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ψ*-Locally Closed Sets and ψ*-Locally Closed Continuous Functions in Topological Spaces

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Abstract – In this paper, we introduce ψ*-locally closed sets and different notions of generalizations of continuous functions in a topological space and study some of their properties. Several examples are given to illustrate the behavior of these new classes of functions. Also, we define ψ*-submaximal spaces.

Keywords – *ψ*-locally closed sets; ψ*LC-continuous functions; ψ*LC-irresolute functions; ψ*-submaximal spaces.*

1. Introduction

Allah and Nawar [1] introduced the concept of ψ*-closed sets. The notion of locally closed sets in a topological space was introduced by Bourbaki [2]. Ganster and Reilly [5] further studied the properties of locally closed sets and defined the LC-continuity and LC-irresoluteness. Gnanambal [6] introduced the concept of α-locally closed sets and αLC-continuous functions and investigated some of their properties. In this paper, we introduce ψ*LC-sets, ψ*LC*-sets and ψ*LC**-sets by using the notion of ψ*-closed and ψ*-open sets and study some of their properties. Finally, we also introduce and study different classes of weaker forms of continuity and irresoluteness and some of their properties in topological spaces.

2. Preliminaries

Throughout this paper (X, τ), (Y, σ) and (Z, η) represent non-empty topological spaces on which no separation axioms are assumed unless or otherwise mentioned. For a subset A of a space (X, τ), $cl(A)$ and $int(A)$ denote the closure of A and the interior of A, respectively.

Let us recall the following definitions, which are useful in the sequel.

Definition 2.1 A subset A of a topological space (X, τ) is called:

(1) Generalized α-closed [7] (briefly gα-closed) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in (X, τ). The complement of gα-closed set is called gα-open.

(2) ψ*-closed [1] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is gα-open in (X, τ). The complement of ψ*-closed set is called ψ*-open.

(3) Locally closed (briefly LC) set [5] if $A = G \cap F$, where G is open and F is closed in (X, τ).

(5) α-Locally closed (briefly αLC) set [6] if $A = G \cap F$, where G is α-open and F is α-closed in (X, τ).

Definition 2.2 A topological space (X, τ) is called:

(1) submaximal space [3] if every dense subset of (X, τ) is open in (X, τ).

(2) α-submaximal space [6] if every dense subset of (X, τ) is α-open in (X, τ).

(3) door space [4] if every subset of (X, τ) is either open or closed in (X, τ).

(4) $T_{1/5}^{\psi^*}$ space [1] if every ψ*-closed set is α-closed.

Definition 2.3 A function $f : (X, \tau) \to (Y, \sigma)$ is called:

(1) LC-continuous [5] if $f^{-1}(V)$ is locally closed set in (X, τ) for each closed set V of (Y, σ).

(2) αLC-continuous [6] if $f^{-1}(V)$ is α-locally closed set in (X, τ) for each closed set V of (Y, σ).

(3) LC-irresolute [5] if $f^{-1}(V)$ is locally closed set in (X, τ) for each locally closed set V of (Y, σ).

(4) αLC-irresolute [6] if $f^{-1}(V)$ is α-locally closed set in (X, τ) for each α–locally closed set V of (Y, σ).

3. ψ*-Locally Closed Sets

In this section, we introduce three weak types of locally closed sets denoted by ψ*LC(X, τ), ψ*LC*(X, τ) and ψ*LC**(X, τ) each of which contains LC(X, τ) and obtain some of their properties. Also, we introduce ψ*-submaximal spaces and obtain some of their properties.

Definition 3.1 A subset A of a topological space (X, τ) is called an ψ*-locally closed set (briefly, ψ*LC-set) if $A = G \cap F$, where G is ψ*-open and F is ψ*-closed in (X, τ). The class of all ψ*-locally closed subsets of (X, τ) is denoted by ψ*LC(X, τ).

Remark 3.1 The following are well known

(i) A subset A of (X, τ) is ψ*LC-set if and only if it's complement $X–A$ is the union of an ψ*-open and an ψ*-closed set.

