APPROXIMATE CONTROLLABILITY OF NEUTRAL INTEGRODIFFERENTIAL INCLUSIONS VIA RESOLVENT OPERATORS

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ABSTRACT. In this work, a set of sufficient conditions are established for the approximate controllability for neutral integrodifferential inclusions in Banach spaces. The theory of fractional power and α-norm is used because of the spatial derivatives in the nonlinear term of the system. Bohnenblust-Karlin's fixed point theorem is used to prove our main results. Further, this result is extended to study the approximate controllability for nonlinear functional control system with nonlocal conditions. An example is also given to illustrate our main results.

1. INTRODUCTION

This paper is mainly focused on the approximate controllability for neutral integrodifferential inclusions in Banach spaces of the form

$$\frac{d}{dt}[x(t) - G(t, x(h_1(t)))] \in -Ax(t) \int_0^t Q(t-s)x(s)ds + F(t, x(h_2(t))) + Bu(t),$$
(1.1)

$$x(0) = x_0, \quad t \in J = [0, b],$$
(1.2)

where $-A$ is the infinitesimal generator of an analytic semigroup on a Banach space X. $Q(t) : X_\alpha \to X_\alpha, t \in J$ is a closed linear operator and B is a bounded linear operator from a Banach space U into X. The function $F : J \times X_\alpha \to 2^{X_\alpha} \setminus \{\emptyset\}$ is a nonempty, bounded, closed and convex multivalued map and the functions G, h_1, h_2 are specified later. Here $0 = t_0 < t_1 < t_2 < ... < t_n < t_n + 1 = b$.

Control theory is an important branch of engineering and mathematics that deals with the behavior of dynamical systems. Controllability is one of the basic concepts in mathematical control theory and it is classified as exact and approximate controllability. Exact controllability enables to drive the system to arbitrary final state while approximate controllability means that the system can be steered to arbitrary small neighborhood of the final state. For the past two decades, authors in [1, 12, 18, 27, 29, 30, 31, 32, 35, 36, 39, 40, 38] investigated the controllability problem for abstract linear control systems in infinite dimensional spaces. Integrodifferential equations can be used to model the various existing problems in the field of electronics, fluid dynamics, biological models and chemical kinetics. Because of such enormous applications, it has been extensively used by the mathematicians.

Initially, in [19], Grimmer et al. proved the existence of solution of the integrodifferential evolution equations by the use of resolvent operator. Since then, many authors studied the existence of solution using resolvent operators which is an alternative for the semigroup operator in the case of integrodifferential equations, see [9, 15, 22, 26, 30]. The impulsive differential equation is a suitable one to model the evolutionary processes from different fields subject to certain perturbations whose duration is negligible when compared to the duration of the whole process. For more detail on these concepts, refer [3, 4, 24, 28] and the references therein.

In many real world problems, the nonlinear terms involve spatial derivatives. In such occasions, we cannot discuss the problem in the whole Banach space X since we normally take $X = L^2([0,\pi])$ and hence the third variable in the nonlinear terms are defined on $X_{\frac{1}{2}}$. So, we restrict the equation in a Banach space $X_\alpha \subset X$ instead of X. We use the fractional power operators and α- norm to show the results, which were used in the papers [17, 33, 10]. Fu et al. [17] studied the existence of solutions for neutral integrodifferential equations with nonlocal conditions. Recently in [33], Mokkedem et al. investigated the approximate controllability of semi-linear neutral integrodifferential systems with infinite delay. Inspired by the above works, in this paper, we establish a set of sufficient conditions for the approximate controllability for neutral impulsive integrodifferential inclusions of the form (1.1)-(1.2).

This paper is organized as follows. In section 2, some necessary concepts and important definitions about the resolvent operators, multivaled map are given. In section 3, a set of sufficient conditions for the approximate controllability for neutral integrodifferential inclusions in Banach spaces are established. In section 4, the approximate controllability for neutral integrodifferential inclusions with nonlocal conditions in Banach spaces is studied. An example is also given in section 5 to illustrate the theory of the abstract main result.

2. Preliminaries

In this section, we introduce some important notations and lemmas concerning the fractional operator and the multi-valued map required in order to prove our results.

Let X be a Banach space with norm $\|\cdot\|$ and here we assume that $-A : D(A) \subseteq X \to X$ is the infinitesimal generator of a compact analytic semigroup $(T(t))_{t>0}$. We denote Y as the Banach space formed from $D(A)$ with the graph norm $\|y\|_Y = \|Ay\| + \|y\|$, for $y \in D(A)$. Let $\mathcal{L}(X)$ is the Banach space of all linear bounded operators L from X into X with norm $\|L\|_{\mathcal{L}(X)} = \sup\{\|L(y)\| : \|y\| = 1\}$. By $\rho(A)$, we denote the resolvent set of a linear operator A and let $0 \in \rho(A)$. Now we define the fractional power A^α for $0 < \alpha \le 1$ as a closed linear operator on its domain $D(A^\alpha)$. Also, the subspace $D(A^\alpha)$ is dense in X and the expression $\|x\|_\alpha = \|A^\alpha x\|$, $x \in D(A^\alpha)$, defines a norm on $D(A^\alpha)$. We denote the space $D(A^\alpha)$ as X_α with the norm $\|\cdot\|_\alpha$. For each $0 < \alpha < 1$, X_α is a Banach space, $X_\alpha \hookrightarrow X_\beta$ for $0 < \beta \le \alpha \le 1$ and the imbedding is compact whenever $\mathcal{R}(\lambda, A)$, the resolvent operator of A is compact. Let $\|A^{-\beta}\| \le M^*$, with M^* a positive constant. We denote by $\mathcal{C}$, the Banach space $C(J, X)$ endowed with supnorm given by

$$\|x\|_C \equiv \sup_{t \in J} \|A^\alpha x(t)\|, \text{ for } x \in \mathcal{C}.$$

The reader may refer [34] for the concepts of semigroup operators. With the help of [19, 20, 21], we now give some essential properties about the resolvent operators.

Definition 2.1. *A family of bounded linear operators $\mathcal{R}(t) \in \mathcal{L}(X)$ for $t \in [0, b]$ is called a resolvent operators for*

$$\frac{d}{dt}x(t) = - Ax(t) + \int_0^b Q(t-s)x(s)ds, \tag{2.1}$$

$$x(0) = x_0 \in X, \tag{2.2}$$

if

(**i**) *$\mathcal{R}(0) = I$ and $\|\mathcal{R}(t)\| \le N_1 e^{\omega t}$ for some $N_1 > 0, \omega \in \mathbb{R}$,*
(**ii**) *for all $x \in X$, $\mathcal{R}(t)x$ is continuous for $t \in [0, b]$,*
(**iii**) *$\mathcal{R}(t) \in \mathcal{L}(Y)$, for $t \in [0, b]$. For $x \in Y$, $\mathcal{R}(t)x \in C^1([0, b], X) \bigcap C([0, b], Y)$ and for $t \ge 0$ such that*

$$\frac{d}{dt}\mathcal{R}(t)x = - A\mathcal{R}(t)x + \int_0^t Q(t-s)\mathcal{R}(s)xds$$

$$= - \mathcal{R}(t)Ax + \int_0^t \mathcal{R}(t-s)Q(s)xds.$$

By [21], the operators A and $Q(\cdot)$ satisfies the following conditions:

(**A$_1$**) A generates an analytic semigroup on X. $Q(t)$ is a closed operator on X with domain at least $D(A)$ a.e $t \ge 0$ with $Q(t)x$ strongly measurable for each $x \in D(A)$ and $\|Q(t)\|_{1,0} \le q(t)$, $q \in L^1(0, \infty)$ with $q^*(\lambda)$ absolutely convergent for $Re\lambda > 0$, where $b^*(\lambda)$ denotes the Laplace transform of $q(t)$.
(**A$_2$**) $\rho(\lambda) := (\lambda I - A_0 - Q^*(\lambda))^{-1}$ exists as a bounded operator on X which is analytic for λ in the region $\Lambda = \{\lambda \in \mathbb{C} : |arg\lambda| \le \frac{\pi}{2} + \delta\}$, where

$0 < \delta < \frac{\pi}{2}$. In Λ if $|\lambda| \geq \varepsilon > 0$, there exists a constant $M = M(\varepsilon) > 0$ so that $\|\rho(\lambda)\| \leq \frac{M}{|\lambda|}$.

