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PREDICTING RESIDUAL LIFETIMES OF DYNAMIC r-out-of-n SYSTEMS

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Abstract: An important reliability index is *the residual lifetime* (RL). The RL may be used for designing engineering systems, maintenance policy programs as well as for comparison and prediction of system lifetimes. In this paper, dynamic r-out-of-n systems are considered as an subclass of engineering coherent systems. Given the first s component failure times, the RL of the system is predicted and the *mean residual lifetime* (MRL) is evaluated. Illustrative examples are also given.

Key words: Mean residual lifetime; Prediction; Sequential order statistics; Weibull distribution

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1. Introduction

In analysing of engineering systems, it is usually assumed that component lifetimes are independent and identically distributed (i.i.d.). A well-known example is *r-out-of-n systems* in which the system lifetimes coincide to the r-th smallest component lifetime, denoted by $X_{r:n}$. Therefore, prediction of $X_{r:n}$ on the basis of observed first s failures, i.e. $X_{1:n}, \cdots, X_{s:n}$, $1 \leq s < r$, is essential in these systems. To do this, theory of order statistics (OS) is utilized; See e.g., Barlow and Proschan [3], Billinton and Allan [4], David and Nagaraja [9], Raqab and Nagaraja [15] and references therein. Cramer and Kamps [6] introduced the concept of *dynamic (or sequential) r-out-of-n systems* as a generalization of the (usual) r-out-of-n systems. In such systems, n i.i.d. components begin to work at time $t = 0$ and failure of any component effects on the remaining component lifetimes. For example, a component failure may cause more loading (or pressure) on the surviving components and hence the component *residual lifetimes* (RLs) decrease stochastically. The main idea is that the common distribution functions (DFs) of the surviving components change to reflect the effects of the failed components. Therefore, usual OS are not adequate for modelling dynamic r-out-of-n system lifetimes. But the concept of sequential order statistics (SOS) provides an appropriate approach. Notice that lifetimes of dynamic r-out-of-n systems coincide to the r-th SOS, denoted by $X^*_{r:n}$. For more information, see, e.g., Baratnia and Doostparast [2], Burkschat and Navarro [5], Cramer and Kamps [6, 7, 8], Kamps [11] and references therein.

In this paper, dynamic r-out-of-n systems are considered. Using the observed first s failures $X^*_{1:n}, \cdots, X^*_{s:n}$, $1 \leq s < r$, the *mean residual lifetime* (MRL) of the system is used as an unbiased prediction for the system RL. The RLs and the MRLs have practical applications in engineering system designings, maintenance policy programs, comparison and prediction purposes; See e.g., Aitchison and Dunsmore [1], Barlow and Proschan [3] and Billinton and Allan [4]. Some studies on the MRLs may be found in literature; For example, see Madadi et al. [13], Navarro and Eryilmaz [14], Raqab [16], Salehi et al. [17] and references therein. In particular, for a useful study on component RLs in dynamic r-out-of-n systems one may refer to Gurler [10].

As a motivation, suppose that an oil transmission-pipeline manager plans to add a new station with four pumps in order to increase the pressure on oil throughout the pipeline. The station works

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properly if at least two pumps among the four pumps be active. Notice that due to the required pressure on oil for proper transmission, defined by the manager, failure of a pump in the new station causes more loading on the surviving pumps. In summarize, we have a dynamic 3-out-of-4 system. Now let, one of the pupms in the new station is failed at the current time. Under this condition, predicting the RL of the station lifetime for proper transmission is critical. With this in mind, the rest of this paper is organized as follows: In Section 2, dynamic r-out-of-n systems are considered and the concept of SOS is reviewed in details. In Sections 3, the future component failure times are predicted by unbiased predictors. Finally in Section 4, the *maximum likelihood prediction* (MLP) for future component failure times is derived. Various illustrative examples are also given. Section 5 concludes.