(ii) Every ψ*-open (resp. ψ*-closed) subset of (X, τ) is an ψ*LC-set.

Theorem 3.1 Every locally closed set is an ψ*LC-set but not conversely.

Proof. The proof follows from the fact that every closed (resp. open) set is an ψ*-closed (resp. ψ*-open).

Example 3.1 Let X = {a, b, c, d} with τ = {X, ϕ, {a}, {b}, {a, b}, {a, b, c}}. Then a subset B = {a, b, d} $\notin$ LC(X) but $B \in \psi$*LC (X).

Theorem 3.2 Every α-locally closed set is an ψ*LC-set but not conversely.

Proof. The proof follows from the fact that every α-closed (resp. α-open) set is an ψ*-closed (resp. ψ*-open).

Example 3.2 Let X = {a, b, c} with τ = {X, ϕ, {a, b}}. Then a subset B = {a, c} $\notin$ αLC(X) but $B \in \psi$*LC (X).

Definition 3.2 A subset A of a topological space (X, τ) is called an ψ*LC*-set if $A = G \cap F$, where G is ψ*-open and F is closed in (X, τ).

The class of all ψ*LC*-subsets of (X, τ) is denoted by ψ*LC*(X, τ).

Definition 3.3 A subset A of a topological space (X, τ) is called an ψ*LC**-set if $A = G \cap F$, where G is open and F is ψ*-closed in (X, τ).

The class of all ψ*LC**-subsets of (X, τ) is denoted by ψ*LC**(X, τ).

Theorem 3.3 If a subset A of (X, τ) is locally closed, then it is ψ*LC(X, τ), ψ*LC*(X, τ) and ψ*LC**(X, τ).

Proof. Let $A = G \cap F$, where G is open and F is closed in (X, τ). Since every open set is ψ*-open and every closed set is ψ*-closed, it follows that A is ψ*LC(X, τ), ψ*LC*(X, τ) and ψ*LC**(X, τ).

The converse of the above theorem need not be true as seen from the following example.

Example 3.3 In Example 3.2, we have LC(X) = {X, ϕ, {c}, {a, b}}, ψ*LC(X) = {X, ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}}, ψ*LC*(X) = {X, ϕ, {a}, {b}, {c}, {a, b}} and ψ*LC**(X) = {X, ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}}. Here, {a} is ψ*LC(X, τ), ψ*LC*(X, τ) and ψ*LC**(X, τ) but not LC(X, τ).

Theorem 3.4 If a subset A of (X, τ) is ψ*LC*(X, τ), then it is ψ*LC(X, τ).

Proof. Let A be an ψ*LC*-set and every closed set is ψ*-closed in (X, τ), we have $A = G \cap F$, where G is ψ*-open and F is ψ*-closed in (X, τ). Therefore, $A \in \psi$*LC (X, τ).

The converse of the above theorem need not be true as seen from the following example.

Example 3.4 In Example 3.2, we have {a, c} $\in$ ψ*LC(X) but {a, c} $\notin$ ψ*LC*(X).

Theorem 3.5 Every ψ*LC**(X, τ) is ψ*LC(X, τ).

Proof. Let A be an ψ*LC**-set and every open set is ψ*-open in (X, τ), we have $A = G \cap F$, where G is ψ*-open and F is ψ*-closed in (X, τ). Therefore, $A \in$ ψ*LC (X, τ).

The converse of the above theorem need not be true as seen from the following example.

Example 3.5 Let X = {a, b, c} with τ = {X, ϕ, {b}}. Then a subset B = {b, c} $\in$ ψ*LC(X) but $B \notin$ ψ*LC**(X).

Theorem 3.6 Every αLC(X, τ) (resp. αLC*(X, τ), αLC**(X, τ)) is ψ*LC(X, τ) (resp. ψ*LC*(X, τ), ψ*LC**(X, τ)).

Proof. Since every α-open set is ψ*-open and every α-closed set is ψ*-closed, the proof follows.

The converse of the above theorem need not be true as seen from the following example.