($\mathbf{A_3}$) $A\rho(\lambda) \in \mathcal{L}(X)$ for $\lambda \in \Lambda$ and are analytic on Λ into $\mathcal{L}(X)$. $B^*(\lambda) \in \mathcal{L}(Y, X)$ and $Q^*(\lambda)\rho(\lambda) \in \mathcal{L}(Y, X)$ for $\lambda \in \Lambda$. Given $\varepsilon > 0$, there exists $M = M(\varepsilon) > 0$ so that for $\lambda \in \Lambda$ with $|\lambda| \geq \varepsilon$, $\|A\rho(\lambda)\|_{1,0} + \|Q^*(\lambda)\rho(\lambda)\|_{1,0} \leq \frac{M}{\lambda}$, and $\|Q^*(\lambda)\|_{1,0} \to 0$ as $|\lambda| \to \infty$ in Λ. In addition, $\|A\rho(\lambda)\| \leq \frac{M}{|\lambda|^n}$ for some $n > 0, \lambda \in \Lambda$ with $|\lambda| \leq \varepsilon$. Further, there exists $D \subset D(A^2)$ which is dense in Y such that $A_0(D)$ and $Q^*(\lambda)(D)$ are contained in Y and $\|Q^*(\lambda)x\|_1$ is bounded for each $x \in D$, $\lambda \in \Lambda$, $|\lambda| \geq \varepsilon$.

With the help of above conditions, there exists a resolvent operator $\mathcal{R}(t)$ for the linear system (2.1)-(2.2) given by

$$\mathcal{R}(0) = I$$

and

$$\mathcal{R}(t)x = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t}(\lambda I - A - Q^*(\lambda))^{-1}x d\lambda, \ t > 0,$$

By the assumption (A_2),

$$R(t)x = \frac{1}{2\pi i} \int_\Gamma \rho(\lambda)x d\lambda, \ t > 0,$$

where Γ is a contour of the type used to obtain an analytic semigroup. We can select contour Γ, included in the region Λ, consisting of $\Gamma_1, \Gamma_2,$ and Γ_3, where

$$\Gamma_1 = \{re^{i\phi} : r \geq 1\}, \ \Gamma_2 = \{e^{i\theta} : -\phi \leq \theta \leq \phi\},$$
$$\Gamma_3 = \{re^{i\phi} : r \geq 1\}, \frac{\pi}{2} < \phi < \frac{\pi}{2} + \delta,$$

oriented so that $\text{Im}(\lambda)$ is increasing on Γ_1 and Γ_2. Moreover, $\mathcal{R}(t)$ is also analytic and there exist $N, C_\alpha > 0$ such that

$$\|\mathcal{R}(t)\| \leq N \text{ and } \|A^\alpha \mathcal{R}(t)\| \leq \frac{C_\alpha}{t^\alpha}, \ 0 < t < b, \ 0 \leq \alpha \leq 1.$$

Lemma 2.2. [17]. *$A\mathcal{R}(t)$ is continuous for $t > 0$ in the uniform operator topology of $\mathcal{L}(X)$.*

In this work, we resquire that A^α be commutative with $\mathcal{R}(t)$ for any $0 \leq \alpha \leq 1$, that is, for any $x \in D(A^\alpha)$,

$$A^\alpha \mathcal{R}(t)x = \mathcal{R}(t)A^\alpha x \tag{2.3}$$

Even though in some references [23, 11] have used it, this commutation is not always valid. But this commutation can be proved in many cases. Take $Q(t-s) = q(t-s)A$ with $b(t)$ a scalar function defined on $(0, +\infty)$, then the linear system (2.1)-(2.2)

becomes

$$\frac{d}{dt}x(t) = -Ax(t) + \int_0^b q(t-s)Ax(s)ds, \tag{2.4}$$

$$x(0) = x_0 \in X, \tag{2.5}$$

Now we apply the following conditions on (2.4)-(2.5) from [21],

($\mathbf{A'_1}$) A generates an analytic semigroup on X. In particular,

$$\Lambda_1 = \{\lambda \in \mathbb{C} : |arg\lambda| < (\frac{\pi}{2}) + \delta_1\},\ 0 < \delta_1 < \frac{\pi}{2}$$

is contained in the resolvent set of A and $\|(\lambda I - A)^{-1}\| \leq M/|\lambda|$ on Λ_1 for some constant $M > 0$. The scalar function $q(\cdot)$ is in $L^1(0,\infty)$ with $q^*(\lambda)$ absolutely convergent for $Re\lambda > 0$, where $q^*(\lambda)$ denotes the Laplace transform of $q(t)$.

($\mathbf{A'_2}$) There exists $\Lambda = \{\lambda \in \mathbb{C} : |arg\lambda| < (\frac{\pi}{2}) + \delta_2\},\ 0 < \delta_2 < \frac{\pi}{2}$, so that $\lambda \in \Lambda$ implies $g_1(\lambda) = 1 + q^*(\lambda)$ exists and is not zero. Further $\lambda g_1^{-1}(\lambda) \in \Lambda_1$ for $\lambda \in \Lambda$.

($\mathbf{A'_3}$) In $\Lambda, q^*(\lambda) \to 0$ as $|\lambda| \to \infty$.

With the help of above conditions, the resolvent operator $\mathcal{R}(t)$ is analytic. Hence (2.3) holds in this case.

Now we introduce some basic definitions and results of multivalued maps. For more details on multivalued maps, see the books of [37, 13].

Definition 2.3. [9]. *A multivalued map F satisfies the following conditions:*

(i) *A multivalued map $F : X \to 2^X \setminus \{\emptyset\}$ is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$. F is bounded on bounded sets if $F(C) = \bigcup_{x \in C} F(x)$ is bounded in X for any bounded set C of X, i.e., $\sup_{x \in C} \left\{ \sup\{\|y\| : y \in G(x)\} \right\} < \infty$.*

(ii) *F is called upper semicontinuous (u.s.c. for short) on X if for each $x_0 \in X$, the set $F(x_0)$ is a nonempty closed subset of X, and if for each open set C of X containing $F(x_0)$, there exists an open neighborhood V of x_0 such that $F(V) \subseteq C$.*

(iii) *F is called completely continuous if $F(C)$ is relatively compact for every bounded subset C of X.*

(iv) *If the multivalued map F is completely continuous with nonempty values, then F is u.s.c., if and only if F has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in Fx_n$ imply $y_* \in Fx_*$. F has a fixed point if there is a $x \in X$ such that $x \in F(x)$.*

Remark 2.4. *In this paper, $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subset of X.*

Definition 2.5. *A function $x \in \mathcal{C}$ is said to be a mild solution of system* (1.1)-(1.2) *if $x(0) = x_0$, and there exists $f \in L^1(J, X)$ such that $f(t) \in F(t, x(h_2(t)))$ on $t \in J$ and the integral equation*

$$x(t) = \mathcal{R}(t)[x_0 - G(t, x(h_1(0)))] + G(t, x(h_1(t))) + \int_0^t \mathcal{R}(t - s)AG(s, x(h_1(s)))ds$$

$$+ \int_0^t \mathcal{R}(t - s) \int_0^s Q(s - \tau)G(\tau, x(h_1(\tau)))d\tau ds + \int_0^t \mathcal{R}(t - s)f(s)ds$$

$$+ \int_0^t \mathcal{R}(t - s)Bu(s)ds,$$

is satisfied.

Now, it is convenient to introduce two appropriate operators and basic assumptions on these operators:

$$\Gamma_0^b = \int_0^b \mathcal{R}(b - s)BB^*\mathcal{R}^*(b - s)ds : X \to X,$$

$$R(a, \Gamma_0^b) = (aI + \Gamma_0^b)^{-1} : X \to X,$$

where B^* denotes the adjoint of B and $\mathcal{R}^*(t)$ is the adjoint of $\mathcal{R}(t)$. It is clear that the operator Γ_0^b is a linear bounded operator.

