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Solution of KdV and Boussinesq using Darboux Transformation

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Abstract: Two Darboux transformations of the Korteweg-de Vries (KdV) equation and Boussinesq equation are constructed through the Darboux method. Soliton solutions of these two equations are presented by applying the Darboux transformations.

Keywords: Darboux transformation, multisoliton solution, spectral problem, generalized Korteweg-de Vries equations, Boussinesq equation.

1 Introduction

In this paper the solution nonlinear partial differential equations using the Darboux transformation method is presented. This chapter is organized as follow:

- (•) Section 2 : Darboux transformation method (DT).
- (•) Section 3: The singular manifold method illustrated with the deduction of Lax pair for KdV and Hirota-Satsuma equations.
- (•) Section 4: Solution of KdV and Boussinesq using Darboux Transformation.
- (•) Section 5: Summary

2 Darboux transformation method

Darboux transformation (DT) is one of the methods used in solving nonlinear evolution equation and applies to a linear algebraic formulation called; Lax system of equations, associated with the nonlinear evolution equation [1], [2]. Using the Darboux transformation, explicit multi soliton solutions (one to three soliton solutions) are obtained from a trivial initial solution. We will present hereafter one and two soliton solutions.

2.1 One soliton solution

Basically, the Darboux transformation is a linear algebra formulation of the Lax pair solution. Consider the Sturm-Liouville equation [3]

$$-\psi_{xx}+u\psi=\lambda\psi \tag{1}$$

where ψ is the stream velocity and u is the potential and λ is an eigen value.

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At $\lambda = \lambda_1$ eq. (1) solution is denoted by $\psi_1(x,\lambda_1)$ and Darboux transformation of the arbitrary solution of (1) is defined by;

$$\psi[1] = \left(\frac{d}{dx} - \sigma\right)\psi = \left(\frac{d}{dx} - \frac{\psi_1'}{\psi_1}\right)\psi = \frac{W(\psi_1,\psi)}{\psi_1} \tag{2}$$

where $W(\psi_1,\psi) = \psi_1\psi' - \psi_1'\psi$ is the usual wronskian determinant, $\psi_1$and ψ satisfy solutions of equation (1) for λ, λ_1.

The function $\psi[1]$ satisfy eq. (1);

$$-\psi_{xx}[1] + u[1]\,\psi[1] = \lambda\ \psi[1] \tag{3}$$

In parallel we evaluate the Darboux transformation for potential u

$$u[1] = u - 2\sigma_x = u_0 - 2\frac{d^2}{dx^2}ln(\psi_1) \tag{4}$$

where u_0 is an initial (seed) solution of eq. (1), u[1] is a new solution for the same equation.

2.2 Two soliton solution

The DT if applied to (3) once more, produces a new solvable Sturm-Liouville equation. For the second step of this procedure we have;

$$\psi[2] = \left(\frac{d}{dx} - \frac{\psi'_2[1]}{\psi_2[1]}\right)\psi[1] = \left(\frac{d}{dx} - \frac{\psi'_2[1]}{\psi_2[1]}\right)\left(\frac{d}{dx} - \frac{\psi'_1}{\psi_1}\right)\psi \tag{5}$$

where $\psi_2[1]$ is a solution of the eq. (3) with $\lambda = \lambda_2$, generated by some fixed solution $\psi_2(x,\lambda_2)$ of eq.(1);

$$\psi_2[1] = \psi_{2x} - \frac{\psi_{1x}}{\psi_1}\psi_2 \ = \ \frac{W(\psi_1,\psi_2)}{\psi_1} \tag{6}$$

The potential corresponding to $\psi[2]$ is then evaluated as

$$u[2] = u[1] - 2\frac{d^2}{dx^2}ln\psi_2[1] = u_0 - 2\frac{d^2}{dx^2}lnW(\psi_1,\psi_2) \tag{7}$$

The formula (7) can be generalized to for an Nth -times repeated Darboux transformation, expressed completely in the required Reviews Completed regarding initial equation (1) solution. We present in the following sections the derivation of Lax pair through the singular manifold method.

3 The Singular Manifold Method

The Singular Manifold Method (SMM) is a series solution of nonlinear partial differential equations [4]. The SMM allows us to drive Bäcklund transformations, Lax pairs, Darboux transformations for a partial differential equation. The Singular Manifold Method [5] series solution of the PDE is written as:

$$u(z_1,z_2,\ldots,z_n) = \int_{j=0}^{\infty} u_j(z_1,z_2,\ldots z_n)\,\phi(z_1,z_2,\ldots,z_n)^{j-\alpha} \tag{8}$$

where z_n are the independent variables, the $u_j(z_1,z_2,\ldots\ldots,z_n)$ represent analytic functions and $\phi(z_1,z_2,\ldots\ldots,z_n)$ are the eigen functions or geometric manifold and α is a real number expressing the difference between the highest differential order for nonlinear terms and the highest degree in linear term. Through this method we derive the Lax pairs for KdV and Hirota Satsuma equations.