Example 3.6 In Example 3.2, we have αLC(X) = αLC*(X) = αLC**(X) {X, ϕ, {c}, {a, b}}, ψ*LC*(X) = {X, ϕ, {a}, {b}, {c}, {a, b}} and ψ*LC(X) = ψ*LC**(X) = P(X). Here, {a} is ψ*LC(X, τ) (resp. ψ*LC*(X, τ) and ψ*LC**(X, τ)) but not an αLC(X, τ) (resp. αLC*(X, τ) and αLC**(X, τ)).

Theorem 3.7 If $A \in$ ψ*LC(X, τ) and B is ψ*-open set in (X, τ), then $A \cap B \in$ ψ*LC(X, τ).

Proof. Since $A \in$ ψ*LC (X, τ), there exist an ψ*-open G and an ψ*-closed set F such that $A = G \cap F$. Now, $A \cap B = (G \cap \mathrm{B}) \cap F$. Since $G \cap B$ is ψ*-open and F is ψ*-closed, it follows that $A \cap B \in$ ψ*LC (X, τ).

Remark 3.2 ψ*LC- sets and ψ*LC**-sets are independent of each other as seen from the examples.

Example 3.7 In Example 3.1, the set A = {a, b, d} is ψ*LC-set but not an ψ*LC**-set.

Example 3.8 In Example 3.2, the set A = {a, c} is ψ*LC**-set but not an ψ*LC-set.

Theorem 3.8 For a subset A of a topological space (X, τ), the following are equivalent:
(1) $A \in$ ψ*LC*(X, τ),
(2) $A = G \cap$ αcl(A) for some ψ*-open set G,
(3) αcl(A) – A is ψ*-closed,

(4) $A \cup (X - \alpha cl(A))$ is ψ^*-open.

Proof: (1) → (2). Let $A \in \psi^*LC^*(X, \tau)$. Then there exist an ψ^*-open set G and a closed set F of (X, τ) such that $A = G \cap F$. Since $A \subseteq G$ and $A \subseteq \alpha cl(A)$. Therefore, we have $A \subseteq G \cap \alpha cl(A)$.

Conversely, since $\alpha cl(A) \subseteq F$, $G \cap \alpha cl(A) \subseteq G \cap F = A$, which implies that $A = G \cap \alpha cl(A)$.

(2) → (1). Since G is ψ^*-open and $\alpha cl(A)$ is closed $G \cap \alpha cl(A) \in \psi^*LC^*(X, \tau)$, which implies that $A \in \psi^*LC^*(X, \tau)$.

(3) → (4). Let $F = \alpha cl(A) - A$. Then F is ψ^*-closed by the assumption and $X - F = X \cap (\alpha cl(A) - A)^c = A \cup (X - \alpha cl(A))$. But $X - F$ is ψ^*-open. This shows that $A \cup (X - \alpha cl(A))$ is ψ^*-open.

(4) → (3). Let $U = A \cup (X - \alpha cl(A))$. Since U is ψ^*-open set, $X - U$ is ψ^*-closed. $X - U = X - (A \cup (X - \alpha cl(A))) = \alpha cl(A) \cap (X - A) = \alpha cl(A) - A$. Thus, $\alpha cl(A) - A$ is ψ^*-closed set.

(4) → (2). Let $G = A \cup (X - \alpha cl(A))$. Thus, G is ψ^*-open. We prove that $A = G \cap \alpha cl(A)$ for some ψ^*-open G. Since, $G \cap \alpha cl(A) = (A \cup (X - \alpha cl(A)) \cap \alpha cl(A) = (\alpha cl(A) \cap A) \cup (\alpha cl(A) \cap (X - \alpha cl(A)) = A$. Therefore, $A = G \cap \alpha cl(A)$.

(2) → (4). Let $A = G \cap \alpha cl(A)$ for some ψ^*-open G. Then we prove that $A \cup (X - \alpha cl(A))$ is ψ^*-open. Now, $A \cup (X - \alpha cl(A)) = G \cap (\alpha cl(A)) \cap (X - \alpha cl(A)) = G$, which is ψ^*-open. Thus, $A \cup (X - \alpha cl(A))$ is ψ^*-open.