To study the approximate controllability of system (1.1)-(1.2), we impose the following condition:

($\mathbf{H_0}$) $aR(a, \Gamma_0^b) \to 0$ as $a \to 0^+$ in the strong operator topology.

In view of [30], Hypothesis ($\mathbf{H_0}$) holds if and only if the linear system

$$x'(t) + Ax(t) \in \int_0^t Q(t - s)x(s)ds + Bu(t), \quad t \in [0, b], \tag{2.6}$$

$$x(0) = x_0 \tag{2.7}$$

is approximately controllable on $[0, b]$.

We use the following well known results to prove our results.

Lemma 2.6. [25, Lasota and Opial] *Let J be a compact real interval, $BCC(X)$ be the set of all nonempty, bounded, closed and convex subset of X and F be a multivalued map satisfying $F : J \times X \to BCC(X)$ is measurable to t for each fixed $x \in X$, u.s.c. to x for each $t \in J$, and for each $x \in \mathcal{C}$ the set*

$$S_{F,x} = \{f \in L^1(J, X) : f(t) \in F(t, x(t)), \ t \in J\}$$

is nonempty. Let $\mathscr{F}$ be a linear continuous from $L^1(J, X)$ to $\mathcal{C}$, then the operator

$$\mathscr{F} \circ S_F : \mathcal{C} \to BCC(\mathcal{C}), \ x \to (\mathscr{F} \circ S_F)(x) = \mathscr{F}(S_{F,x}),$$

is a closed graph operator in $\mathcal{C} \times \mathcal{C}$.

Lemma 2.7. [5, Bohnenblust and Karlin]. *Let $\mathcal{D}$ be a nonempty subset of X, which is bounded, closed, and convex. Suppose $G : \mathcal{D} \to 2^X \setminus \{\emptyset\}$ is u.s.c. with closed, convex values, and such that $G(\mathcal{D}) \subseteq \mathcal{D}$ and $G(\mathcal{D})$ is compact. Then G has a fixed point.*

3. Approximate controllability results

In this section, first the existence of mild solutions for system (1.1)-(1.2) is proved by using Bohnenblust-Karlin fixed point theorem. And then, we show under certain assumptions, the approximate controllability of (2.6)-(2.7) implies the approximate controllability of (1.1)-(1.2). To prove the results, we need the following hypotheses and let $\alpha \in (0, 1)$.

($\mathbf{H_1}$) $\mathcal{R}(t)$ is a compact operator for each $t > 0$.

($\mathbf{H_2}$) $(Q(t))_{t \in J}$ is a family of operators from Y to X such that $Q(t) \in \mathcal{L}(X_{\alpha+\beta}, X)$ for each $t \in J$. Then, there exists a constant $M_1 > 0$ such that

$$\|Q(t)\|_{\alpha+\beta,0} \leq M_1.$$

($\mathbf{H_3}$) There exist a constant $\beta \in (0, 1)$ with $\alpha + \beta = 1$, such that $G : J \times X_\alpha \to X_{\alpha+\beta}$ satisfies the Lipschitz condition, i.e., there exists a constant $L_g > 0$ such that

$$\|G(t_1, x_1) - G(t_2, x_2)\|_{\alpha+\beta} \leq L_g(|t_1 - t_2| + \|x_1 - x_2\|_\alpha)$$

for any $0 \leq t_1, t_2 \leq b$, $x_1, x_2 \in X_\alpha$, and the inequality

$$\|G(t, x)\|_{\alpha+\beta} \leq L_g(\|x\|_\alpha + 1)$$

holds for any $(t, x) \in [0, b] \times X_\alpha$.

($\mathbf{H_4}$) The multivalued map $F : J \times X_\alpha \to BCC(X_\alpha)$ satisfies the following conditions.

   (**i**) For each $t \in J$, the function $F(t, \cdot) : X_\alpha \to BCC(X_\alpha)$ is u.s.c; and for each $x \in X_\alpha$, the function $F(\cdot, x)$ is measurable.

   (**ii**) For each $x \in \mathcal{C}$, the set

$$S_{F,x} = \left\{ f \in L^1(J, X_\alpha) : f(t) \in F(t, x(h_2(t))), t \in J \right\}$$

   is non-empty.

($\mathbf{H_5}$) For each positive number r and $x \in \mathcal{C}$ with $\|x\|_\mathcal{C} \leq r$, there exists $L_{f,r}(\cdot) \in L^1(J, \mathbb{R}^+)$ such that

$$\sup_{\|x\| \leq r} \|F(t, x)\|_\alpha \leq L_{f,r}(s)ds, \quad \text{for a.e.} t \in J$$

where $\|F(t, x)\| = \sup\{\|f\| : f(t) \in F(t, x(h_2(t)))\}$.

($\mathbf{H_6}$) The function $s \to L_{f,r}(s) \in L^1([0, t], \mathbb{R}^+)$ and there exists a $\delta > 0$ such that

$$\lim_{r \to \infty} \frac{\int_0^t L_{f,r}(s)ds}{r} = \delta < +\infty.$$

($\mathbf{H_7}$) $h_i \in C(J, J)$, $i = 1, 2$.

It will be shown that the system (1.1)-(1.2) is approximately controllable, if for all $a > 0$, there exists a continuous function $x(\cdot)$ such that

$$
\begin{aligned}
x(t) =& \mathcal{R}(t)[x_0 - G(t, x(h_1(0)))] + G(t, x(h_1(t))) + \int_0^t \mathcal{R}(t-s)AG(s, x(h_1(s)))ds \\
&+ \int_0^t \mathcal{R}(t-s) \int_0^s Q(s-\tau)G(\tau, x(h_1(\tau)))d\tau ds + \int_0^t \mathcal{R}(t-s)f(s)ds \\
&+ \int_0^t \mathcal{R}(t-s)Bu(s, x)ds, \quad t \in J, \quad f \in S_{F,x},
\end{aligned} \tag{3.1}
$$

$$
u(t, x) = B^* \mathcal{R}^*(b-t)R(a, \Gamma_0^b)p(x(\cdot)), \tag{3.2}
$$

where

$$
\begin{aligned}
p(x(\cdot)) =& x_b - \mathcal{R}(b)[x_0 - G(b, x(h_1(0)))] - G(b, x(h_1(b))) - \int_0^b \mathcal{R}(b-s)AG(s, x(h_1(s)))ds \\
&- \int_0^b \mathcal{R}(b-s) \int_0^s Q(s-\tau)G(\tau, x(h_1(\tau)))d\tau ds - \int_0^b \mathcal{R}(b-s)f(s)ds
\end{aligned}
$$

Theorem 3.1. *Suppose that the hypotheses* $(\mathbf{H_0})$-$(\mathbf{H_7})$ *are satisfied. Assume also*

$$
\left(1 + \frac{1}{a}N^2 M_B^2 b\right)\left[NM^*L_g + M^*L_g + \frac{b^\beta C_{1-\beta}}{\beta}L_g + \frac{b^{2-\alpha}C_\alpha}{1-\alpha}M_1 L_g + N\gamma\right] < 1, \tag{3.3}
$$

where $M_B = \|B\|$, *then the system* (1.1)-(1.2) *has a solution on* J.

Proof. The main aim of this theorem is to find conditions for solvability of system (1.1)-(1.2) for $a > 0$. We show that, using the control $u(t, x)$, the operator $\Gamma : \mathcal{C} \to 2^{\mathcal{C}}$, defined by

$$
\begin{aligned}
\Upsilon(x) = \Big\{\varphi \in& \mathcal{C} : \varphi(t) = \mathcal{R}(t)[x_0 - G(t, x(h_1(0)))] + G(t, x(h_1(t))) \\
&+ \int_0^t \mathcal{R}(t-s)AG(s, x(h_1(s)))ds \\
&+ \int_0^t \mathcal{R}(t-s) \int_0^s Q(s-\tau)G(\tau, x(h_1(\tau)))d\tau ds \\
&+ \int_0^t \mathcal{R}(t-s)f(s)ds + \int_0^t \mathcal{R}(t-s)Bu(s, x)ds, t \in J\Big\},
\end{aligned}
$$

has a fixed point x, which is a mild solution of system (1.1)-(1.2).