3.1 Derivation of KdV equation, Lax pair

As an example we study the Korteweg–de Vries equation. The combined knowledge of the Painlevé test (8) and the singular manifold method provides a systematic procedure to obtain the Lax pair, Darboux transformation and solutions for the PDE under consideration. To explain it, let us consider the KdV equation in the form [6];

$$u_t + u_{xxx} + 6uu_x = 0. \tag{9}$$

As in eq. (8), α is evaluated from the difference between the highest differential order for nonlinear terms and the highest degree in linear term, we find here α= 2 (uu_x and u_{xxx} difference in differential order). In this case the series expansion (8) reduces to;

$$u(x,t) = \int_{j=0}^{\infty} u_j\,(x,t)\,\phi\,(x,t)^{j-2} = \; u_0\phi^{-2} + \; u_1\phi^{-1} + u_2 \tag{10}$$

This solution *u(x, t)* will be substituted in eq. (9) as well as all its derivatives. Differentiate eq. (10) w.r.t (x) once gives;

$$u_x = \frac{u_{0x}}{\phi^2} - 2\frac{u_0\phi_x}{\phi^3} + \frac{u_{1x}}{\phi} - \frac{u_1\phi_x}{\phi^2} + u_{2x} \tag{11}$$

The term uu_x is obtained multiplying eq. (10) by (11) giving;

$$uu_x = (\; u_0\phi^{-2} + \; u_1\phi^{-1} + u_2)(\frac{u_{0x}}{\phi^2} - 2\frac{u_0\phi_x}{\phi^3} + \frac{u_{1x}}{\phi} - \frac{u_1\phi_x}{\phi^2} + u_{2x})$$

Expanding the two brackets product;

$$\begin{aligned} uu_x = & \frac{u_{0x}u_0}{\phi^4} - 2\frac{u_0{}^2\phi_x}{\phi^5} + \frac{u_{1x}u_0}{\phi^3} - \frac{u_1u_0\phi_x}{\phi^4} + \frac{u_0u_{2x}}{\phi^2} + \frac{u_{0x}u_1}{\phi^3} - 2\frac{u_0u_1\phi_x}{\phi^4} + \frac{u_{1x}u_1}{\phi^2} - \frac{u_1{}^2\phi_x}{\phi^3} + \frac{u_1u_{2x}}{\phi} \\ & + \frac{u_2u_{0x}}{\phi^2} - 2\frac{u_2u_0\phi_x}{\phi^3} + \frac{u_2u_{1x}}{\phi} - \frac{u_2u_1\phi_x}{\phi^2} + u_{2x}u_2. \end{aligned} \tag{12}$$

(1) To get the term u_{xxx} , differentiate eq. (11) twice w.r.t (x)

$$\begin{aligned} u_{xxx} = & \frac{u_{0xxx}}{\phi^2} - 6\frac{u_{0xx}\phi_x}{\phi^3} - 6\frac{u_{0x}\phi_{xx}}{\phi^3} + 18\frac{u_{0x}\phi_x{}^2}{\phi^4} - \frac{2u_0\phi_{xxx}}{\phi^3} + 18\frac{u_0\phi_x\phi_{xx}}{\phi^4} - 24\frac{u_0\phi_x{}^3}{\phi^5} + \frac{u_{1xxx}}{\phi} \\ & - 3\frac{u_{1xx}\phi_x}{\phi^2} - 3\frac{u_{1x}\phi_{xx}}{\phi^2} + 4\frac{u_{1x}\phi_x{}^2}{\phi} - \frac{u_1\phi_{xxx}}{\phi^2} + \frac{6u_1\phi_x\phi_{xx}}{\phi^3} + \frac{2u_{1x}\phi_x{}^2}{\phi^3} - 6\frac{u_1\phi_x{}^3}{\phi^4} + u_{2xxx} \end{aligned} \tag{13}$$

(2) Differentiate eq. (10) w.r.t (t) once

$$u_t = \frac{u_{0t}}{\phi^2} - 2\frac{u_0\phi_t}{\phi^3} + \frac{u_{1t}}{\phi} - \frac{u_1\phi_t}{\phi^2} + u_{2t} \tag{14}$$

Substitute from eq. (12), eq. (13), and eq. (14) in eq. (9);

$$\begin{aligned} u_t + u_{xxx} + 3uu_x = & (u_{2t} + u_{2xxx} + 3u_{2x}u_2) + \left(u_{1t} + u_{1xxx} + 4u_{1x}\phi_x{}^2 + 3u_1u_{2x} + 3u_2u_{1x}\right)\phi^{-1} \\ & + (u_{0t} - u_1\phi_t + u_{0xxx} - 3u_{1xx}\phi_x - 3u_{1x}\phi_{xx} - u_1\phi_{xxx} + 3u_0u_{2x} + 3u_{1x}u_1 + 3u_2u_{0x} - 3u_2u_1\phi_x)\phi^{-2} \\ & + \left(-2u_0\phi_t - 6u_{0xx}\phi_x - 6u_{0x}\phi_{xx} - 2u_0\phi_{xxx} + 2u_{1x}\phi_x{}^2 + 3u_{1x}u_0 + 3u_{0x}u_1 - 3u_1{}^2\phi_x - 6u_2u_0\phi_x + 6u_1\phi_x\phi_{xx}\right)\phi^{-3} \\ & + \left(18u_{0x}\phi_x{}^2 + 18u_0\phi_x\phi_{xx} - 6u_1\phi_x{}^3 + 3u_{0x}u_0 - 3u_1u_0\phi_x - 6u_0u_1\phi_x\right)\phi^{-4} + \left(-24u_0\phi_x{}^3 - 6u_0{}^2\phi_x\right)\phi^{-5} \\ & = 0 \end{aligned} \tag{15}$$

