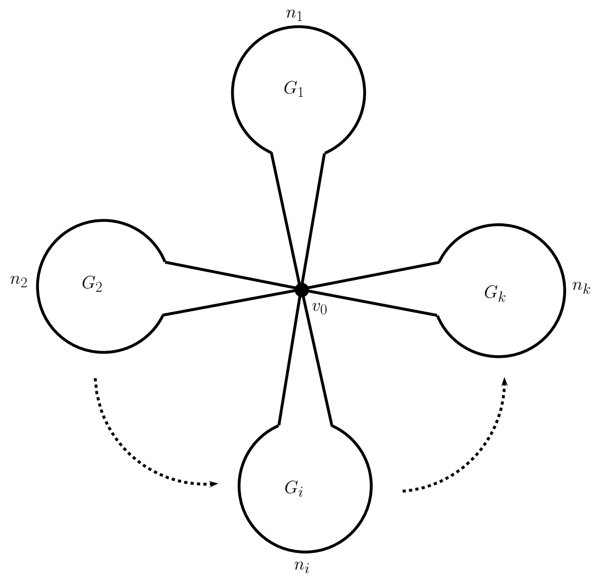

Figure 6. Vertex amalgamation of G_i's

Theorem 3.15. *Let G be the graph obtained by merging graphs $\{G_i\}_{i=1}^{k}$, on a common root vertex v_0. Then Algorithm 3.14 correctly computes the betweenness value of merged vertex v_0 in $O(n'm')$ times where n' is the order of the largest component graph G_i and m' the number of edges in G_i.*

Proof. Proof of *Algorithm 3.14* follows from Proposition 3.13. □

Algorithm 3.16. *Algorithm for Betweenness Computation of Non-terminal Vertex v in G_i in a Vertex Amalgamation*

Require: *Vertex disjoint graphs $G_1, G_2, \ldots, G_k$ of order $n_1, n_2, \ldots, n_k$, v_0 the merged vertex, vertex v a non-terminal vertex in G_i.*
Ensure: *Betweenness of vertex v, $B_G(v)$*
1: *Compute betweenness of v in G_i, $B_{G_i}(v)$.*
2: *Compute betweenness of v induced by v_0 in G_i $B_{G_i}(v, v_0)$.*
3: *Find $B_{Contribution} = \sum_{j=1,\ldots,k; j\neq i}(n_j - 1)$.*
4: *Determine $B_G(v) = B_{G_i}(v) + B_{G_i}(v, v_0) * B_{Contribution}$*

Theorem 3.17. *Let G be the graph obtained by merging graphs $\{G_i\}_{i=1}^{k}$ on a common root vertex v_0. Then Algorithm 3.16 correctly computes the betweenness value of non-terminal vertex v in $O(n_i m_i)$ times where n_i is the order of the component graph G_i and m_i the number of edges in G_i.*

Proof. Proof of *Algorithm 3.16* follows from Proposition 3.13. □

Proposition 3.18. *Let $G_1, \ldots, G_k$ be vertex disjoint graphs of order $n_1, \ldots, n_k$. Consider the graph G obtained on joining the vertices $v_i \in G_i$ to a single vertex v_0 by means of edges and $v(\neq v_i) \in G_i$, then*

$$\begin{aligned} B_G(v_0) &= \sum_{i<j} n_i n_j, \\ B_G(v_i) &= B_{G_i}(v_i) + (n_i - 1)\big(\sum_{j\neq i} n_j + 1\big) \text{ and} \\ B_G(v) &= B_{G_i}(v) + B_{G_i}(v, v_0)\sum_{j\neq i}(n_j - 1). \end{aligned}$$

The above result is significant in view of the fact that networks are often connected by joining to a common node.

If G is a rooted graph, the graph $G^{(n)}$ obtained on identifying the root of n-copies of G is called a *one-point union of n copies of G* [9]. We consider the following examples.

1. Windmill graph

The windmill graph $K_n^{(m)}$ is the one - point union of m copies of the complete graph K_n; $K_n^{(m)}$ has $(n-1)m+1$ vertices and $m\binom{n}{2}$ edges. For example, $K_5^{(4)}$ is given in Figure 7.