2. Sequential order statistics

Following Burkschat and Navarro [5], suppose $F_1, \cdots, F_n$ be lifetime DFs with respective reliability and inverse functions $\bar{F}_1, \cdots, \bar{F}_n$ and $F_1^{[-1]}, \cdots, F_n^{[-1]}$ in which

$$F_1^{[-1]}(1) \leq \cdots \leq F_n^{[-1]}(1).$$

Consider a system consisting of n independent components with the common DF F_i $(1 \leq i \leq n)$ if $n-i+1$ components are jointly work at time $t=0$. Suppose that n components in the system start to work at time $t=0$ and component lifetimes are denoted by $X_1^{[1]}, \cdots, X_n^{[1]}$, then $X_1^{[1]}, \cdots, X_n^{[1]}$ are i.i.d. according to F_1, abbreviated by $X_1^{[1]}, \cdots, X_n^{[1]} \overset{iid}{\sim} F_1$. The first failure time is then

$$X^*_{1:n} = \min_{1 \leq j \leq n} \left\{ X_j^{[1]} \right\}.$$

Given $X^*_{1:n} = t_1$, DFs of the lifetimes of $n-1$ remaining components change to F_2 (instead of F_1) which is truncated from the left at the point t_1, i.e. $F_2(.\mid t_1)$, where $\bar{F}_2(x|t_1) = \bar{F}_2(x)/\bar{F}_2(t_1)$, for $x > t_1$. Let $X_1^{[2]}, \cdots, X_{n-1}^{[2]}$ denote component lifetimes in this time. Then $X_1^{[2]}, \cdots, X_{n-1}^{[2]} \overset{iid}{\sim} F_2(.\mid t_1)$ and the second component failure time will be

$$X^*_{2:n} = \min_{1 \leq j \leq n-1} \left\{ X_j^{[2]} \right\}.$$

By proceed this way and induction, if the k-th failure $(k \geq 2)$ occurs at time $t_k(> t_{k-1})$, i.e. $X^*_{k:n} = t_k$, then DFs of the lifetimes of $n-k$ surviving components in the system change to F_{k+1} (instead of F_k) and it is truncated from the left at the point t_k, i.e. $F_{k+1}(.\mid t_k)$, where $\bar{F}_{k+1}(x \mid t_k) = \bar{F}_{k+1}(x)/\bar{F}_{k+1}(t_k)$ for $x > t_k$. If the component lifetimes be represented by $X_1^{[k+1]}, \cdots, X_{n-k}^{[k+1]}$, then $X_1^{[k+1]}, \cdots, X_{n-k}^{[k+1]} \overset{iid}{\sim} F_{k+1}(.\mid t_k)$ and the $(k+1)$-th failure time is given by

$$X^*_{k+1:n} = \min_{1 \leq j \leq n-k} \left\{ X_j^{[k+1]} \right\}, \qquad 1 \leq k \leq n-1.$$

The random variables $X^*_{1:n} \leq \cdots \leq X^*_{n:n}$ are called SOS based on $F_1, \cdots, F_n$.
Notice that, the lifetime of the dynamic r-out-of-n system (T) concides to r-the component failure time, i.e. $T = X^*_{r:n}$. It is easy to verify that SOS form a Markov process with transition probabilities

$$P\left(X^*_{s:n} \leq t \mid X^*_{s-1:n} = x\right) = 1 - \left(\frac{\overline{F}_s(t)}{\overline{F}_s(x)}\right)^{n-s+1}, \qquad t \geq x; \tag{2.1}$$

See Cramer and Kamps [6, pp. 537] and Kamps [11, pp. 4]. Cramer and Kamps [6, Lem 2.1] and [8, Lem 2.4] derived the joint density function of $X^*_{1:n}, \cdots, X^*_{s:n}$ $(1 \leq s \leq n)$ and the corresponding marginal DF of $X^*_{s:n}$ as follow, respectively.

Lemma 1. *The joint density function of* $X^*_{1:n}, \cdots, X^*_{s:n}$ $(1 \le s \le n)$ *is*

$$f_{X^*_{1:n},\cdots,X^*_{s:n}}(x_1,\cdots,x_s) = \frac{n!}{(n-s)!}\prod_{j=1}^{s-1} f_j(x_j)\left(\frac{1-F_j(x_j)}{1-F_{j+1}(x_j)}\right)^{n-j} f_s(x_s)\big(1-F_s(x_s)\big)^{n-s},$$

where f_i *is density function of* F_i $(1 \le i \le n)$ *and* $x_1 \le \cdots \le x_s$.