Comparing both sides of this equation;

(1) Coefficients of ϕ^{-5};

$$-24u_0{\phi_x}^3 - 12{u_0}^2\phi_x = 0;$$

$$u_0 = -2{\phi_x}^2 \tag{16}$$

Differentiating eq. (16) twice;

$$\begin{aligned} u_{0x} &= -4\phi_x\phi_{xx} \\ u_{0xx} &= -4{\phi_{xx}}^2 - 4\phi_x\phi_{xxx} \end{aligned} \tag{17}$$

(2) Coefficients of ϕ^{-4}

$$18u_{0x}{\phi_x}^2 + 18u_0\phi_x\phi_{xx} - 6u_1{\phi_x}^3 + 6u_{0x}u_0 - 18u_0u_1\phi_x = 0 \tag{18}$$

Substitute from eq. (17) in the coefficients of ϕ^{-4} (eq. 18) which reduce to the form;

$$-72{\phi_x}^3\phi_{xx} - 36{\phi_x}^3\phi_{xx} - 6u_1{\phi_x}^3 + 48{\phi_x}^3\phi_{xx} + 36u_1{\phi_x}^3 = 0$$

Simplifying we obtain;

$$u_1 = 2\phi_{xx}. \tag{19}$$

(3) Coefficients of ϕ^{-3}

$$-2u_0\phi_t - 6u_{0xx}\phi_x - 6u_{0x}\phi_{xx} - 2u_0\phi_{xxx} + 6u_{1x}{\phi_x}^2 + 6u_{1x}u_0 + 6u_{0x}u_1 - 6{u_1}^2\phi_x - 12u_2u_0\phi_x + 6u_1\phi_x\phi_{xx} = 0. \tag{20}$$

Differentiating eq. (19) once;

$$u_{1x} = 2\phi_{xxx} \tag{21}$$

and the coefficients of ϕ^{-3} (eq. (20)) reduce to the form;

$$4\phi_t{\phi_x}^2 - 12\phi_x{\phi_{xx}}^2 + 16{\phi_x}^2\phi_{xxx} + 24u_2{\phi_x}^3 = 0$$

Simplifying results in;

$$u_2 = -\frac{4\phi_{xxx}}{6\phi_x} + \frac{{\phi_{xx}}^2}{2{\phi_x}^2} - \frac{\phi_t}{6\phi_x}$$

u_2 is rewritten as follows;

$$u_2 = -\frac{\phi_{xxx}}{2\phi_x} + \frac{{\phi_{xx}}^2}{4{\phi_x}^2} - \frac{1}{6}\left(\frac{\phi_{xxx}}{\phi_x} - \frac{3{\phi_{xx}}^2}{2{\phi_x}^2} + \frac{\phi_t}{\phi_x}\right) \tag{22}$$

The last term in eq. (22) is set equal to λ,

$$\lambda = \frac{1}{6}\left(\frac{\phi_{xxx}}{\phi_x} - \frac{3{\phi_{xx}}^2}{2{\phi_x}^2} + \frac{\phi_t}{\phi_x}\right) = \frac{1}{6}\frac{\phi_t}{\phi_x} + \frac{1}{6}\left\{\frac{\phi_{xxx}}{\phi_x} - \frac{3{\phi_{xx}}^2}{2{\phi_x}^2}\right\} \tag{23}$$

where the last term in eq. (23) is the Schwarzian derivative and λ rewrites as:

$$6\lambda = \frac{\phi_t}{\phi_x} + \{\phi;x\}$$

and eq. (22) is rewritten as;

$$u_2 + \lambda = -\frac{\phi_{xxx}}{2\phi_x} + \frac{{\phi_{xx}}^2}{4{\phi_x}^2} \tag{24}$$

This equation can be written as;

$$u_2+\lambda=\frac{-1}{2}\frac{\partial}{\partial x}\left(\frac{\phi_{xx}}{\phi_x}\right)-\frac{{\phi_{xx}}^2}{4{\phi_x}^2} \tag{25}$$

Let;

$$\phi_x=\psi^2 \tag{26}$$

this equation rewrites as;

$$\frac{\phi_{xx}}{\phi_x}=2\frac{\psi_x}{\psi} \tag{27}$$

Substitute from eq. (27) in eq. (25);

$$u_2+\lambda=-\frac{\partial}{\partial x}\left(\frac{\psi_x}{\psi}\right)-(\frac{\psi_x}{\psi})^2 \tag{28}$$

