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A NEW NUMERICAL METHOD FOR SOLVING DELAY INTEGRAL EQUATIONS WITH VARIABLE BOUNDS BY USING GENERALIZED MOTT POLYNOMIALS

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ABSTRACT

In this study, the delay integral equations with variable bounds are considered and their approximate solutions are obtained by using a new numerical method based on matrices, collocation points and the generalized Mott polynomials including a parameter- β . An error analysis technique consisting of the residual function is performed. The numerical examples are applied to illustrate the practicability and usability of the method. The behavior of the solutions is monitored in terms of the parameter- β . The accuracy of the method is scrutinized for different values of N and also the numerical results are discussed in figures and tables.

Keywords: Mott polynomials, Matrix method, Collocation points, Error analysis

1. INTRODUCTION

In recent years, integral equations have been encountered in variety applied sciences, such as mathematics, engineering, thermodynamic, molecular properties, electromagnetics, Stokes flow, heat and mass transfer, and micromechanics [1-5,16-21]. Most of the time, integral equations and their different types are too hard to be solved analytically. Therefore, the numerical methods are required. For this aim, Sezer [6] has obtained the Taylor polynomial solution of Volterra integral equations. Kürkçü et al. [7-9] have applied a numerical method based on Dickson polynomials to integro-differential-(difference) equations and to model problems. Baykuş and Sezer [10] have established a practical Taylor matrix method for obtaining the solutions of Fredholm integro-differential equations with piecewise intervals. The other polynomial methods can be found in [11-15]. Also, in order to solve the Volterra integral equations for the second kind, Adomian's decomposition method has been employed by Babolian and Davary [16]. Variational iteration method, collocation method, Walsh functions method and Coiflet-Galerkin method have been introduced to solve the integral equations and their some types [17-21].

The generalized Mott polynomials [22-26] include a real parameter- β . Thereby, we can use β to monitor the behavior of the approximate solutions of different types of Eq. (1). By considering β in the method, our aim in this study is to employ a new matrix-collocation method based on the generalized Mott polynomials to solve the delay integral equations with variable bounds represented by (see [12])

$$\sum_{k=0}^{m_1} P_k(x)\, y(\delta_k x+\tau_k) = g(x) + \sum_{r=0}^{m_2} \lambda_r \int_{u_r(x)}^{v_r(x)} K_r(x,t)\, y(\mu_r t+\gamma_r)\,dt\,,\ a \le x,t \le b\,, \tag{1}$$

or briefly

$$D(x) = g(x) + I(x),$$

where

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$$D(x)=\sum_{k=0}^{m_1} P_k(x)\,y(\delta_k x+\tau_k) \text{ and } I(x)=\sum_{r=0}^{m_2}\lambda_r \int_{u_r(x)}^{v_r(x)} K_r(x,t)\,y(\mu_r t+\gamma_r)\,dt,$$

and also the functions $y(x)$, $P_k(x)$, $g(x)$, $v_r(x)$ and $u_r(x)$ are defined on $[a,b]$; $a \le u_r(x) < v_r(x) \le b$, $K_r(x,t)$ is also an analytic function on $[a,b]\times[a,b]$. δ_k, τ_k, λ_r, μ_r and γ_r are suitable constants. We seek the generalized Mott polynomial solution of Eq. (1) in the form

$$y(x) \cong y_N(x)=\sum_{n=0}^{N} y_n M_n(x,\beta), \tag{2}$$

where y_n $(n=0,1,\ldots,N;\ N\in N)$ are the Mott coefficients to be obtained, and $M_n(x,\beta)$ is the generalized Mott polynomials [22-26]. We also let the standard collocation points that used in the matrices as follows:

$$x_i=a+\left(\frac{b-a}{N}\right)i,\ i=0,1,\ldots,N, \tag{3}$$

such that $a=x_0<x_1<\ldots<x_N=b$.

2. BASIC PROPERTIES OF MOTT POLYNOMIALS

In this section, we mention about some basic properties of the Mott Polynomials, which are denoted by $M_n(x)$ in this study. In 1932, Mott [22] introduced these polynomials while observing the behaviors of electrons for a problem in the theory of electrons. Then, Erdélyi et al. [23, p. 251] constructed an explicit formula of the polynomials as follows:

$$\begin{aligned} M_n(x) &= \left(-\frac{x}{2}\right)^n (n-1)! \sum_{l=0}^{\lfloor n/2 \rfloor} \frac{x^{-2l}}{l!(n-l)!(n-2l-1)!} \\ &= (n!)^{-1}\left(-\frac{x}{2}\right)^n {}_3F_0\left(-n,\frac{1}{2}-\frac{n}{2},1-\frac{n}{2};-4x^{-2}\right), \end{aligned}$$

where ${}_3F_0$ is the generalized hypergeometric function.