Theorem 3.9 If $A, B \in \psi^*LC^*(X, \tau)$, then $A \cap B \in \psi^*LC^*(X, \tau)$.

Proof. From the assumption, there exist ψ^*-open sets G and H such that $A = G \cap \alpha cl(A)$ and $B = H \cap \alpha cl(B)$. Then $A \cap B = (G \cap H) \cap (\alpha cl(A) \cap \alpha cl(B))$. Since $G \cap H$ is ψ^*-open set and $\alpha cl(A) \cap \alpha cl(B)$ is closed. Therefore, $A \cap B \in \psi^*LC^*(X, \tau)$.

Theorem 3.10 If $A \in \psi^*LC(X, \tau)$ and B is ψ^*-open set in (X, τ), then $A \cap B \in \psi^*LC(X, \tau)$.

Proof. Let $A \in \psi^*LC^*(X, \tau)$. Then $A = G \cap F$ where G is ψ^*-open and F is ψ^*-closed So, $A \cap B = (G \cap B) \cap F$. Since $G \cap B$ is ψ^*-open and F is ψ^*-closed, it follows that $A \cap B \in \psi^*LC(X, \tau)$.

Theorem 3.11 If $A \in \psi^*LC^*(X, \tau)$ and B is ψ^*-open (or closed) set in (X, τ), then $A \cap B \in \psi^*LC^*(X, \tau)$.

Proof. Since $A \in \psi^*LC^*(X, \tau)$, there exist an ψ^*-open G and a closed set F such that $A = G \cap F$. Now, $A \cap B = (G \cap B) \cap F$. Since $G \cap B$ is ψ^*-open and F is closed, it follows that $A \cap B \in \psi$*LC*(X, τ).

In this case, B being a closed set, we have $A \cap B = (G \cap F) \cap B = G \cap (F \cap B)$. Since G is ψ*-open set and $F \cap B$ is closed, $A \cap B \in \psi$*LC*(X, τ).

Theorem 3.12 If $A \in \psi$*LC**(X, τ) and B is ψ*-closed (or open) set in (X, τ), then $A \cap B \in \psi$*LC**(X, τ).

Proof. Since $A \in \psi$*LC**(X, τ), there exist an open set G and an ψ*-closed set F such that $A = G \cap F$. Now, $A \cap B = G \cap (F \cap B)$. Since G is open and $(F \cap B)$ is ψ*-closed, it follows that $A \cap B \in \psi$*LC**(X, τ).

In this case, B being an open set, we have $A \cap B = (G \cap F) \cap B = (G \cap B) \cap F$. Since $G \cap B$ is open set and F is ψ*-closed, then $A \cap B \in \psi$*LC**(X, τ).

Theorem 3.13 Let (X, τ) and (Y, σ) be two topological spaces
(i) If $A \in \psi$*LC(X, τ) and $B \in \psi$*LC(Y, σ), then $A \times B \in \psi$*LC$(X \times Y, \tau \times \sigma)$.
(ii) If $A \in \psi$*LC*(X, τ) and $B \in \psi$*LC*(Y, σ), then $A \times B \in \psi$*LC*$(X \times Y, \tau \times \sigma)$.
(iii) If $A \in \psi$*LC**(X, τ) and $B \in \psi$*LC**(Y, σ), then $A \times B \in \psi$*LC**$(X \times Y, \tau \times \sigma)$,

Proof. Let $A \in \psi$*LC(X, τ) and $B \in \psi$*LC(Y, σ). Then there exist ψ*-open sets M and N of (X, τ) and (Y, σ) and ψ*-closed sets F and K of X and Y respectively, such that $A = M \cap F$ and $B = N \cap K$. Then $A \times B = (M \times N) \cap (F \times K)$ holds. Hence, $A \times B \in \psi$*LC$(X \times Y, \tau \times \sigma)$.

(ii) and (iii) The proofs are similar to (i).

Definition 3.4 A topological space (X, τ) is said to be ψ*-submaximal if every dense subset in it is ψ*-open.

Theorem 3.14 Every submaximal space is ψ*-submaximal.