Expand the differentiation;

$$u_2+\lambda=-\left(\frac{\psi_{xx}\psi-{\psi_x}^2}{\psi^2}\right)-\frac{{\psi_x}^2}{\psi^2}$$

Simplifying;

$$u_2+\lambda=-\left(\frac{\psi_{xx}}{\psi}\right)$$

So, the first Lax pair is expressed as;

$$(u_2+\lambda)\psi=-\psi_{xx}. \tag{29}$$

This first Lax pair is a Sturm - Liouville equation in $u_2$and ψ. To get the second Lax pair equation we start from eq. (23); $\lambda=\frac{1}{6}\left(\frac{\phi_{xxx}}{\phi_x}-\frac{3{\phi_{xx}}^2}{2{\phi_x}^2}+\frac{\phi_t}{\phi_x}\right)$. Rewriting this equation, we get;

$$\lambda=\frac{1}{6}(\frac{\phi_{xxx}}{\phi_x}-\frac{3{\phi_{xx}}^2}{2{\phi_x}^2}+\frac{\phi_t}{\phi_x}). \tag{30}$$

Separate the term ϕ_t;

$$\phi_t=-\phi_{xxx}+\frac{3{\phi_{xx}}^2}{2\phi_x}+6\lambda\phi_x. \tag{31}$$

Differentiate eq. (31) w.r.t (x) once;

$$\phi_{xt}=-\phi_{xxxx}+\frac{3\phi_{xx}\phi_{xxx}}{\phi_x}-\frac{3{\phi_{xx}}^3}{2{\phi_x}^2}+6\lambda\phi_{xx}. \tag{32}$$

As this equation contains nonlinear terms, we linearize them through the associated function $\phi_x=\psi^2$. Differentiating both sides leads to;

$$\begin{aligned}\phi_{xt}&=2\psi\psi_t\\ \phi_{xx}&=2\psi\psi_x\\ \phi_{xxx}&=2{\psi_x}^2+2\psi\psi_{xx}\end{aligned} \tag{33}$$

From first Lax pair eq. (29) ϕ_{xxx} rewrites as;

$$\phi_{xxx}=2{\psi_x}^2-2(u_2+\lambda)\psi^2$$

Differentiate it once w.r.t (x);

$$\phi_{xxxx}=4\psi_x\psi_{xx}-4\psi\psi_x(u_2+\lambda)-2\psi^2u_{2x} \tag{34}$$

Replacing for ψ_{xx} term with eq. (29) and substituting in eq. (34) yields;

$$\phi_{xxxx} = -8\psi_x\psi(u_2+\lambda) - 2\psi^2 u_{2x}. \tag{35}$$

Substitute from eq. (35), eq. (33) in eq. (32);

$$2\psi\psi_t = 8\psi_x\psi(u_2+\lambda) + 2\psi^2 u_{2x} + \frac{12\psi\psi_x({\psi_x}^2-(u_2+\lambda)\psi^2)}{\psi^2} - \frac{12(\psi\psi_x)^3}{\psi^4} + 12\lambda\psi\psi_x$$

Simplifying we obtain;

$$\psi_t = -2\psi_x u_2 + \psi u_{2x} + 4\lambda\psi_x, \tag{36}$$

this is the second Lax pair with u_2 being a solution of the KdV; eq. (9) as long as λ is independent of time. As a result, equations (29), (36) represent the Lax pair for KdV, with a spectral parameter $\lambda = \frac{\phi_t}{\phi_x}$.

4 Applications of Darboux transformations to non-linear evolution equations

We here solve two nonlinear evolution equations; KdV and Boussinesq using Darboux transformation.

4.1 Solution of KdV equation using Darboux transformation

The Korteweg-de Vries (KdV) equation is a nonlinear partial differential equation modeling many physical phenomena such as waves on shallow water surfaces, gas dynamics, hydromagnetic, plasma physics, and blood flow in arteries. This equation writes as;

$$u_t + 6uu_x + u_{xxx} = 0 \tag{37}$$

Their Lax pair system of equations;

$$\psi_{xx} = -(u+\lambda)\psi \tag{38}$$

$$\psi_t = -4\psi_{xxx} - 6u\psi_x - 3u_x\psi \tag{39}$$

where $\phi_x = \psi^2$.

4.1.1 One soliton solution

From first DT;

$$\psi[1] = \psi_x - \sigma\psi, \sigma = \frac{\psi_1'}{\psi_1}. \tag{40}$$

This satisfies the first Lax pair equation;

$$\psi_{xx}[1] = -(u[1]+\lambda)\psi[1]$$

Replacing for $\psi[1]$ we obtain;

$$\psi_{xx}[1] = -(u[1]+\lambda)\psi_x + \sigma(u[1]+\lambda)\psi. \tag{41}$$

Differentiate eq. (40) w.r.t (x) twice;

$$\psi_{xx}[1] = \psi_{xxx} - \sigma_{xx}\psi - 2\sigma_x\psi_x - \sigma\psi_{xx}. \tag{42}$$