We now show that Υ satisfies all the conditions of Lemma 2.7. To simplify the result, we subdivide the proof into five steps.

Step 1. Γ is convex for each $x \in \mathcal{C}$.

In fact, if φ_1, φ_2 belong to $\Upsilon(x)$, then there exist f_1, $f_2 \in S_{F,x}$ such that for each $t \in [0, b]$, we have

$$
\begin{aligned}
\varphi_i(t) = {}&\mathcal{R}(t)[x_0 - G(t, x(h_1(0)))] + G(t, x(h_1(t))) + \int_0^t \mathcal{R}(t-s)AG(s, x(h_1(s)))ds \\
&+ \int_0^t \mathcal{R}(t-s)\int_0^s Q(s-\tau)G(\tau, x(h_1(\tau)))d\tau ds + \int_0^t \mathcal{R}(t-s)f_i(s)ds \\
&+ \int_0^t \mathcal{R}(t-\eta)BB^*\mathcal{R}^*(b-t)R(a, \Gamma_0^b) \times \Big[x_b - \mathcal{R}(b)[x_0 - G(b, x(h_1(0)))] \\
&- G(b, x(h_1(b))) - \int_0^b \mathcal{R}(b-s)AG(s, x(h_1(s)))ds \\
&- \int_0^b \mathcal{R}(b-s)\int_0^s Q(s-\tau)G(\tau, x(h_1(\tau)))d\tau ds - \int_0^b \mathcal{R}(b-s)f_i(s)ds\Big](\eta)d\eta
\end{aligned}
$$

Let $\lambda \in [0, 1]$. Then for each $t \in J$, we get

$$
\begin{aligned}
\lambda\varphi_1(t) + (1-\lambda)\varphi_2(t) = {}&\mathcal{R}(t)[x_0 - G(t, x(h_1(0)))] + G(t, x(h_1(t))) \\
&+ \int_0^t \mathcal{R}(t-s)AG(s, x(h_1(s)))ds + \int_0^t \mathcal{R}(t-s)\int_0^s Q(s-\tau)G(\tau, x(h_1(\tau)))d\tau ds \\
&+ \int_0^t \mathcal{R}(t-s)[\lambda f_1(s) + (1-\lambda)f_2(s)]ds + \int_0^t \mathcal{R}(t-\eta)BB^*\mathcal{R}^*(b-t)R(a, \Gamma_0^b) \\
&\times \Big[x_b - \mathcal{R}(b)[x_0 - G(b, x(h_1(0)))] - G(b, x(h_1(b))) \\
&- \int_0^b \mathcal{R}(b-s)AG(s, x(h_1(s)))ds \\
&- \int_0^b \mathcal{R}(b-s)\int_0^s Q(s-\tau)G(\tau, x(h_1(\tau)))d\tau ds \\
&- \int_0^b \mathcal{R}(b-s)[\lambda f_1(s) + (1-\lambda)f_2(s)](s)ds\Big](\eta)d\eta.
\end{aligned}
$$

It is easy to see that $S_{F,x}$ is convex since F has convex values. So, $\lambda f_1 + (1-\lambda)f_2 \in S_{F,x}$. Thus,

$$\lambda\varphi_1 + (1-\lambda)\varphi_2 \in \Upsilon(x).$$

Step 2. For $r > 0$, let $\mathcal{B}_r = \{x \in \mathcal{C} : \|x\|_C \le r\}$. Certainly, $\mathcal{B}_r$ is a bounded, closed and convex set of $\mathcal{C}$. We claim that there exists a positive number r such that $\Upsilon(\mathcal{B}_r) \subseteq \mathcal{B}_r$.

If this is not true, then for each positive number r, there exists a function $x^r \in \mathcal{B}_r$, but $\Upsilon(x^r) \neq \mathcal{B}_r$, i.e., $\|\Upsilon(x^r)\|_C \equiv \sup\left\{\|\varphi^r\|_C : \varphi^r \in (\Upsilon x^r)\right\} > r$ and

$$\varphi^r(t) = \mathcal{R}(t)[x_0 - G(t, x^r(h_1(0)))] + G(t, x^r(h_1(t)))$$
$$+ \int_0^t \mathcal{R}(t-s)AG(s, x^r(h_1(s)))ds$$
$$+ \int_0^t \mathcal{R}(t-s)\int_0^s Q(s-\tau)G(\tau, x^r(h_1(\tau)))d\tau ds$$
$$+ \int_0^t \mathcal{R}(t-s)f^r(s)ds + \int_0^t \mathcal{R}(t-s)Bu^r(s,x)ds,$$

for some $f^r \in S_{F,x^r}$. Using $(\mathbf{H_1})$-$(\mathbf{H_7})$, we have

$$r < \|\Upsilon(x^r)(t)\|_\alpha$$
$$\leq \left\|\mathcal{R}(t)[x_0 - G(t, x^r(h_1(0)))]\right\|_\alpha + \left\|G(t, x^r(h_1(t)))\right\|_\alpha$$
$$+ \left\|\int_0^t \mathcal{R}(t-s)AG(s, x^r(h_1(s)))ds\right\|_\alpha$$
$$+ \left\|\int_0^t \mathcal{R}(t-s)\int_0^s Q(s-\tau)G(\tau, x^r(h_1(\tau)))d\tau ds\right\|_\alpha$$
$$+ \left\|\int_0^t \mathcal{R}(t-s)f^r(s)ds\right\|_\alpha + \left\|\int_0^t \mathcal{R}(t-\eta)BB^*\mathcal{R}^*(b-t)R(a, \Gamma_0^b)\Big[x_b\right.$$
$$- \mathcal{R}(b)[x_0 - G(b, x^r(h_1(0)))] - G(b, x^r(h_1(b)))$$
$$- \int_0^b \mathcal{R}(b-s)AG(s, x^r(h_1(s)))ds - \int_0^b \mathcal{R}(b-s)\int_0^s Q(s-\tau)G(\tau, x^r(h_1(\tau)))d\tau ds$$
$$- \int_0^b \mathcal{R}(b-s)f^r(s)ds\Big](\eta)d\eta\Big\|_\alpha$$
$$\leq N[\|x_0\|_\alpha + M^*L_g(1+r)] + M^*L_g(1+r) + \int_0^t \|A^{1-\beta}\mathcal{R}(t-s)A^\beta G(s, x^r(h_1(s)))\|_\alpha ds$$
$$+ \int_0^t \|A^\alpha \mathcal{R}(t-s)\|\int_0^s \|Q(s-\tau)G(\tau, x^r(h_1(\tau)))\|d\tau ds$$
$$+ N\int_0^t L_{f,r}(s)ds + \frac{1}{a}N^2M_B^2b \times \Big[N[\|x_0\|_\alpha + M^*L_g(1+r)] + M^*L_g(1+r)$$
$$+ \int_0^t \|A^{1-\beta}\mathcal{R}(t-s)A^\beta G(s, x^r(h_1(s)))\|_\alpha ds$$
$$+ \int_0^t \|A^\alpha \mathcal{R}(t-s)\|\int_0^s \|Q(s-\tau)G(\tau, x^r(h_1(\tau)))\|d\tau ds + N\int_0^t L_{f,r}(s)ds\Big]$$

$$\leq \quad \left[1 + \frac{1}{a}N^2 M_B^2 b\right]\left(N[\|x_0\|_\alpha + M^* L_g(1+r)]\right.$$

$$+ M^* L_g(1+r) + \frac{b^\beta C_{1-\beta}}{\beta} L_g(1+r) + \frac{b^{2-\alpha} C_\alpha}{1-\alpha} M_1 L_g(1+r)$$

$$\left. + N \int_0^t L_{f,r}(s)ds\right)$$

Dividing both sides of the above inequality by r and taking the limit as $r \to \infty$, using $\mathbf{H_3}$, we get

$$\left(1 + \frac{1}{a}N^2 M_B^2 b\right)\left[NM^* L_g + M^* L_g + \frac{b^\beta C_{1-\beta}}{\beta} L_g + \frac{b^{2-\alpha} C_\alpha}{1-\alpha} M_1 L_g + N\gamma\right] \geq 1$$