Proof. Let (X, τ) be a submaximal space and A be a dense subset of (X, τ). Then A is open. But every open set is ψ*-open and so A is ψ*-open. Therefore, (X, τ) is ψ*-submaximal.

The converse of the above theorem need not be true as seen from the following example.

Example 3.9 Let $X = \{a, b, c, d\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then the space (X, τ) is ψ*-submaximal but not submaximal. However, the set $A = \{a, b, d\}$ is dense in (X, τ), but it is not open in X. Therefore, (X, τ) is not submaximal.

Theorem 3.15 Every α-submaximal space is ψ*-submaximal.

Proof. Let (X, τ) be an α-submaximal space and A be a dense subset of (X, τ). Then A is α-open. But every α-open set is ψ*-open and so A is ψ*-open. Therefore, (X, τ) is ψ*-submaximal.

Theorem 3.16 A topological space (X, τ) is ψ^*-submaximal if and only if $\psi^*LC^*(X, \tau) = P(X)$.

Proof. Necessity: Let $A \in P(X)$ and $U = A \cup (X - \alpha cl(A))$. Then $\alpha cl(U) = X$. Since (X, τ) is ψ^*-submaximal, U is ψ^*-open. By Theorem 3.8, $A \in \psi^*LC^*(X, \tau)$ and so $P(X) = \psi^*LC^*(X, \tau)$.

Sufficiency: Let A be a dense subset of (X, τ). Then $A \cup (X - \alpha cl(A)) = A$. Since $A \in \psi^*LC^*(X, \tau)$, by Theorem 3.8, A is ψ^*-open in (X, τ). Hence, (X, τ) is ψ^*-submaximal.

4. ψ^*LC-Continuous Functions in Topological Spaces

In this section, we introduce the concepts of ψ^*LC-continuous, ψ^*LC*-continuous and ψ^*LC**-continuous functions which are weaker than LC-continuous functions.

Definition 4.1 A function $f: (X, \tau) \to (Y, \sigma)$ is called ψ^*LC-continuous (resp. ψ^*LC*-continuous, ψ^*LC**-continuous) if $f^{-1}(V) \in \psi^*LC(X, \tau)$ (resp. $f^{-1}(V) \in \psi^*LC^*(X, \tau)$, $f^{-1}(V) \in \psi^*LC^{**}(X, \tau)$) for each closed set V of (Y, σ).

Theorem 4.1 Let $f: (X, \tau) \to (Y, \sigma)$ be a function. Then we have the following
(i) If f is LC-continuous, then f is ψ^*LC-continuous, ψ^*LC*-continuous and ψ^*LC**-continuous.
(ii) If f is ψ^*LC*-continuous or ψ^*LC**-continuous, then f is ψ^*LC-continuous.

Proof. (i) Let f be a LC-continuous and V be an open set of (Y, σ). Then $f^{-1}(V)$ is locally closed in (X, τ). Since every locally closed set is ψ^*LC-set, ψ^*LC*-set and ψ^*LC**-set, it follows that f is ψ^*LC-continuous, ψ^*LC*-continuous and ψ^*LC**-continuous.

(ii) Let $f: (X, \tau) \to (Y, \sigma)$ be an ψ^*LC*-continuous or ψ^*LC**-continuous function. Since every ψ^*LC*-set is ψ^*LC-set and every ψ^*LC**-set is ψ^*LC-set. Therefore, the proof follows.

The converse of the above theorem need not be true as seen from the following example.

Example 4.1 Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{b\}\}$ and $\sigma = P(Y)$. Let $f: (X, \tau) \to (Y, \sigma)$ be the identity function. Now, $LC(X, \tau) = \{X, \phi, \{b\}, \{a, c\}\}$, $\psi^*LC(X, \tau) = \psi^*LC^*(X, \tau) = \psi^*LC^{**}(X, \tau) = P(X)$ and $LC(Y, \sigma) = \psi^*LC(Y, \sigma) = \psi^*LC^*(Y, \sigma) = \psi^*LC^{**}(Y, \sigma) = P(Y)$. Then f is not LC-continuous, since for the closed set $\{b, c\}$, $f^{-1}\{b, c\} = \{b, c\}$ is not locally closed in X, but it is ψ^*LC-continuous, ψ^*LC*-continuous and ψ^*LC**-continuous.