Differentiate the first Lax pair in eq. (38) w.r.t (*x)* once;

$$\psi_{xxx} = -\left(u+\lambda\right)\psi_x - u_x\psi \tag{43}$$

Substitute from (43), (38) in (42);

$$\psi_{xx}\left[1\right] = -\left(u+\lambda+2\sigma_x\right)\psi_x + \left(-u_x-\sigma_{xx}+\sigma u+\sigma\lambda\right)\psi \tag{44}$$

Compare eq. (41) with eq. (44), coefficients of ψ_x;

$$u\left[1\right] = u\left[0\right]+2\sigma_x, \tag{45}$$

where $u[0]$ is a seed solution. Consider a seed solution;

$$u\left[0\right] = 0 \tag{46}$$

In this case Lax pair system reduces to;

$$\psi_{xx} = -\lambda\,\psi \tag{47}$$

$$\psi_t = -4\psi_{xxx} \tag{48}$$

Solve these equations (47), (48) together, we get;

$$\psi_1\left(x,t\right) = exp\left(0.5\left(k_1x-k_1{}^3t\right)\right)+exp\left(-0.5\left(k_1x-k_1{}^3t\right)\right) \tag{49}$$

where $\lambda = -\frac{k_1{}^2}{4}$. Substituting from eq. (46), eq. (49) in eq. (45) gives:

$$u\left[1\right] = -0.5\,\frac{\left(k_1{}^2e^{0.5\left(k_1\,x-k_1{}^3t\right)}+k_1{}^2e^{-0.5\left(k_1\,x-k_1{}^3t\right)}\right)}{e^{0.5\left(k_1\,x-k_1{}^3t\right)}+e^{-0.5\left(k_1\,x-k_1{}^3t\right)}}+0.5\left(\frac{k_1e^{0.5\left(k_1\,x-k_1{}^3t\right)}-k_1e^{-0.5\left(k_1\,x-k_1{}^3t\right)}}{e^{0.5\left(k_1\,x-k_1{}^3t\right)}+e^{-0.5\left(k_1\,x-k_1{}^3t\right)}}\right)^2$$

Regrouping exponential terms give;

$$u\left[1\right] = \frac{k_1{}^2}{2}sech^2\left(0.5\left(k_1x-k_1{}^3t\right)\right) \tag{50}$$

This is an exact solution for the KdV equation. Fig. 1 shows this solution for various times and parameter k_1. As time increases from $t=1$ to $t=10$ the wave moves keeping its amplitude. Increasing the parameter $k_1$from 0.5 to 2 largely increases the amplitude as it appears in Fig.1.c and d.

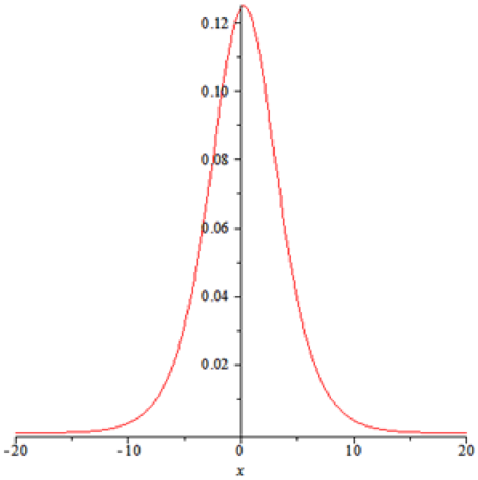
4.1.2 Comparison with previous work

We do in Fig. 2 compare our results using Darboux transformation with the tanh-coth method [8] results. They are found similar.

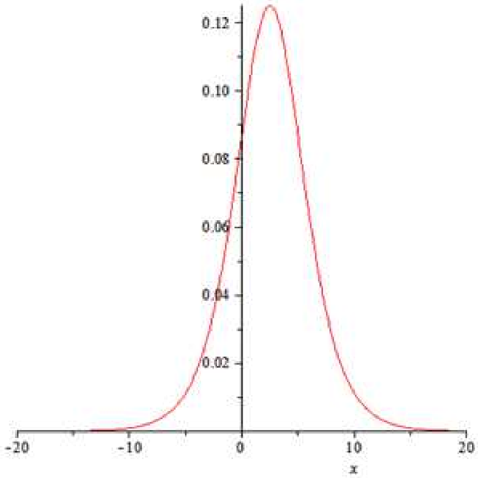
4.2 Boussinesq equation

4.2.1 Historical background

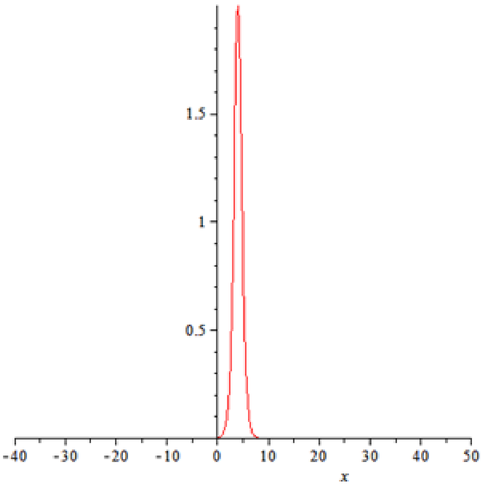
Boussinesq approximation for water waves takes into account the vertical structure of the horizontal and vertical flow velocity. This results in non-linear partial differential equations, called Boussinesq-type equations, which incorporate