Lemma 2. *For* $1 \le s \le n$, *let* $F^*_{s:n}(t) = P(X^*_{s:n} \le t)$ *is the marginal DF of* $X^*_{s:n}$. *Then*

$$F^*_{1:n}(t) = 1 - \big(\overline{F}_1(t)\big)^n, \qquad \forall t \in \mathbb{R},$$

and for $2 \le s \le n$

$$F^*_{s:n}(t) = \begin{cases} 1, & t > t_0^s, \\ F^*_{s-1:n}(t) - \displaystyle\int_{-\infty}^{t}\left(\frac{\overline{F}_s(t)}{\overline{F}_s(x)}\right)^{n-s+1} dF^*_{s-1:n}(x), & t > t_0^s, \end{cases}$$

where $t_0^s = \inf\{t : F_s(t) = 1\}$.

3. Conditional prediction of RLs; An unbiased prediction

The predictor $\hat{Y}$ for the future random variable Y is unbiased if $E(\hat{Y} - Y) = 0$. For example, $\hat{Y} = E(Y \mid data)$ is an unbiased prediction for Y; See e.g., Aitchison and Dunsmore [1]. In this section, the conditional means of RLs for dynamic r-out-of-n systems are derived as an unbiased prediction for remaining system lifetime. Since $F_1, \cdots, F_n$ are lifetime DFs, $t_0^s = \infty$ for $1 \le s \le n$ and Lemma 2 gives

$$E\left(X^*_{1:n}\right) = \int_0^\infty \left(1 - F^*_{1:n}(t)\right)dt = \int_0^\infty \big(\overline{F}_1(t)\big)^n dt, \tag{3.1}$$

and for $2 \le s \le n$

$$\begin{aligned} E\left(X^*_{s:n}\right) &= \int_0^\infty \left(1 - F^*_{s:n}(t)\right)dt \\ &= E\left(X^*_{s-1:n}\right) + \int_0^\infty \left[\int_0^t \left(\frac{\overline{F}_s(t)}{\overline{F}_s(x)}\right)^{n-s+1} dF^*_{s-1:n}(x)\right] dt. \end{aligned} \tag{3.2}$$

Theorem 1. *For* $1 \le s \le n-1$ *and* $1 \le j \le n-s$, *let* $T^{[x]}_{s,j} := \left[X^*_{s+j:n} \mid X^*_{s:n} = x\right]$, *where* $X^*_{1:n}, \cdots, X^*_{n:n}$ *are SOS on basis of* $F_1, \cdots, F_n$. *Then,*

$$F_{T^{[x]}_{s,1}}(t) = 1 - \left(\frac{\overline{F}_{s+1}(t)}{\overline{F}_{s+1}(x)}\right)^{n-s}, \qquad t \ge x, \tag{3.3}$$

and for $2 \le j \le n-s$,

$$F_{T^{[x]}_{s,j}}(t) = F_{T^{[x]}_{s,j-1}}(t) - \int_x^t \left(\frac{\overline{F}_{s+j}(t)}{\overline{F}_{s+j}(y)}\right)^{n-s-j+1} dF_{T^{[x]}_{s,j-1}}(y), \qquad t \ge x,$$

where $F_{T^{[x]}_{s,j}}$ *stands for DF of* $T^{[x]}_{s,j}$, *i.e.*

$$F_{T^{[x]}_{s,j}}(y) = P\left(X^*_{s+j:n} \le y \mid X^*_{s:n} = x\right), \qquad y \ge x, \quad 1 \le s \le n-1, \quad 1 \le j \le n-s.$$

Proof. Equation (3.3) follows from Equation (2.1). For $2 \le j \le n-s$ and $t \in (0,\infty)$

$$F_{T^{[x]}_{s,j}}(t) = P\left(X^*_{s+j:n} \le t \mid X^*_{s:n} = x\right) = \int_{-\infty}^{\infty} P\left(X^*_{s+j:n} \le t \mid X^*_{s:n} = x, X^*_{s+j-1:n} = y\right) dF_{T^{[x]}_{s,j-1}}(y). \tag{3.4}$$