Example 4.2 Let $X = Y = \{a, b, c, d\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\sigma = \{Y, \phi, \{c\}\}$. Then the identity function $f: (X, \tau) \to (Y, \sigma)$ is ψ^*LC-continuous but not ψ*LC**-continuous, since for the closed set {a, b, d} of (Y, σ), f^{-1}{a, b, d} = {a, b, d} is not ψ*LC**-set in X but it is ψ*LC- set in X.

Theorem 4.2 Let f: (X, τ) → (Y, σ) and g: (Y, σ) → (Z, η) be any two functions. Then:
(i) g o f is ψ*LC-continuous if g is continuous and f is ψ*LC-continuous,
(ii) g o f is ψ*LC*-continuous if g is continuous and f is ψ*LC*-continuous,
(iii g o f is ψ*LC**-continuous if g is continuous and f is ψ*LC**-continuous,

Proof. Let V be a closed set in (Z, η) and g be a continuous function. Then $g^{-1}(V)$ is closed set in (Y, σ) and since f is ψ*LC-continuous, we get $f^{-1}(g^{-1}(V))$ is ψ*LC- set in (X, τ). Thus, g o f is ψ*LC-continuous.

(ii) – (iii) Similarly.

Definition 4.2 A function f: (X, τ) → (Y, σ) is called ψ*LC-irresolute (resp. ψ*LC*-irresolute, ψ*LC**-irresolute) if f^{-1} (V) ∈ ψ*LC(X, τ) (resp. f^{-1} (V) ∈ ψ*LC*(X, τ), $f^{-1}(V)$ ∈ ψ*LC**(X, τ)) for V ∈ ψ*LC(Y, σ) (resp. V ∈ ψ*LC*(Y, σ), V ∈ ψ*LC**(Y, σ).

Example 4.3 Let X = Y = {a, b, c} with τ = {X, ϕ, {a}} and σ = {Y, ϕ, {a}, {b, c}}. Then the identity function f: (X, τ) → (Y, σ) is ψ*LC- irresolute, ψ*LC*- irresolute and ψ*LC**- irresolute.

Theorem 4.3 If a function f: (X, τ) → (Y, σ) is LC- irresolute, then f is ψ*LC-irresolute, ψ*LC*- irresolute and ψ*LC**- irresolute.

Proof. Let f be a LC- irresolute and V be a LC-set of (Y, σ). Then f^{-1} (V) is LC(X, τ). Since every LC-set is a ψ*LC-set, ψ*LC*-set and ψ*LC**-set, it follows that f is ψ*LC- irresolute, ψ*LC*- irresolute and ψ*LC**- irresolute.

The converse of the above theorem need not be true as seen from the following example.

Example 4.4 As in Example 4.1, the function f is not LC- irresolute, since for the locally closed set {b, c}, f^{-1}{b, c} = {b, c} is not locally closed in X. However, f is ψ*LC- irresolute, ψ*LC- irresolute and ψ*LC**- irresolute.

Theorem 4.4 If a function f: (X, τ) → (Y, σ) is ψ*LC- irresolute (resp. ψ*LC*-irresolute and ψ*LC**- irresolute), then f is ψ*LC-continuous, ψ*LC*-continuous and ψ*LC**- continuous.

Proof. Since every LC-set is ψ*LC-set, ψ*LC*-set and ψ*LC**-set, the proof follows.

The converse of the above theorem need not be true as seen from the following example.

Example 4.5 Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{b\}\}$. Define a function $f: (X, \tau) \to (Y, \sigma)$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then f is ψ^*LC-continuous, ψ^*LC*- continuous and ψ^*LC**- continuous but not ψ^*LC- irresolute, ψ^*LC*- irresolute and ψ^*LC**- irresolute, since for the ψ^*LC-set (resp. ψ^*LC*-set and ψ^*LC**-set) $\{a, b\}$, $f^{-1}\{a, b\} = \{a, b\}$ is not ψ^*LC-set (resp. ψ^*LC*-set and ψ^*LC**-set) in (X, τ).