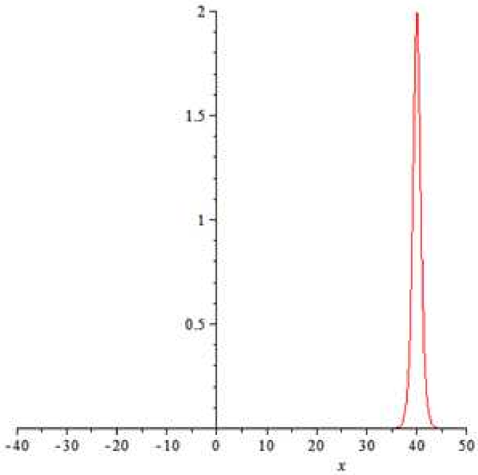
(a). u[1] for $u[0] = 0$, $k_1 = 0.5, t = 1$.



(b). u[1] for $u[0] = 0$, $k_1 = 0.5, t = 10$.



(c). u[1] for $u[0] = 0$, $k_1 = 2, t = 1$.



(d). u[1] for $u[0] = 0$, $k_1 = 2, t = 10$.

Fig. 1: Soliton solution $u[1] = \frac{{k_1}^2}{2} sech^2\left(0.5\left(k_1 x - {k_1}^3 t\right)\right)$ for KdV equation with seed solution $u[0] = 0$.

frequency dispersion as opposite to the shallow water equations, which are not frequency-dispersive. In coastal engineering, Boussinesq-type equations are frequently used in computer models for the simulation of water waves in shallow seas and harbors. While the Boussinesq approximation is applicable to fairly long waves - that is, when the wavelength is large compared to the water depth - the Stokes expansion is more appropriate for short waves when the wavelength is of the same order as the water depth, or shorter. Boussinesq equation writes as follows;

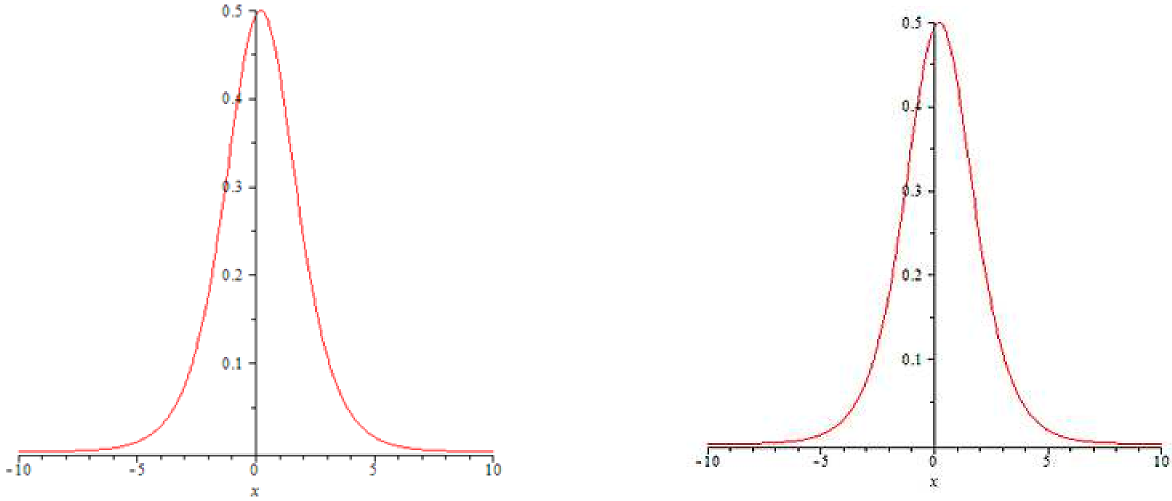
$$3\varepsilon u_{tt} + (6uu_x + u_{xxx})_x = 0 \quad, \varepsilon = \pm 1 \tag{51}$$

Its Lax pair [9] write as;

$$\psi_{xxx} = \lambda \psi - \frac{3}{2} u \psi_x - w\psi \tag{52}$$

$$\psi_t = \beta \psi_{xx} + \beta u \psi \tag{53}$$

where $w_x = \frac{3}{4\beta}(\beta u_{xx} + u_t)$, $\beta^2 = \varepsilon$, ψ is an eigen function weighting u and w series expansion.



(a). $u(x,t) = \frac{k_1{}^2}{2} sech^2\left(0.5\left(k_1 x - k_1{}^3 t\right)\right),\ k_1 = 1, t = 0.2.$

(b). $u(x,t) = \frac{w}{2} sech^2\left(0.5\sqrt{w}\,(x - wt)\right),\ w = 1, t = 0.2.$

Fig. 2: Comparison between $u(x,t)$ at w=1, u[0] = 0, t = 0.2 using D.T method and Tanh-Coth method.