If $t < x$, the integrand in Equation (3.4) vanishes. Otherwise the Markovian property of SOS, ascending order between SOS and Equation (2.1) imply that

$$\begin{aligned} F_{T^{[x]}_{s,j}}(t) &= \int_x^t P\left(X^*_{s+j:n} \le t \mid X^*_{s+j-1:n} = y\right) dF_{T^{[x]}_{s,j-1}}(y) \\ &= \int_x^t \left[1 - \left(\frac{\overline{F}_{s+j}(t)}{\overline{F}_{s+j}(y)}\right)^{n-s-j+1}\right] dF_{T^{[x]}_{s,j-1}}(y) \\ &= F_{T^{[x]}_{s,j-1}}(t) - \int_x^t \left(\frac{\overline{F}_{s+j}(t)}{\overline{F}_{s+j}(y)}\right)^{n-s-j+1} dF_{T^{[x]}_{s,j-1}}(y), \end{aligned}$$

and the proof is completed.
Suppose that $(r-j)$-th failure time in a dynamic r-out-of-n system occurs at time x. Then, the median and the mean of RL of the system, i.e. $T^{[x]}_{r-j,j}$, are known as *conditional median predictor* and *conditional mean predictor*, respectively; See, e.g., Raqab and Nagaraja [15] and Aitchison and Dunsmore [1]. Notice that, the MRL of the system is

$$E\left(T^{[x]}_{r-j,j}\right) - x.$$

In the sequel illustrative examples, let Φ_a stands for DF of the normal distribution with mean 0 and variance $1/2a$.

Example 1. Consider a dynamic 3-out-of-4 system in which

$$F_j(t) = 1 - \exp\left\{-jt^2\right\}, \qquad t \ge 0, \quad j = 1,2,3.$$

Lemma 2 yeilds for $t \ge 0$,

$$F^*_{1:4}(t) = 1 - \exp\left\{-4t^2\right\}, \tag{3.5}$$
$$F^*_{2:4}(t) = 1 - 3\exp\left\{-4t^2\right\} + 2\exp\left\{-6t^2\right\}, \tag{3.6}$$
$$F^*_{3:4}(t) = 1 - 9\exp\left\{-4t^2\right\} + 8\exp\left\{-6t^2\right\} + 12t^2\exp\left\{-6t^2\right\}. \tag{3.7}$$

From Equations (3.1), (3.2) and (3.5)-(3.7), the mean time to failures are

$$E\left(X^*_{1:4}\right) \approx 0.44, \qquad E\left(X^*_{2:4}\right) \approx 0.61, \qquad E\left(X^*_{3:4}\right) \approx 0.73.$$

For $x \ge 0$ and $t \ge x$, Theorem 1 implies

$$\begin{aligned} F_{T^{[x]}_{1,1}}(t) &= 1 - \exp\left\{-6(t^2 - x^2)\right\}, \\ F_{T^{[x]}_{2,1}}(t) &= 1 - \exp\left\{-6(t^2 - x^2)\right\}, \\ F_{T^{[x]}_{1,2}}(t) &= 1 - \exp\left\{-6(t^2 - x^2)\right\}\left(1 + 6(t^2 - x^2)\right). \end{aligned}$$

Then, integration by parts concludes

$$\begin{aligned} E\left(T^{[x]}_{1,1}\right) &= \sqrt{\frac{\pi}{6}}\exp\{6x^2\}\left(1 - \Phi_6(x)\right), \\ E\left(T^{[x]}_{2,1}\right) &= \sqrt{\frac{\pi}{6}}\exp\{6x^2\}\left(1 - \Phi_6(x)\right), \\ E\left(T^{[x]}_{1,2}\right) &= \sqrt{\frac{\pi}{6}}\exp\{6x^2\}\left(\frac{3}{2} - 6x^2\right)\left(1 - \Phi_6(x)\right) + \frac{x}{2}. \end{aligned}$$