This contradicts with the condition (3.3). Hence, for some $r > 0$, $\Upsilon(\mathcal{B}_r) \subseteq \mathcal{B}_r$.
Step 3. Υ sends bounded sets into equicontinuous sets of $\mathcal{C}$. For each $x \in \mathcal{B}_r$, $\varphi \in \Upsilon(x)$, there exists a $f \in S_{F,x}$ such that for $\varepsilon > 0$ and $0 < t_1 < t_2 \leq b$, then

$$\|\varphi(t_1) - \varphi(t_2)\| = \|\mathcal{R}(t_1) - \mathcal{R}(t_2)\|\|x_0 - G(t, x(h_1(0)))\|_\alpha$$

$$+ \|G(t_1, x(h_1(t))) - G(t_2, x(h_1(t)))\|_\alpha$$

$$+ \left\|\int_0^{t_1-\varepsilon} [\mathcal{R}(t_1 - s) - \mathcal{R}(t_2 - s)]AG(s, x(h_1(s)))ds\right\|_\alpha$$

$$+ \left\|\int_{t_1-\varepsilon}^{t_1} [\mathcal{R}(t_1 - s) - \mathcal{R}(t_2 - s)]AG(s, x(h_1(s)))ds\right\|_\alpha$$

$$+ \left\|\int_{t_1}^{t_2} \mathcal{R}(t_2 - s)AG(s, x(h_1(s)))ds\right\|_\alpha$$

$$+ \left\|\int_0^{t_1-\varepsilon} [\mathcal{R}(t_1 - s) - \mathcal{R}(t_2 - s)]\int_0^s Q(s-\tau)G(\tau, x(h_1(\tau)))d\tau ds\right\|_\alpha$$

$$+ \left\|\int_{t_1-\varepsilon}^{t_1} [\mathcal{R}(t_1 - s) - \mathcal{R}(t_2 - s)]\int_0^s Q(s-\tau)G(\tau, x(h_1(\tau)))d\tau ds\right\|_\alpha$$

$$+ \left\|\int_{t_1}^{t_2} \mathcal{R}(t_2 - s)\int_0^s Q(s-\tau)G(\tau, x(h_1(\tau)))d\tau ds\right\|_\alpha$$

$$+ \left\|\int_0^{t_1-\varepsilon} [\mathcal{R}(t_1 - s) - \mathcal{R}(t_2 - s)]f(s)ds\right\|_\alpha$$

$$+ \left\|\int_{t_1-\varepsilon}^{t_1} [\mathcal{R}(t_1 - s) - \mathcal{R}(t_2 - s)]f(s)ds\right\|_\alpha + \left\|\int_{t_1}^{t_2} \mathcal{R}(t_2 - s)f(s)ds\right\|_\alpha$$

$$+ \left\|\int_0^{t_1-\varepsilon} [\mathcal{R}(t_1 - \eta) - \mathcal{R}(t_2 - \eta)]Bu(\eta, x)d\eta\right\|_\alpha$$

$$+ \left\|\int_{t-\varepsilon}^{t_1} \mathcal{R}[(t_1 - \eta) - \mathcal{R}(t_2 - \eta)]Bu(\eta, x)d\eta\right\|_\alpha + \left\|\int_{t_1}^{t_2} \mathcal{R}(t_2 - \eta)Bu(\eta, x)d\eta\right\|_\alpha$$

$$\leq \|\mathcal{R}(t_1) - \mathcal{R}(t_2)\| \|[x_0 - G(t, x(h_1(0)))]\|_\alpha + M^* L_g[|t_1 - t_2| + \|x(h_1(t)) - x(h_1(t))\|_\alpha$$

$$+ L_g \int_0^{t_1-\varepsilon} \|A^{1-\beta}\| [\mathcal{R}(t_1 - s) - \mathcal{R}(t_2 - s)](1 + \|x(s)\|_\alpha) ds$$

$$+ L_g \int_{t_1-\varepsilon}^{t_1} \|A^{1-\beta}\| [\mathcal{R}(t_1 - s) - \mathcal{R}(t_2 - s)](1 + \|x(s)\|_\alpha) ds + L_g \int_{t_1}^{t_2} \frac{C_{1-\beta}}{(t-s)^{1-\beta}} (1 + \|x(s)\|_\alpha) ds$$

$$+ b M_1 L_g \int_0^{t_1-\varepsilon} \|A^\alpha\| [\mathcal{R}(t_1 - s) - \mathcal{R}(t_2 - s)](1 + \|x(s)_\alpha) ds$$

$$+ b M_1 L_g \int_{t_1-\varepsilon}^{t_1} \|A^\alpha\| [\mathcal{R}(t_1 - s) - \mathcal{R}(t_2 - s)](1 + \|x(s)_\alpha\|) ds$$

$$+ b M_1 L_g \int_{t_1}^{t_2} \frac{C_\alpha}{(t-s)^\alpha} (1 + \|x(s)_\alpha\| ds + \int_0^{t_1-\varepsilon} [\mathcal{R}(t_1 - s) - \mathcal{R}(t_2 - s)] L_{f,r}(s) ds$$

$$+ \int_{t_1-\varepsilon}^{t_1} [\mathcal{R}(t_1 - s) - \mathcal{R}(t_2 - s)] L_{f,r} ds + N \int_{t_1}^{t_2} L_{f,r}(s) ds$$

$$+ M_B \int_0^{t_1-\varepsilon} [\mathcal{R}(t_1 - \eta) - \mathcal{R}(t_2 - \eta)] \|u(\eta, x)\|_\alpha d\eta$$

$$+ M_B \int_{t-\varepsilon}^{t_1} \mathcal{R}[(t_1 - \eta) - \mathcal{R}(t_2 - \eta)] \|u(\eta, x)\|_\alpha d\eta + N M_B \int_{t_1}^{t_2} \|u(\eta, x)\|_\alpha d\eta$$

The right-hand side of the above inequality tends to zero independently of $x \in \mathcal{B}_r$ as $(t_1 - t_2) \to 0$ and ε sufficiently small, since the compactness of the resolvent operator $\mathcal{R}(t)$ implies the continuity in the uniform operator topology. Thus $\Upsilon(x^r)$ sends $\mathcal{B}_r$ into equicontinuous family of functions.

Step 4. The set $\Pi(t) = \{\varphi(t) : \varphi \in \Upsilon(\mathcal{B}_r)\}$ is relatively compact in X_α.

Let $t \in (0, b]$ be fixed and ε a real number satisfying $0 < \varepsilon < t$. For $x \in \mathcal{B}_r$, we define

$$\varphi_\varepsilon(t) = \mathcal{R}(t)[x_0 - G(t, x(h_1(0)))] + G(t, x(h_1(t))) + \int_0^{t-\varepsilon} \mathcal{R}(t - s) A G(s, x(h_1(s))) ds$$

$$+ \int_0^{t-\varepsilon} \mathcal{R}(t - s) \int_0^s Q(s - \tau) G(\tau, x(h_1(\tau))) d\tau ds + \int_0^{t-\varepsilon} \mathcal{R}(t - s) f(s) ds$$

$$+ \int_0^{t-\varepsilon} \mathcal{R}(t - s) B u(s, x) ds, t \in J.$$

Since $\mathcal{R}(t)$ is a compact operator, the set $\Pi_\varepsilon(t) = \{\varphi_\varepsilon(t) : \varphi_\varepsilon \in \Upsilon(\mathcal{B}_r)\}$ is relatively compact in X_α for each ε, $0 < \varepsilon < t$. Moreover, for each $0 < \varepsilon < t$, we have

$$\|\varphi(t) - \varphi_\varepsilon(t)\| \leq \left\| \int_{t-\varepsilon}^t \mathcal{R}(t - s) A G(s, x(h_1(s))) ds \right\|_\alpha$$

$$+\left\| \int_{t-\varepsilon}^{t} \mathcal{R}(t-s) \int_{0}^{s} Q(s-\tau)G(\tau, x(h_1(\tau)))d\tau ds \right\|_{\alpha}$$

$$+\left\| \int_{t-\varepsilon}^{t} \mathcal{R}(t-s)f(s)ds \right\|_{\alpha} + \left\| \int_{t-\varepsilon}^{t} \mathcal{R}(t-s)Bu(s,x)ds \right\|_{\alpha}$$

$$\leq \quad \frac{C_{1-\beta}}{\beta}\varepsilon^{\beta}L_g(1+r) + \frac{C_{\alpha}}{1-\alpha}\varepsilon^{2-\alpha}M_1L_g(1+r) + N\gamma + NM_B \int_{t-\varepsilon}^{t} \|u(s,\eta)\|_{\alpha}d\eta.$$

Hence there exist relatively compact sets arbitrarily close to the set $\Pi(t) = \{\varphi(t) : \varphi \in \Upsilon(\mathcal{B}_r)\}$, and the set $\widetilde{\Pi}(t)$ is relatively compact in X_{α} for all $t \in [0,b]$. Since it is compact at $t = 0$, hence $\Pi(t)$ is relatively compact in X_{α} for all $t \in [0,b]$.