Theorem 4.5 Let $f: (X, \tau) \to (Y, \sigma)$ and $g: (Y, \sigma) \to (Z, \eta)$ be any two functions. Then:
(i) g o f is ψ^*LC-continuous if g is ψ^*LC-continuous and f is ψ^*LC-irresolute,
(ii) g o f is ψ^*LC*-continuous if g is ψ^*LC**-continuous and f is ψ^*LC*-irresolute,
(iii) g o f is ψ^*LC**-continuous if g is ψ^*LC**-continuous and f is ψ^*LC**-irresolute,
(iv) g o f is ψ^*LC- irresolute if f and g are ψ^*LC-irresolute,
(v) g o f is ψ^*LC*- irresolute if f and g are ψ^*LC*-irresolute,
(vi) g o f is ψ^*LC**- irresolute if f and g are ψ^*LC**-irresolute.

Proof. (i) Let V be a closed set in (Z, η) and g be an ψ^*LC-continuous function. Then $g^{-1}(V)$ is ψ^*LC- set in (Y, σ) and since f is ψ^*LC-irresolute, we get $f^{-1}(g^{-1}(V))$ is ψ^*LC- set in (X, τ). Thus, g o f is ψ^*LC-continuous.

(ii) – (iii) Similar to (i).

(iv) Let V be an ψ^*LC-set in (Z, η) and g be an ψ^*LC- irresolute function. Then $g^{-1}(V)$ is ψ^*LC- set in (Y, σ) and since f is ψ^*LC-irresolute, we get $f^{-1}(g^{-1}(V))$ is ψ^*LC- set in (X, τ). Thus, g o f is ψ^*LC-irresolute.

(v) – (vi) Similar to (iv).

Theorem 4.6 Let $\{Z_i : i \in \tau\}$ be a cover of X, where X is finite set and A be a subset of X. Suppose $\{Z_i : i \in \tau\}$ is ψ^*LC- set in X and the collection of ψ^*LC- set is closed under finite unions. If $A \cap Z_i \in \psi^*LC^{**}(Z_i, \tau / Z_i)$ for each $i \in \tau$, then $A \in \psi^*LC^{**}(X, \tau)$.

Proof. Let $i \in \tau$ and since $A \cap Z_i \in \psi^*LC^{**}(Z_i, \tau / Z_i)$. Then there exist an open set U_i of (X, τ) and ψ^*-closed set F_i of $(Z_i, \tau / Z_i)$ such that $A \cap Z_i = (U_i \cap Z_i) \cap F_i = U_i \cap (Z_i \cap F_i)$. Therefore, $A = \bigcup\{A \cap Z_i : i \in \tau\} = \bigcup\{U_i : i \in \tau\} \cap (\bigcup\{Z_i \cap F_i : i \in \tau\})$ and hence $A \in \psi^*LC^{**}(X, \tau)$.

Theorem 4.7 Let $f: (X, \tau) \to (Y, \sigma)$ be an ψ^*-irresolute injective map. Then
(i) If $B \in \psi^*LC(Y, \sigma)$, then $f^{-1}(B) \in \psi^*LC(X, \tau)$,
(ii) If X is a $T^{\psi^*}_{1/5}$- space and $B \in \psi^*LC(Y, \sigma)$, then $f^{-1}(B) \in \alpha LC(X, \tau.)$

Proof. (i) Let $B \in \psi^*LC(Y, \sigma)$. Then there exist ψ^*-open set G and ψ^*-closed set F such that $B = G \cap F$, $f^{-1}(B) = f^{-1}(G) \cap f^{-1}(F)$. Since f is ψ^*-irresolute,

$f^{-1}(G)$ and $f^{-1}(F)$ are ψ^*-open and ψ^*-closed sets in X respectively. Hence, $f^{-1}(B) \in \psi^*LC(X, \tau)$.