□

4. Likelihood prediction of RLs

In this section, the *maximum likelihood prediction* (MLP) for $X^*_{r:n}$ $(r \geq 2)$ on the basis of the first s $(1 \leq s < r)$ SOS is derived. Let $s = r - j$ and $1 \leq j \leq r-1$. For $x_1 \leq \cdots \leq x_s \leq x_r$, the Markovian property of SOS yields

$$\begin{aligned} L(x_1, \cdots, x_s, x_r) &:= f_{X^*_{1:n}, \cdots, X^*_{s:n}, X^*_{r:n}}(x_1, \cdots, x_s, x_r) \\ &= f_{X^*_{1:n}, \cdots, X^*_{s:n}}(x_1, \cdots, x_s) f_{X^*_{r:n}|X^*_{s:n}}(x_r \mid x_s), \end{aligned} \tag{4.1}$$

where $f_{X^*_{1:n}, \cdots, X^*_{s:n}}$ is given by Lemma 1 and $f_{X^*_{r:n}|X^*_{s:n}}$ is density function of $T^{[x_s]}_{s,j}$ which can be obtain from Theorem 1. The function $L(x_1, \cdots, x_s, x_r)$ is called *predictive likelihood function* (PLF). The value of x_r maximizing the PLF (4.1) is called MLP of the system lifetime and denoted by $\hat{X}^*_{r:n,ML}$. Therefore, the MLP of the system RL when the s-th component failure accures at time x_s is $\hat{X}^*_{r:n,ML} - x_s$. In the sequel, some examples are given in two cases.

Case I. Without nuisance parameters

If the PLF (4.1) is known, then the mode of $f_{X^*_{r:n}|X^*_{s:n}}$ is MLP of the system lifetime. Thus

$$\hat{X}^*_{r:n,ML} = \arg\max_{x_r} f_{X^*_{r:n}|X^*_{s:n}}(x_r|x_s).$$

EXAMPLE 2. In Example 1, density function of $T^{[x]}_{1,2}$ is

$$f_{T^{[x]}_{1,2}}(t) = 72t\left(t^2 - x^2\right)\exp\left\{-6\left(t^2 - x^2\right)\right\}, \qquad 0 \leq x \leq t.$$

Therefore, the MLP of the system lifetime when the first failure occurs at time x is

$$\hat{X}^*_{3:4,ML} = \sqrt{\frac{12x^2 + 3 + \sqrt{144x^4 + 24x^2 + 9}}{24}}.$$

□

Case II. With nuisance parameters

Suppose that the PLF (4.1) consists of some nuisance parameters, say $\underline{\theta} := (\theta_1, \cdots, \theta_m)$ where $\underline{\theta} \in \Theta$, then one must maximizes also Equation (4.1) with respected to $\underline{\theta}$. Therefore, the MLP of the system lifetime is the value of x_r which maximizes the profile PLF, denoted by $L_p(x_1, \cdots, x_s, x_r)$, that obtained by plug-in the estimated values of $\underline{\theta}$ in the PLF (4.1). Therefore

$$\hat{X}^*_{r:n,ML} = \arg\max_{x_r} L_p(x_1, \cdots, x_s, x_r),$$

where

$$L_p(x_1, \cdots, x_s, x_r) = \sup_{\underline{\theta} \in \Theta} L(x_1, \cdots, x_s, x_r).$$

EXAMPLE 3. In a dynamic 4-out-of-5 system, let $X^*_{1:5} = x_1$, $X^*_{2:5} = x_2$ are observed and

$$F_j(t) = 1 - \exp\{-j\lambda t\}, \qquad j = 1, 2, 3, 4,$$

where $\lambda > 0$ is an unknown parameter. Lemma 1 yields

$$f_{X^*_{1:5}, X^*_{2:5}}(x_1, x_2) = 40\lambda^2 \exp\{-\lambda(8x_2 - 3x_1)\}, \qquad 0 \leq x_1 \leq x_2. \tag{4.2}$$