Theorem 3.19. *The betweenness centrality of windmill graph $K_n^{(m)}$ is given by*

$$B(v) = \begin{cases} (n-1)^2\dbinom{m}{2}, & \textit{for central vertex,} \\ 0, & \textit{for any other vertex.} \end{cases}$$

Proof. The central vertex of $K_n^{(m)}$ is a cut vertex and the removal of which disconnects into m components of K_{n-1}. Therefore, by Proposition 3.13, the betweenness centrality of the central vertex is given by $\mathcal{C}(n-1, n-1, \ldots m \text{ times}) = (n-1)^2\binom{m}{2}$ where $\mathcal{C}(n_1, n_2, \ldots, n_k) = \sum\limits_{i<j} n_i n_j$. Since other vertices are vertices of the induced subgraph K_{n-1}, their betweenness centrality is zero. □

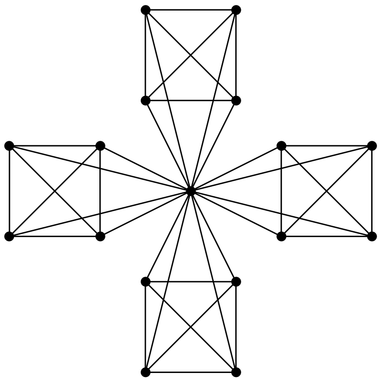

Figure 7. Windmill graph $K_5^{(4)}$

The graph $K_3^{(t)}$ is called a *friendship graph* or *Dutch t-windmill graph*. For example, $K_3^{(5)}$ is given in Figure 8. Every pair of vertices in $K_3^{(t)}$ has exactly one common neighbour. $K_3^{(2)}$ is the butterfly graph and $K_2^{(n)}$ is the star $S_{1,n}$.

Corollary 3.20. *The betweenness centrality of friendship graph $K_3^{(n)}$ is given by*

$$B(v) = \begin{cases} 4\dbinom{n}{2}, & \textit{for central vertex,} \\ 0, & \textit{for any other vertex.} \end{cases}$$

2. Vertex-amalgamation of cycles

Now let us consider the case when different cycles merged at a vertex. See Figure 9.

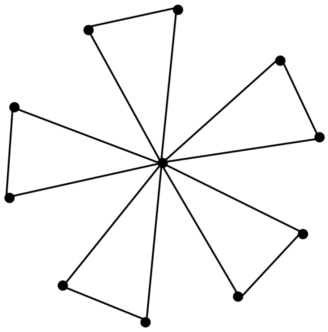

Figure 8. Friendship graph $K_3^{(5)}$

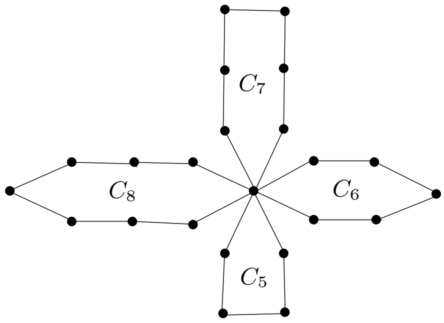

Figure 9. One point union of different cycles

Proposition 3.21. *Let the graph G be the union of m_1 copies of odd cycle C_{n_1} and m_2 copies of even cycle C_{n_2} with a common vertex v_0. In any cycle C, let v_i denotes the vertex such that $d(v_i, v_0) = i$, then,*

$$B(v_0) = \frac{m_1(n_1-1)(n_1-3)}{8} + \frac{m_2(n_2-2)^2}{8} + (n_1-1)^2\binom{m_1}{2} + (n_2-1)^2\binom{m_2}{2} + (n_1-1)(n_2-1)m_1m_2, \text{ and}$$

$$B(v_i) = \begin{cases} \frac{(n_1-1)(n_1-3)}{8} + \binom{\frac{n_1-1-2i}{2}}{} \Big[(m_1-1)(n_1-1) + m_2(n_2-1)\Big], & \text{for } v_i \in C_{n_1},\ 1 \le i \le \frac{n_1-1}{2}, \\ \frac{(n_2-2)^2}{8} + \left(\frac{n_2-1-2i}{2}\right) \Big[m_1(n_1-1) + (m_2-1)(n_2-1)\Big], & \text{for } v_i \in C_{n_2},\ 1 \le i < \frac{n_2}{2}, \\ \frac{(n_2-2)^2}{8}, & \text{for } v_i \in C_{n_2},\ i = \frac{n_2}{2}. \end{cases}$$