(ii) Let $B \in \psi^*LC(Y, \sigma)$. Then there exist ψ^*-open set G and ψ^*-closed set F such that $B = G \cap F$, $f^{-1}(B) = f^{-1}(G) \cap f^{-1}(F)$. Since f is ψ^*-irresolute, $f^{-1}(G)$ and $f^{-1}(F)$ are ψ^*-open and ψ^*-closed sets in X respectively. From hypothesis $f^{-1}(G)$ and $f^{-1}(F)$ are α-open and α-closed sets in X. Hence, $f^{-1}(B) \in \alpha LC(X, \tau)$.

Theorem 4.8 Any function defined in a door space is ψ^*-continuous (resp. ψ^*-irresolute).

Proof. Let f: $(X, \tau) \to (Y, \sigma)$ be a function where (X, τ) is a door space and $A \in (Y, \sigma)$ (resp. $A \in \psi^*LC((Y, \sigma)$). Then $f^{-1}(A)$ is either open or closed. Since every open or closed set is ψ^*-open or ψ^*-closed respectively and hence $f^{-1}(A) \in \psi^*LC(X, \tau)$. Therefore, f is ψ^*-continuous (resp. ψ^*-irresolute).

Theorem 4.9 If X is a $T_{1/5}^{\psi^*}$- space, then $\psi^*LC(X, \tau) = \alpha LC(X, \tau)$.

Proof. Let $A \in \psi^*LC(X, \tau)$. Then there exist ψ^*-open set G and ψ^*-closed set F such that $A = G \cap F$. Since X is a $T_{1/5}^{\psi^*}$-space, then G and F are α-open and α-closed sets respectively and hence $A \in \alpha LC(X, \tau)$. The above implies $\psi^*LC(X, \tau) \subseteq \alpha LC(X, \tau)$.

On the other hand, let $A \in \alpha LC(X, \tau)$. Then $A = G \cap F$, G is α-open set and F is α-closed. But every α-open (resp. α-closed) is ψ^*-open (resp. ψ^*-closed) Hence, G is ψ^*-open set F is ψ^*-closed set. The above implies $\alpha LC(X, \tau) \subseteq \psi^*LC(X, \tau)$. Therefore, $\psi^*LC(X, \tau) = \alpha LC(X, \tau)$.

Theorem 4.10 Every αLC-continuous function is ψ^*LC-continuous.

Proof. Obvious.

The converse of the above theorem need not be true as shown in the following example.

Example 4.6 Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \phi, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{b\}\}$. Then the identity function f: $(X, \tau) \to (Y, \sigma)$ is not αLC-continuous since $\{a, c\} \in C(Y)$ but $f^{-1}(\{a, c\}) = \{a, c\} \notin \alpha LC(X)$. However, f is ψ^*LC-continuous.

Theorem 4.11 If f: $(X, \tau) \to (Y, \sigma)$ is ψ^*LC-continuous and X is a $T_{1/5}^{\psi^*}$- space, then f is αLC-continuous.

Proof. Let G be an open set of and f be an ψ^*LC-continuous. Then $f^{-1}(G)$ is ψ^*LC-set in X. Since X is a $T_{1/5}^{\psi^*}$- space, every ψ^*-open (resp. ψ^*-closed) is α-open (resp. α-closed) in X. Then $f^{-1}(G)$ is ψ*LC-set in Y and hence f is αLC-continuous.

Theorem 4.12 If f: $(X, \tau) \rightarrow (Y, \sigma)$ is αLC-irresolute and g: $(Y, \sigma) \rightarrow (Z, \eta)$ is ψ*LC-continuous and Y is a $T_{1/5}^{\psi^*}$- space, then $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is αLC- continuous.

Proof. Let F be a closed set of Z and g be an ψ*LC-continuous. Then $g^{-1}(F)$ is ψ*LC-set in Y. Since Y is a $T_{1/5}^{\psi^*}$- space, $g^{-1}(F)$ is αLC-set in Y. Since f is αLC-irresolute, then $f^{-1}(g^{-1}(F))$ is αLC-set in X. Therefore, $(g \circ f)^{-1}(F)$ is αLC-set in X and $g \circ f$ is αLC- continuous.

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