4.2.2 One soliton solution

Starting from eq. (40);

$$\psi[1] = \psi_x - \sigma\psi, \quad \sigma = \frac{\psi_1'}{\psi_1} \tag{54}$$

The function of $\psi[1]$ satisfy the equation (54), so;$\psi_{xxx}[1] = \lambda\ \psi[1] - \frac{3}{2}u[1]\psi_x[1] - w[1]\,\psi[1]$ (53). Differentiate $\psi[1]$ w.r.t x.

$$\psi_x[1] = \psi_{xx} - \sigma_x\psi - \sigma\psi_x. \tag{55}$$

Substitute from eq. (55) in eq. (54);

$$\psi_{xxx}[1] = \lambda\psi_x - \lambda\sigma\psi - \frac{3u[1]}{2}(\psi_{xx} - \sigma_x\psi - \sigma\psi_x) - w[1](\psi_x - \sigma\psi)$$

$$\psi_{xxx}[1] = -\frac{3u[1]}{2}\psi_{xx} + \left(\lambda + \frac{3\sigma u[1]}{2} - w[1]\right)\psi_x + \left(-\lambda\sigma + \frac{3\sigma_x u[1]}{2} + w[1]\,\sigma\right)\psi \tag{56}$$

Differentiate (55) w.r.t x twice,

$$\psi_{xx}[1] = \psi_{xxx} - \sigma_{xx}\psi - 2\sigma_x\psi_x - \sigma\psi_{xx}$$

Then;

$$\psi_{xxx}[1] = \psi_{xxxx} - \sigma_{xxx}\psi - 3\sigma_{xx}\psi_x - 3\sigma_x\psi_{xx} - \sigma\psi_{xxx} \tag{57}$$

Differentiate first Lax pair eq. (52) w.r.t (x);

$$\psi_{xxxx} = (\lambda - \frac{3u_x}{2} - w)\psi_x - \frac{3}{2}u\psi_{xx} - w_x\psi \tag{58}$$

Substitute from eq. (58) in eq. (57);

$$\psi_{xxx}[1]=(\lambda-\frac{3u_x}{2}-w-3\sigma_{xx}+\frac{3\sigma u}{2})\psi_x+\left(-\frac{3u}{2}-3\sigma_x\right)\psi_{xx}+(-w_x-\sigma_{xxx}-\sigma\lambda+\sigma w)\psi \tag{59}$$

compare the coefficients of eq. (59) with eq. (57); Coefficients of ψ_{xx};

$$u[1]=u[0]+2\sigma_x \tag{60}$$

which this solution is a new solution for the Boussinesq equation (51).

4.2.3 Solitary wave solution of the Boussinesq equation and its Lax pair

Consider the seed solution in the form;

$$u[0]=0 \tag{61}$$

So the Lax pair equations (52, 53) will be;

$$\psi_{xxx}=\lambda\psi \tag{62}$$

$$\psi_t=\psi_{xx} \tag{63}$$

Solve these equations (62), (63) together, we get;

$$\psi_1(x,t)=exp\left(0.5\sqrt[3]{\lambda}(-1+I\sqrt{3}\right)x+\frac{\lambda^{\frac{2}{3}}}{4}\left(-2-2I\sqrt{3}\right)t)+exp\left(-0.5\sqrt[3]{\lambda}(-1+I\sqrt{3}\right)x+\frac{\lambda^{\frac{2}{3}}}{4}\left(-2+2I\sqrt{3}\right)t) \tag{64}$$

as $\sigma=\frac{\psi_1'}{\psi_1}$ hence,

$$\sigma=-\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}\tan(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}\left(-x+\sqrt[3]{\lambda}\,t\right)-0.5\sqrt[3]{\lambda} \tag{65}$$

substitute from (65), (61) in (60);

$$u[1]=\frac{-3}{2}\lambda^{\frac{2}{3}}\,(1+tan(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}\left(-x+\sqrt[3]{\lambda}\,t\right)^2) \tag{66}$$

This recurrent solution of Boussinesq equation (51) is plotted in Fig.3 for various times and λ. In Fig. (3-a, 3-b) the sharp vertical wave $u(x,t)$ downward response to a vertical force is depicted. The amplitude of the response decay with time from 0 to 5 sec and the wave amplitude distribute [10-14]. The hump's amplitudes increase by increasing as shown in Fig. (3-c, 3-d).

5 Conclusion

In this paper, we derive KdV and Boussinesq Lax pair. We then we solve two integrable systems KdV and Boussinesq equations using Darboux transformation and plotted the results for different seeds solutions.