Proof. Since m_1 copies of odd cycles C_{n_1} and m_2 copies of even cycles C_{n_2} have a common vertex v_0, v_0 is a cut vertex of G and each cycle contributes its own betweenness centrality to v_0. Again v_0 lies on the path joining vertices of different cycles. Each pair of cycles contributes a betweenness centrality $(n_i-1)^2$ and there are $\binom{m_i}{2}$ pairs of such cycles giving $(n_i-1)^2\binom{m_i}{2}$ to the centrality. Considering the three possible combinations odd-odd, even-even and odd-even pairs of cycles, we get the centrality of v_0 as their sum. Let v_i denotes the vertices at a distance i from v_0. If $v_i \in C_{n_1}$, an odd cycle, for $i \le \frac{n_1-1}{2}$ then from each vertex lying between v_i and $v_{\frac{n_1-1}{2}}$, there is a path passing through v_i to each vertex of the remaining cycles and consequently there is an increase of $\left(\frac{n_1-1-2i}{2}\right)[(m_1-1)(n_1-1) + m_2(n_2-1)]$ for $B(v_i)$. If $v_i \in C_{n_2}$, an even cycle, then from each vertex lying between v_i and $v_{\frac{n}{2}}$, there is a path passing through

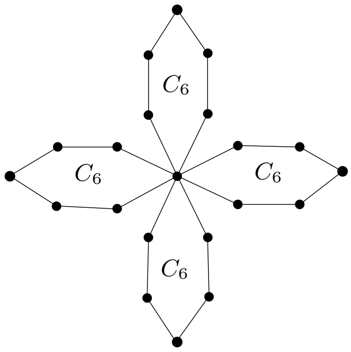

Figure 10. $C_6^{(4)}$ **and** $C_5^{(4)}$

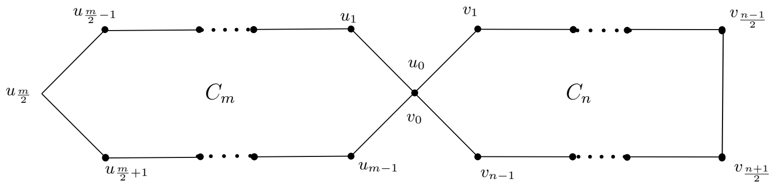

Figure 11. An even cycle C_m and an odd cycle C_n merged at a vertex

v_i to each vertex of the remaining cycles and from the extreme vertex $v_{\frac{n}{2}}$ there are two paths and one of which passes through v_i. Hence there is an increase of centrality $\left(\frac{n_2-1-2i}{2}\right)[m_1(n_1-1)+(m_2-1)(n_2-1)]$ for $B(v_i)$. Since there is no path passing through the extreme vertex $v_{\frac{n_2}{2}}$, its centrality remains the same as that of C_{n_2}, that is, $\frac{(n_2-2)^2}{8}$. □

Corollary 3.22. *Let $C_n^{(m)}$ be the one point union of m copies of C_n. See Figure 10. If v_0 is the common vertex and $v_i \in C_n^{(m)}$ such that $d(v_i, v_0) = i$, then the betweenness centrality of $C_n^{(m)}$ is given by*

Case 1: *When C_n is even*

$$B(v_i) = \begin{cases} \dfrac{m(n-2)^2}{8} + \dbinom{m}{2}(n-1)^2, & \text{when } i = 0, \\ \dfrac{(n-2)^2}{8} + \dfrac{(m-1)(n-1)(n-1-2i)}{2}, & \text{when } 1 \le i < \dfrac{n}{2}, \\ \dfrac{(n-2)^2}{8}, & \text{when } i = \dfrac{n}{2}. \end{cases}$$