Theorem 1 also concludes for $x_2 \le t$

$$F_{T_{2,2}^{[x_2]}}(t) = 1 + 8\exp\{-9\lambda(t-x_2)\} - 9\exp\{-8\lambda(t-x_2)\}, \tag{4.3}$$

and then

$$f_{T_{2,2}^{[x_2]}}(t) = 72\lambda\Big(\exp\{-8\lambda(t-x_2)\} - \exp\{-9\lambda(t-x_2)\}\Big), \qquad x_2 \le t. \tag{4.4}$$

Upon substituting Equations (4.2) and (4.4) into Equation (4.1), one can see that

$$\begin{aligned} &L(x_1,x_2,x_4;\lambda) = \\ &2880\lambda^3\exp\{-\lambda(8x_2-3x_1)\}\Big(\exp\{-8\lambda(x_4-x_2)\} - \exp\{-9\lambda(x_4-x_2)\}\Big), \\ &\hspace{8cm} 0 \le x_1 \le x_2 \le x_4. \end{aligned}$$

Note that the predictive log-likelihood function (PLLF) based on the available data x_1 and x_2 is

$$\begin{aligned} &l := l(x_4,\lambda;x_1,x_2) = \ln(L(x_1,x_2,x_4;\lambda)) = \\ &\ln(2880) + 3\ln(\lambda) - \lambda(8x_2-3x_1) + \ln\Big(\exp\{-8\lambda(x_4-x_2)\} - \exp\{-9\lambda(x_4-x_2)\}\Big). \end{aligned} \tag{4.5}$$

The likelihood equations are

$$\begin{cases} \dfrac{\partial l}{\partial x_4} = \dfrac{\lambda(-8\exp\{\lambda(x_4-x_2)\}+9)}{\exp\{\lambda(x_4-x_2)\}-1} = 0, \\ \dfrac{\partial l}{\partial \lambda} = \dfrac{3}{\lambda} - (8x_2-3x_1) + \dfrac{(x_4-x_2)(-8\exp\{\lambda(x_4-x_2)\}+9)}{\exp\{\lambda(x_4-x_2)\}-1} = 0, \end{cases} \tag{4.6}$$

where $0 \le x_1 \le x_2 \le x_4$ and $\lambda > 0$. Unfortunately, the system equations (4.6) can not be solved analytically. Therefore, the MLP of the system lifetime may be derived numerically by maximizing the PLLF (4.5). Notice that,

$$\begin{aligned} \frac{\partial^2 l}{\partial x_4^2} &= -\frac{\lambda^2\exp\{\lambda(x_4-x_2)\}}{(\exp\{\lambda(x_4-x_2)\}-1)^2}, \\ \frac{\partial^2 l}{\partial\lambda\partial x_4} &= \frac{\exp\{\lambda(x_4-x_2)\}(-8\exp\{\lambda(x_4-x_2)\}+17-\lambda(x_4-x_2))-9}{(\exp\{\lambda(x_4-x_2)\}-1)^2}, \\ \frac{\partial^2 l}{\partial\lambda^2} &= -\frac{3}{\lambda^2} - \frac{\exp\{\lambda(x_4-x_2)\}(x_4-x_2)^2}{(\exp\{\lambda(x_4-x_2)\}-1)^2}. \end{aligned}$$

Therefore, the Hessian matrix of the PLLF (4.5) is given by

$$H = \begin{bmatrix} \dfrac{\partial^2 l}{\partial x_4^2} & \dfrac{\partial^2 l}{\partial\lambda\partial x_4} \\ \\ \dfrac{\partial^2 l}{\partial\lambda\partial x_4} & \dfrac{\partial^2 l}{\partial\lambda^2} \end{bmatrix}. \tag{4.7}$$

If H be strictly negative definite for all $x_4 \ge x_2$ and $\lambda > 0$, then $\hat{X}^*_{4:5,ML}$ derived numerically from the likelihood equations (4.6) is the unique MLP of the system lifetime; See Khuri [12, pp. 285].

□

5. Conclusion

This paper dealt with the problem of predicting RLs and estimating the corresponding means for dynamic r-out-of-n systems. Conditionally on $X^*_{s:n} = x$, an explicit expression for mean of $X^*_{s+j:n}$ was derived. Given the first s component failure times, future component failures were predicted. These findings may be used for comparison dynamic r-out-of-n system lifetimes and maintenance policy programs.

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