Step 5. Υ has a closed graph.

Let $x_n \to x_*$ as $n \to \infty$, $\varphi_n \in \Upsilon(x_n)$, and $\varphi_n \to \varphi_*$ as $n \to \infty$. We will show that $\varphi_* \in \Upsilon(x_*)$. Since $\varphi_n \in \Upsilon(x_n)$, there exists a $f_n \in S_{F,x_n}$ such that

$$\varphi_n(t) = \mathcal{R}(t)[x_0 - G(t, x_n(h_1(0)))] + G(t, x_n(h_1(t)))$$

$$+ \int_{0}^{t} \mathcal{R}(t-s)AG(s, x_n(h_1(s)))ds$$

$$+ \int_{0}^{t} \mathcal{R}(t-s) \int_{0}^{s} Q(s-\tau)G(\tau, x_n(h_1(\tau)))d\tau ds + \int_{0}^{t} \mathcal{R}(t-s)f_n(s)ds$$

$$+ \int_{0}^{t} \mathcal{R}(t-\eta)BB^*\mathcal{R}^*(b-t)R(a, \Gamma_0^b)$$

$$\times \left[x_b - \mathcal{R}(b)[x_0 - G(b, x_n(h_1(0)))] - G(b, x_n(h_1(b))) \right.$$

$$- \int_{0}^{b} \mathcal{R}(b-s)AG(s, x_n(h_1(s)))ds$$

$$- \int_{0}^{b} \mathcal{R}(b-s) \int_{0}^{s} Q(s-\tau)G(\tau, x_n(h_1(\tau)))d\tau ds$$

$$\left. - \int_{0}^{b} \mathcal{R}(b-s)f_n(s)ds \right](\eta)d\eta.$$

We must prove that there exists a $f_* \in S_{F,x_*}$ such that

$$\varphi_*(t) = \mathcal{R}(t)[x_0 - G(t, x_*(h_1(0)))] + G(t, x_*(h_1(t)))$$

$$+ \int_{0}^{t} \mathcal{R}(t-s)AG(s, x_*(h_1(s)))ds$$

$$+ \int_0^t \mathcal{R}(t-s) \int_0^s Q(s-\tau)G(\tau, x_*(h_1(\tau)))d\tau ds + \int_0^t \mathcal{R}(t-s)f_*(s)ds$$

$$+ \int_0^t \mathcal{R}(t-\eta)BB^*\mathcal{R}^*(b-t)R(a,\Gamma_0^b)\Big[x_b - \mathcal{R}(b)[x_0 - G(b, x_*(h_1(0)))]$$

$$-G(b, x_*(h_1(b))) - \int_0^b \mathcal{R}(b-s)AG(s, x_*(h_1(s)))ds$$

$$- \int_0^b \mathcal{R}(b-s) \int_0^s Q(s-\tau)G(\tau, x_*(h_1(\tau)))d\tau ds - \int_0^b \mathcal{R}(b-s)f_*(s)ds\Big](\eta)d\eta.$$

Clearly, we have

$$\Big\| \Big(\varphi_n(t) - \mathcal{R}(t)[x_0 - G(t, x_n(h_1(0)))] - G(t, x_n(h_1(t))) - \int_0^t \mathcal{R}(t-s)AG(s, x_n(h_1(s)))ds$$

$$- \int_0^t \mathcal{R}(t-s) \int_0^s Q(s-\tau)G(\tau, x_n(h_1(\tau)))d\tau ds - \int_0^t \mathcal{R}(t-\eta)BB^*\mathcal{R}^*(b-t)R(a,\Gamma_0^b)$$

$$\times \Big[x_b - \mathcal{R}(b)[x_0 - G(b, x_n(h_1(0)))] - G(b, x_n(h_1(b))) - \int_0^b \mathcal{R}(b-s)AG(s, x_n(h_1(s)))ds$$

$$- \int_0^b \mathcal{R}(b-s) \int_0^s Q(s-\tau)G(\tau, x_n(h_1(\tau)))d\tau ds - \int_0^b \mathcal{R}(b-s)f_n(s)ds\Big](\eta)d\eta\Big) - \Big(\varphi_*(t)$$

$$-\mathcal{R}(t)[x_0 - G(t, x_*(h_1(0)))] - G(t, x_*(h_1(t))) - \int_0^t \mathcal{R}(t-s)AG(s, x_*(h_1(s)))ds$$

$$- \int_0^t \mathcal{R}(t-s) \int_0^s Q(s-\tau)G(\tau, x_*(h_1(\tau)))d\tau ds - \int_0^t \mathcal{R}(t-\eta)BB^*\mathcal{R}^*(b-t)R(a,\Gamma_0^b)$$

$$\times \Big[x_b - \mathcal{R}(b)[x_0 - G(b, x_*(h_1(0)))] - G(b, x_*(h_1(b))) - \int_0^b \mathcal{R}(b-s)AG(s, x_*(h_1(s)))ds$$

$$- \int_0^b \mathcal{R}(b-s) \int_0^s Q(s-\tau)G(\tau, x_*(h_1(\tau)))d\tau ds - \int_0^b \mathcal{R}(b-s)f_*(s)ds\Big](\eta)d\eta\Big) \Big\|_C$$

$$\to 0 \text{ as } n \to \infty.$$

Consider the operator $\widetilde{\mathcal{W}} : L^1(J, X) \to \mathcal{C}$,

$$(\widetilde{\mathcal{W}}f)(t) = \int_0^t \mathcal{R}(t-s)\Big[f(s) - BB^*\mathcal{R}^*(b-t)\Big(\int_0^b \mathcal{R}(b-t)f(\eta)d\eta\Big)(s)\Big]ds$$

We can see that the operator $\widetilde{\mathcal{W}}$ is linear and continuous. From Lemma 2.7 again, it follows that $\widetilde{\mathcal{W}} \circ S_F$ is a closed graph operator. Moreover,

$$\Big(\varphi_n(t) - \mathcal{R}(t)[x_0 - G(t, x_n(h_1(0)))] - G(t, x_n(h_1(t)))$$

$$- \int_0^t \mathcal{R}(t - s) A G(s, x_n(h_1(s))) ds$$

$$- \int_0^t \mathcal{R}(t - s) \int_0^s Q(s - \tau) G(\tau, x_n(h_1(\tau))) d\tau ds$$

$$- \int_0^t \mathcal{R}(t - \eta) B B^* \mathcal{R}^*(b - t) R(a, \Gamma_0^b)$$

$$\times \Big[x_b - \mathcal{R}(b)[x_0 - G(b, x_n(h_1(0)))] - G(b, x_n(h_1(b)))$$

$$- \int_0^b \mathcal{R}(b - s) A G(s, x_n(h_1(s))) ds$$

$$- \int_0^b \mathcal{R}(b - s) \int_0^s Q(s - \tau) G(\tau, x_n(h_1(\tau))) d\tau ds$$

$$- \int_0^b \mathcal{R}(b - s) f_n(s) ds \Big] (\eta) d\eta \Big) \in \mathfrak{W}(S_{F, x_n}).$$