Case 2: *When C_n is odd*

$$B(v_i) = \begin{cases} \dfrac{m(n-1)(n-3)}{8} + \dbinom{m}{2}(n-1)^2, & \text{when } i = 0, \\ \dfrac{(n-1)(n-3)}{8} + \dfrac{(m-1)(n-1)(n-1-2i)}{2}, & \text{when } 1 \le i \le \dfrac{n-1}{2}. \end{cases}$$

Corollary 3.23. *Let G be the graph obtained by merging the vertices $u_0 \in C_m$ and $v_0 \in C_n$ and let $u_i \in C_m$ and $v_j \in C_n$ such that $d(u_i, u_0) = i$ and $d(v_j, v_0) = j$. Then the betweenness centrality of G is given by*

Case 1: *If both C_m and C_n are even, then*

$$B(u_i) = \begin{cases} \frac{(m-2)^2}{8} + \frac{(n-2)^2}{8} + (m-1)(n-1), & \textit{when } i = 0, \\ \frac{(m-2)^2}{8} + \frac{(n-1)(m-1-2i)}{2}, & \textit{when } 1 \le i < \frac{m}{2}, \\ \frac{(m-2)^2}{8}, & \textit{when } i = \frac{m}{2}. \end{cases}$$

$B(v_i)$ are obtained by interchanging m and n.

Case 2: *If both C_m and C_n are odd, then*

$$B(u_i) = \begin{cases} \frac{(m-1)(m-3)}{8} + \frac{(n-1)(n-3)}{8} + (m-1)(n-1), & \textit{when } i = 0, \\ \frac{(m-1)(m-3)}{8} + \frac{(n-1)(m-1-2i)}{2}, & \textit{when } 1 \le i \le \frac{m-1}{2}. \end{cases}$$

$B(v_i)$ are obtained by interchanging m and n.

Case 3: *If C_m is even and C_n is odd (see Figure 11), then*

$$B(u_i) = \begin{cases} \frac{(m-2)^2}{8} + \frac{(n-1)(n-3)}{8} + (m-1)(n-1), & \textit{when } i = 0, \\ \frac{(m-2)^2}{8} + \frac{(n-1)(m-1-2i)}{2}, & \textit{when } 1 \le i < \frac{m}{2}, \\ \frac{(m-2)^2}{8}, & \textit{when } i = \frac{m}{2}. \end{cases}$$

$$B(v_j) = \frac{(n-1)(n-3)}{8} + \frac{(m-1)(n-1-2j)}{2}, \ \textit{when } 1 \le j \le \frac{n-1}{2}.$$

Note: If the two cycles are identical, then

Case 1: $m = n$ an even number.

$$B(u_i) = B(v_i), \text{ where}$$

$$B(u_i) = \begin{cases} \frac{(m-2)^2}{4} + (m-1)^2, & \text{when } i = 0, \\ \frac{(m-2)^2}{8} + \frac{(m-1)(m-1-2i)}{2}, & \text{when } 1 \le i < \frac{m}{2}, \\ \frac{(m-2)^2}{8}, & \text{when } i = \frac{m}{2}. \end{cases}$$

Case 2: $m = n =$ an odd number.

$$B(u_i) = B(v_i), \text{ where}$$

$$B(u_i) = \begin{cases} \frac{(m-1)(m-3)}{4} + (m-1)^2, & \text{when } i = 0, \\ \frac{(m-1)(m-3)}{8} + \frac{(m-1)(m-1-2i)}{2}, & \text{when } 1 \le i \le \frac{m-1}{2}. \end{cases}$$

4. Conclusion

Betweenness centrality is a useful metric for analysing graph-structures and networks. When compared to other centrality measures, computation of betweenness centrality is rather difficult as it involves finding the shortest paths between pairs of vertices in a graph. Therefore studying the structure based on graph operations becomes important. Here new graph classes originated by subgraph amalgamation have been studied. This study can be extended to other structures and is therefore helpful for analysing larger classes of graphs.

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