In 1984, Roman [24] obtained the following associated Sheffer sequence and generating function respectively as

$$f(t)=\frac{-2t}{1-t^2} \text{ and } \sum_{k=0}^{\infty}\frac{M_k(x)}{k!}t^k=\exp\left(\frac{x\sqrt{1-t^2}-x}{t}\right).$$

Thus the first few Mott polynomials are

$$M_0(x)=1,\ M_1(x)=-\frac{x}{2},\ M_2(x)=\frac{x^2}{4},\ M_3(x)=-\frac{3x}{4}-\frac{x^3}{8} \text{ and } M_4(x)=\frac{x^2}{2}+\frac{x^4}{16}.$$

In addition, a triangle coefficient matrix of the Mott polynomials can be found in the sequence A137378 of OEIS [25]. In 2014, Kruchinin [26] turned the Mott polynomials to a generalized form including a parameter-β:

$$M_n(x,\beta)=\sum_{p=1}^{n}\sum_{q=0}^{p}(-1)^{p-q+(n+p)/2}\frac{n!\left(1+(-1)^{n+p}\right)}{2p!}\binom{p}{q}\binom{\beta q}{(n+p)/2}x^p,\ n>0,$$

where the Mott polynomials are obtained for $\beta = 0.5$.

On the other hand, one can refer to [26] for more details of the generalized Mott polynomials.

3. DESCRIPTION OF METHOD

3.1. Fundamental Matrix Relations

Now, we can write the solution (2) of Eq. (1) in the matrix form

$$y(x) \cong y_N(x) = \boldsymbol{M}(x,\beta)\boldsymbol{Y},\ \boldsymbol{M}(x,\beta) = \boldsymbol{X}(x)\boldsymbol{K}(\beta),$$

then

$$y(x) = \boldsymbol{X}(x)\boldsymbol{K}(\beta)\boldsymbol{Y}, \tag{4}$$

where

$$\boldsymbol{M}(x,\beta) = \left[M_0(x,\beta) \quad M_1(x,\beta) \quad \ldots \quad M_N(x,\beta)\right],\ \boldsymbol{X}(x) = \left[1 \quad x \quad x^2 \quad \cdots \quad x^N\right],$$
$$\boldsymbol{Y} = \left[y_0 \quad y_1 \quad \cdots \quad y_N\right]^T,$$

and by taking the coefficients of $\boldsymbol{M}(x,\beta)$, $\boldsymbol{K}(\beta)$ is constituted for N=5 as follows:

$$\underbrace{\begin{bmatrix} M_0(x,\beta) \\ M_1(x,\beta) \\ M_2(x,\beta) \\ M_3(x,\beta) \\ M_4(x,\beta) \\ M_5(x,\beta) \end{bmatrix}}_{\boldsymbol{M}^T(x,\beta)} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta^2 & 0 & 0 & 0 \\ 0 & 3\beta^2 - 3\beta & 0 & -\beta^3 & 0 & 0 \\ 0 & 0 & -12\beta^3 + 12\beta^2 & 0 & \beta^4 & 0 \\ 0 & -20\beta^3 + 60\beta^2 - 40\beta & 0 & 30\beta^4 - 30\beta^3 & 0 & -\beta^5 \end{bmatrix}}_{\boldsymbol{K}^T(\beta)} . \underbrace{\begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{bmatrix}}_{\boldsymbol{X}^T(x)} .$$

If $x \to \delta_k x + \tau_k$ is inserted into Eq. (4), then

$$y(\delta_k x + \tau_k) = \boldsymbol{X}(\delta_k x + \tau_k)\boldsymbol{K}(\beta)\boldsymbol{Y}. \tag{5}$$

On the other hand, the matrix relation between $\boldsymbol{X}(\delta_k x + \tau_k)$ and $\boldsymbol{X}(x)$ becomes [11]

$$\boldsymbol{X}(\delta_k x + \tau_k) = \left[1 \quad (\delta_k x + \tau_k) \quad \ldots \quad (\delta_k x + \tau_k)^N\right] = \boldsymbol{X}(x)\boldsymbol{S}(\delta_k, \tau_k), \tag{6}$$

where ($\delta_k = 0$, $\tau_k = 0 \Rightarrow \delta_k^0 = 1, \tau_k^0 = 1$)

$$\boldsymbol{S}(\delta_k,\tau_k)=\begin{bmatrix} \binom{0}{0}\delta_k^0\tau_k^0 & \binom{1}{0}\delta_k^0\tau_k^1 & \binom{2}{0}\delta_k^0\tau_k^2 & \cdots & \binom{N}{0}\delta_k^0\tau_k^N \\ 0 & \binom{1}{1}\delta_k^1\tau_k^0 & \binom{2}{1}\delta_k^1\tau_k^1 & \cdots & \binom{N}{1}\delta_k^1\tau_k^{N-1} \\ 0 & 0 & \binom{2}{2}\delta_k^2\tau_k^0 & \cdots & \binom{N}{2}\delta_k^2\tau_k^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \binom{N}{N}\delta_k^N\tau_k^0 \end{bmatrix} \text{ for } \delta_k\neq 0 \text{ and } \tau_k\neq 0 .$$