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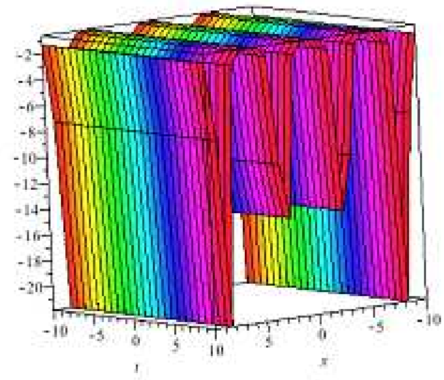
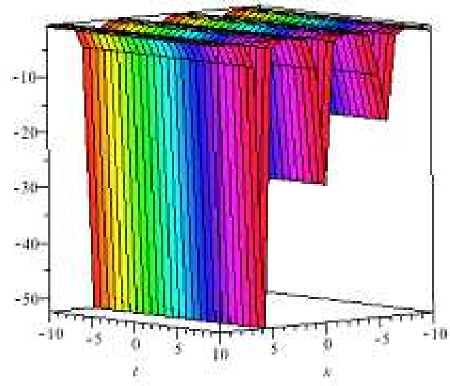
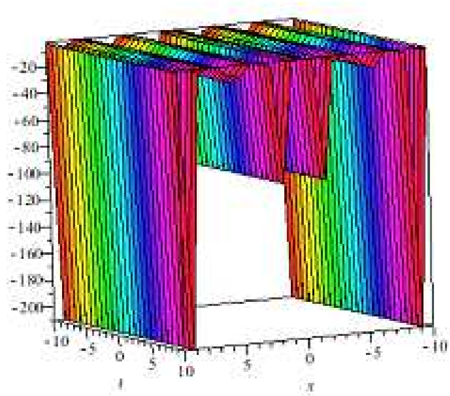


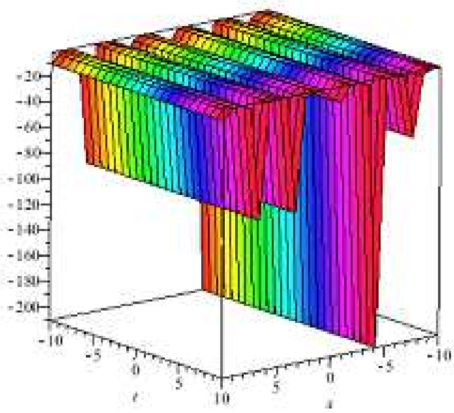
(a). $u[1]$ for $u[0]=0$, $\lambda=0.25, t=0$.



(b). $u[1]$ for $u[0]=0$, $\lambda=0.25, t=5$.



(c). $u[1]$ for $u[0]=0$, $\lambda=1, t=0$.



(d). $u[1]$ for $u[0]=0$, $\lambda=1,\ t=5$.

Fig. 3: Soliton solution for Boussinesq equation for seed solution $u[0]=0$.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The author has made equal contribution. The author read and approved the final manuscript.

References

[1] V. B. Matveev and M. A. Salle, Darboux Transformation and Solitons. Springer. (1991).

[2] C. Rogers and W. K. Schief, Bäcklund and Darboux transformations: geometry and modern applications in soliton theory, Cambridge Texts in Applied Mathematics. (2002).

[3] A. A. Halim, Korteweg-de-Vries equations in problems of fluid dynamic. (2001) 1-10.

[4] P.G. Estevez and J .Prada, Singular manifold method for an equation in 2 + 1 dimensions, Journal of Nonlinear Mathematical Physics. 12 (2005) 266–279.

[5] J.Weiss, M. Tabor and G. Carnevale, The Painlevé property for partial differential equations, J. Math. Phys. 24 (1983) 522-526.

[6] P.G. Estevez and P.R. Gordoa, The Singular Manifold Method: Darboux Transformations and Nonclassical Symmetries, Nonlinear Mathematical Physics. 2 (1995) 334–355.

[7] G. Chaohao, H. Hesheng and Z. Zixiang, Darboux Transformations In Integrable Systems Theory And Their Applications To Geometry, Institute of Mathematics, Fudan University, Shanghai, China. (2005) PP. 2.

[8] L.Wazzan, A modifed tanh–coth method for solving the KdV and the KdV–Burgers' equations, Communications in Nonlinear Science and Numerical Simulation .14 (2009) 443–450.

[9] G. Chaohao, H. Hesheng and Z. Zixiang, Darboux Transformations In Integrable Systems Theory And Their Applications To Geometry, Institute of Mathematics, Fudan University, Shanghai, China.(2005) PP. 85.

[10] R. Sadat, M. Kassem, Explicit Solutions for the (2+ 1)-Dimensional Jaulent–Miodek Equation Using the Integrating Factors Method in an Unbounded Domain. Mathematical and Computational Applications, 23(1)(2018) 1-9.

[11] M. A. Ramadan, M. R. Ali, An effcient hybrid method for solving fredholm integral equations using triangular functions, *New Trends in Mathematical Sciences*,5(1) (2017) 213-224.

[12] M. A. Ramadan, M. R. Ali, Numerical Solution of Volterra-Fredholm Integral Equations Using Hybrid Orthonormal Bernstein and Block-Pulse Functions, Asian Research Journal of Mathematics,4(4) (2017) 1-14.

[13] M. A. Ramadan, M. R. Ali, Application of Bernoulli wavelet method for numerical solution of fuzzy linear Volterra-Fredholm integral equations, Communication in Mathematical Modeling and Applications,2(3)(2017) 40-49.

[14] M. A. Ramadan, M. R. Ali, Solution of integral and Integro-Differential equations system using Hybrid orthonormal Bernstein and block-pulse functions, Journal of abstract and computational mathematics,2(1)(2017) 35-48.