In view of $x_n \to x_*$ as $n \to \infty$, it follows again from Lemma 2.7 that

$$\Big(\varphi_*(t) - \mathcal{R}(t)[x_0 - G(t, x_*(h_1(0)))] - G(t, x_*(h_1(t)))$$

$$- \int_0^t \mathcal{R}(t - s) A G(s, x_*(h_1(s))) ds$$

$$- \int_0^t \mathcal{R}(t - s) \int_0^s Q(s - \tau) G(\tau, x_*(h_1(\tau))) d\tau ds$$

$$- \int_0^t \mathcal{R}(t - \eta) B B^* \mathcal{R}^*(b - t) R(a, \Gamma_0^b)$$

$$\times \Big[x_b - \mathcal{R}(b)[x_0 - G(b, x_*(h_1(0)))] - G(b, x_*(h_1(b)))$$

$$- \int_0^b \mathcal{R}(b - s) A G(s, x_*(h_1(s))) ds$$

$$- \int_0^b \mathcal{R}(b - s) \int_0^s Q(s - \tau) G(\tau, x_*(h_1(\tau))) d\tau ds$$

$$- \int_0^b \mathcal{R}(b - s) f_*(s) ds \Big] (\eta) d\eta \Big) \in \mathscr{W}(S_{F, x_*}).$$

Therefore Υ has a closed graph.

As a consequence of **Steps 1-5** together with the Arzela-Ascoli theorem, we conclude that Υ is a compact multivalued map, u.s.c. with convex closed values.

As a consequence of Lemma 2.7, we can deduce that Υ has a fixed point x which is a mild solution of system (1.1)-(1.2). □

Definition 3.2. *The control system* (1.1) *is said to be approximately controllable on J if $\overline{R(b, x_0)} = X$, where $R(b, x_0) = \{x_b(x_0; u) : u(\cdot) \in L^1(J, U)\}$ is called the reachable set of system* (1.1) *at terminal time b and its closure in X is denoted by $\overline{R(b, x_0)}$; Let $x_b(x_0, u)$ be the state value of* (1.1) *at terminal time b corresponding to the control u and the initial value $x_0 \in X$.*

Frankly speaking, by using the control function u, from any given initial point $x_0 \in X$ we can move the system with the trajectory as close as possible to any other final point $x_b \in X$.

Theorem 3.3. *Suppose that the assumptions $(\mathbf{H_0})$-$(\mathbf{H_7})$ hold. Assume additionally that $(\mathbf{H_a})$ $G : J \times X_\alpha \to X_{\alpha+\beta}$ and $G(t, \cdot)$ is continuous from the weak topology of X_α to the strong topology of X_α and $(\mathbf{H_b})$ there exists $N \in L^1(J, [0, \infty))$ such that $\sup_{x \in X_\alpha} \|F(t, x)\| + \sup_{y \in X_{\alpha+\beta}} \|G(t, y)\| \leq N(t)$ for a.e. $t \in J$, then the system* (1.1)-(1.2) *is approximately controllable on J.*

Proof. Let $\widehat{x}^a(\cdot)$ be a fixed point of Γ in $\mathcal{B}_r$. By Theorem 3.1, any fixed point of Γ is a mild solution of (1.1)-(1.2) under the control

$$\widehat{u}^a(t) = B^* S^*(b, t) R(a, \Gamma_0^b) p(\widehat{x}^a)$$

and satisfies the following inequality

$$\widehat{x}^a(b) = x_b + aR(a, \Gamma_0^b)\Big[x_b - \mathcal{R}(b)[x_0 - G(b, x(h_1(0)))] - G(b, x(h_1(b)))$$

$$- \int_0^b \mathcal{R}(b-s)AG(s, x(h_1(s)))ds - \int_0^b \mathcal{R}(b-s)\int_0^s Q(s-\tau)G(\tau, x(h_1(\tau)))d\tau ds$$

$$- \int_0^b \mathcal{R}(b-s)f(s)ds\Big]. \tag{3.4}$$

Also, $\widehat{x}^\alpha(b) \to \widetilde{x}$ weakly as $\alpha \to 0^+$ and by the assumption (H_a)

$$G(b, \widehat{x}^a(b)) \to G(b, \widetilde{x}),$$

strongly as $a \to 0^+$. Moreover, because of assumption $(\mathbf{H_b})$,

$$\int_0^b \|f(s, \widehat{x}_s^a)\|^2 ds + \int_0^b \|G(s, \widehat{x}_s^a)\|^2 ds \leq \int_0^b N(s)ds.$$

Consequently, the sequences $f(\cdot, x^a)$, $G(\cdot, x^a)$ are bounded. Then there is a subsequence still denoted by $f(\cdot, x^a), G(\cdot, x^a)$ which weakly converges to, say $f(\cdot), g(\cdot)$ in

$L^2(J, X)$. Define

$$
\begin{aligned}
w = x_b &- \mathcal{R}(b)[x_0 - G(b, x(h_1(0)))] - G(b, \widetilde{x}) - \int_0^b \mathcal{R}(b-s)Ag(s)ds - \int_0^b \mathcal{R}(b-s) \\
&\times \int_0^s Q(s-\tau)g(s)d\tau ds - \int_0^b \mathcal{R}(b-s)f(s)ds
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
\|p(\widehat{x}^a) - w\| = &\|G(b, \widehat{x}^a(b)) - G(b, \widetilde{x})\| + \left\| \int_0^b \mathcal{R}(b-s)A[G(s, \widehat{x}_s^a) - g(s)]ds \right\| \\
&+ \left\| \int_0^b \mathcal{R}(b-s) \int_0^s Q(s-\tau)[G(s, \widehat{x}_s^a) - g(s)]d\tau ds \right\| \\
&+ \left\| \int_0^b \mathcal{R}(b-s)[F(s, \widehat{x}^a(s)) - f(s)]ds \right\| \\
\leq &\sup_{0 \leq t \leq b} \|G(t, \widehat{x}^a(t)) - G(t, \widetilde{x})\| + \sup_{0 \leq t \leq b} \left\| \int_0^t \mathcal{R}(t-s)A[G(s, \widehat{x}_s^a) - g(s)]ds \right\| \\
&+ \sup_{0 \leq t \leq b} \left\| \int_0^t \mathcal{R}(t-s) \int_0^s Q(s-\tau)[G(s, \widehat{x}_s^a) - g(s)]d\tau ds \right\| \\
&+ \sup_{0 \leq t \leq b} \left\| \int_0^t \mathcal{R}(t-s)[F(s, \widehat{x}^a(s)) - f(s)]ds \right\|. \qquad (3.5)
\end{aligned}
$$

By using infinite-dimensional version of the Ascoli-Arzela theorem, one can show that an operator $l(\cdot) \to \int_0^{\cdot} S(\cdot, s)l(s)ds : L^1(J, X) \to \mathcal{C}$ is compact. Therefore, we obtain that $\|p(\widehat{x}^a) - w\| \to 0$ as $a \to 0^+$. Moreover, from (3.4) we get

$$
\begin{aligned}
\|\widehat{x}^a(b) - x_b\| \leq &\|aR(a, \Gamma_0^b)(w)\| + \|aR(a, \Gamma_0^b)\| \|p(\widehat{x}^a) - w\| \\
\leq &\|aR(a, \Gamma_0^b)(w)\| + \|p(\widehat{x}^a) - w\|.
\end{aligned}
$$

It follows from assumption $\mathbf{H_0}$ and the estimation (3.5) that $\|\widehat{x}^a(b) - x_b\| \to 0$ as $a \to 0^+$. This proves the approximate controllability of system (1.1)-(1.2). $\square$

4. APPROXIMATE CONTROLLABILITY RESULTS WITH NONLOCAL CONDITIONS

Since the differential equations with nonlocal conditions have better applications than the initial conditions in fields like Physics and Engineering, this type of equations have been widely studied by the various authors. First it was initiated by Byszewski in [6, 7, 8] and then the authors in [2, 14, 16, 27, 30] extended the concepts of nonlocal conditions with different kinds of problems.

Inspired by the above works, in this section, we discuss the approximate controllability for a class of neutral integrodifferential inclusions with nonlocal conditions

in Banach spaces of the form

$$\frac{d}{dt}[x(t) - G(t, x(h_1(t)))] + Ax(t) \in \int_0^t Q(t-s)x(s)ds + F(t, x(h_2(t))) + Bu(t),$$

$$(4.1)$$

$$x(0) = x_0 + g(x), \quad t \in J = [0, b],$$
$$(4.2)$$

where $g : \mathcal{C} \to X_\alpha$ is a continuous function which satisfies the following condition:

($\mathbf{H_9}$) There exists a constant $L > 0$ such that for any $x \in \mathcal{C}([0, b], X_\alpha)$,

$$\|g(x)\|_\alpha \leq L_1 \|x\|_{\mathcal{C}}, \text{ for } x \in \mathcal{C}$$

Definition 4.1. *A function $x \in \mathcal{C}$ is said to be a mild solution of system (4.1)-(4.2) if $x(0) + g(x) = x_0$ and there exists $f \in L^1(J, X)$ such that $f(t) \in F(t, x(t))$ on $t \in J$ and the integral equation*

$$x(t) = \mathcal{R}(t)[x_0 - G(t, x(h_1(0))) + g(x)] + G(t, x(h_1(t)))$$

$$+ \int_0^t \mathcal{R}(t-s)AG(s, x(h_1(s)))ds$$

$$+ \int_0^t \mathcal{R}(t-s) \int_0^s Q(s-\tau)G(\tau, x(h_1(\tau)))d\tau ds + \int_0^t \mathcal{R}(t-s)f(s)ds$$

$$+ \int_0^t \mathcal{R}(t-s)Bu(s)ds,$$

is satisfied.

Theorem 4.2. *Assume that the assumptions of Theorem 3.1 are satisfied. Further, if the hypothesis ($\mathbf{H_9}$) is satisfied, then the system (4.1)-(4.2) is approximately controllable on J provided that*

$$\left(1 + \frac{1}{a}N^2 M_B^2 b\right)\left[NM^*L_g + M^*L_g + \frac{b^\beta C_{1-\beta}}{\beta}L_g + \frac{b^{2-\alpha}C_\alpha}{1-\alpha}M_1 L_g + N\left(\gamma + L_1\right)\right] < 1,$$

where $M_B = \|B\|$.

Proof. For each $a > 0$, we define the operator $\widehat{\Upsilon}_a$ on X by

$$(\widehat{\Upsilon}_a x) = z,$$

where

$$z(t) = \mathcal{R}(t)[x_0 - G(t, x(h_1(0))) + g(x)] + G(t, x(h_1(t)))$$

$$+ \int_0^t \mathcal{R}(t-s)AG(s, x(h_1(s)))ds$$

$$+ \int_0^t \mathcal{R}(t-s)\int_0^s Q(s-\tau)G(\tau, x(h_1(\tau)))d\tau ds + \int_0^t \mathcal{R}(t-s)f(s)ds$$

$$+ \int_0^t \mathcal{R}(t-s)Bu(s)ds$$

$$v(t) = B^* S^*(b,t)R(a, \Upsilon_0^b)p(x(\cdot)),$$

$$p(x(\cdot)) = x_b - \mathcal{R}(b)[x_0 - G(b, x(h_1(0))) + g(x)] - G(b, x(h_1(b)))$$

$$- \int_0^b \mathcal{R}(b-s)AG(s, x(h_1(s)))ds$$

$$- \int_0^b \mathcal{R}(t-s)\int_0^s Q(s-\tau)G(\tau, x(h_1(\tau)))d\tau ds$$

$$- \int_0^b \mathcal{R}(b-s)f(s)ds$$

where $f \in S_{F,x}$.

It can be easily proved that if for all $a > 0$, the operator $\widehat{\Upsilon}_a$ has a fixed point by implementing the technique used in Theorem 3.1. Then, we can show that the second order control system (4.1)-(4.2) is approximately controllable. The proof of this theorem is similar to that of Theorem 3.1 and Theorem 3.3, and hence it is omitted. □

5. APPLICATION

Consider the partial functional integrodifferential equation with control

$$\frac{\partial}{\partial t}\Big[z(t,x) + \int_0^\pi \widetilde{a}(y,x)\Big(z(t\sin t, y) + \sin\Big(\frac{\partial}{\partial y}z(t,y)\Big)\Big)dy\Big] \in \frac{\partial^2}{\partial x^2}z(t,x)$$

$$+ \int_0^t \widetilde{q}(t-s)\frac{\partial}{\partial x^2}z(s,x)ds + \mu(t,x) + \widetilde{c}\Big(t, z\Big(t\cos t, x, \frac{\partial}{\partial x}z(t,x)\Big)\Big),$$
(5.1)

for $0 \le x \le \pi, \quad 0 \le t \le b$, subject to the initial conditions

$$z(t,0) = z(t,\pi) = 0, \quad t \in J, \tag{5.2}$$

$$z(0,x) = z_0(x), \quad 0 \le x \le \pi, \tag{5.3}$$

where $\widetilde{a} : [0,1] \times [0,\pi] \times [0,\pi] \to \mathbb{R}$, $\widetilde{q}(\cdot)$ is a continuous function such that $\|\widetilde{q}(\cdot)\| \le M^*$. Here $\widetilde{c} : [0,1] \times \mathbb{R} \to BCC(\mathbb{R})$ is a continuous function. Now we define the

space $X = L^2([0, \pi])$ and $z_0(x) \in X$. To rewrite the above equation in the abstract form, we define the operator A by

$$Az = -z''$$

with the domain

$$D(A) = \{z(\cdot) \in X : z', z'' \in X, and z(0) = z(\pi) = 0\}.$$

Then $-A$ generates a strongly continuous semigroup $(T(t))_{t>0}$ which is compact, analytic and self-adjoint. Also, A has a discrete spectrum, the eigenvalues are $n^2, n \in \mathbb{N}$, with the corresponding normalized eigenvectors $e_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$, $n = 1, 2, \cdots$. And,

(i) If $z \in D(A)$, then $Az = \sum_{n=1}^{\infty} n^2 \langle z, e_n \rangle e_n$.
(ii) For each $z \in X$, $A^{-1/2}z = \sum_{n=1}^{\infty} \frac{1}{n} \langle z, e_n \rangle e_n$. In particular, $\|A^{-1/2}\| = 1$.
(iii) The operator $A^{1/2}$ is given by

$$A^{1/2}z = \sum_{n=1}^{\infty} n \langle z, e_n \rangle e_n$$

on the space

$$D(A^{1/2}) = \left\{ z(\cdot) \in X, \sum_{n=1}^{\infty} n \langle z, e_n \rangle e_n \in X \right\}.$$

We take $\alpha = \beta = \frac{1}{2}$, $Q(t) = q(t)A$, and put $z(t) = z(t, \cdot)$, that is $z(t)(\tau) = z(t, \tau)$, $t \in J$, $x \in [0, \pi]$ and $u(t) = \mu(t, \cdot)$, here $\mu : I \times [0, \pi] \to [0, \pi]$ is continuous. Define the functions $G : [0, b] \times X_{\frac{1}{2}} \to D(A)$, $F : [0, b] \times X_{\frac{1}{2}} \to 2^{X_{\frac{1}{2}}}$ respectively by

$$G(t, z)(x) = \tilde{c}\left(t, z(t, x), \frac{\partial}{\partial x} z(t, x)\right),$$
$$F(t, z)(x) = \int_0^{\pi} \tilde{a}(y, x)[z(t, y) + \sin(z'(t, y))]dy,$$

and the bounded linear operator $B : U \to X$ by

$$Bu(t)(\tau) = \mu(t, \tau).$$

Assume these functions satisfy the requirement of hypotheses. From the above choices of the functions and evolution operator $A(t)$ with $B = J$, the system (5.1)-(5.2) can be formulated as the system (1.1)-(1.2) in X. Since all hypotheses of Theorem 3.3 are satisfied, approximate controllability of system (5.1)-(5.2) on J follows from Theorem 3.3.

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