It follows from Eqs. (5) and (6) that

$$y(\delta_k x+\tau_k)=\boldsymbol{X}(x)\boldsymbol{S}(\delta_k,\tau_k)\boldsymbol{K}(\beta)\boldsymbol{Y}\,, \tag{7}$$

and

$$y(\mu_r t+\gamma_r)=\boldsymbol{X}(t)\boldsymbol{S}(\mu_r,\gamma_r)\boldsymbol{K}(\beta)\boldsymbol{Y}\,. \tag{8}$$

By (3) and (7), we obtain the matrix relation of $D(x)$ at the collocation points as follows:

$$\boldsymbol{D}=\sum_{k=0}^{m_1}\boldsymbol{P}_k\boldsymbol{X}\boldsymbol{S}(\delta_k,\tau_k)\boldsymbol{K}(\beta)\boldsymbol{Y}\,, \tag{9}$$

where

$$\boldsymbol{D}=\begin{bmatrix} D(x_0)\\ D(x_1)\\ \vdots \\ D(x_N)\end{bmatrix},\ \boldsymbol{P}_k=\begin{bmatrix} P_k(x_0) & 0 & \cdots & 0\\ 0 & P_k(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & P_k(x_N)\end{bmatrix},\ \boldsymbol{X}=\begin{bmatrix} \boldsymbol{X}(x_0)\\ \boldsymbol{X}(x_1)\\ \vdots\\ \boldsymbol{X}(x_N)\end{bmatrix}=\begin{bmatrix} 1 & x_0 & \cdots & x_0^N\\ 1 & x_1 & \cdots & x_1^N\\ \vdots & \vdots & \ddots & \vdots\\ 1 & x_N & \cdots & x_N^N\end{bmatrix}.$$

Let us now construct the matrix relation of the integral part *I*(*x*). Since $K_r(x,t)$ is an analytic function on $[a,b]\times[a,b]$, the matrix form of $K_r(x,t)$ can be formed by using its Taylor expansion as the following [6]:

$$K_r(x,t)=\sum_{p=0}^{N}\sum_{q=0}^{N}k_{pq}^r x^p t^q \Rightarrow K_r(x,t)=\boldsymbol{X}(x)\boldsymbol{K}_r\boldsymbol{X}^T(t),\ a\le x,t\le b\,, \tag{10}$$

where

$$\boldsymbol{K}_r=\left[k_{pq}^r\right];\ k_{pq}^r=\frac{1}{p!q!}\frac{\partial^{p+q}K_r(0,0)}{\partial x^p\partial t^q},\ r=0,1,\ldots,m_2 \text{ and } p,q=0,1,\ldots,N\,.$$

If we substitute the matrix relations (8) and (10) into *I*(*x*), then we have

$$\left.\begin{aligned} \boldsymbol{I}(x)&=\sum_{r=0}^{m_2}\lambda_r\int_{u_r(x)}^{v_r(x)}\boldsymbol{X}(x)\boldsymbol{K}_r\boldsymbol{X}^T(t)\boldsymbol{X}(t)\boldsymbol{S}(\mu_r,\gamma_r)\boldsymbol{K}(\beta)\boldsymbol{Y}dt\\ &=\sum_{r=0}^{m_2}\lambda_r\boldsymbol{X}(x)\boldsymbol{K}_r\boldsymbol{Q}_r(x)\boldsymbol{S}(\mu_r,\gamma_r)\boldsymbol{K}(\beta)\boldsymbol{Y}\end{aligned}\right\} \tag{11}$$

where

$$\boldsymbol{Q}_r(x)=\int_{u_r(x)}^{v_r(x)}\boldsymbol{X}^T(t)\boldsymbol{X}(t)dt=\left[q_{mn}^r(x)\right];\ q_{mn}^r(x)=\frac{\left(v_r(x)\right)^{m+n+1}-\left(u_r(x)\right)^{m+n+1}}{m+n+1},\ m,n=0,1,\ldots,N\ .$$

By (3) and the matrix relation (11), we have the matrix form of *I*(*x*) on the collocation points

$$\boldsymbol{I}=\sum_{r=0}^{m_2}\lambda_r\left(\overline{\boldsymbol{X}}\right)\left(\overline{\boldsymbol{K}_r}\right)\left(\overline{\boldsymbol{Q}_r}\right)\overline{\boldsymbol{S}}\left(\mu_r,\gamma_r\right)\boldsymbol{K}(\beta)\boldsymbol{Y}\ , \tag{12}$$

where

$$\overline{\boldsymbol{X}}=\begin{bmatrix}\boldsymbol{X}(x_0) & 0 & \cdots & 0\\ 0 & \boldsymbol{X}(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \boldsymbol{X}(x_N)\end{bmatrix}_{(N+1)\times(N+1)^2},\ \overline{\boldsymbol{K}_r}=\begin{bmatrix}\boldsymbol{K}_r & 0 & \cdots & 0\\ 0 & \boldsymbol{K}_r & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \boldsymbol{K}_r\end{bmatrix}_{(N+1)^2\times(N+1)^2},$$

$$\overline{\boldsymbol{Q}_r}=\begin{bmatrix}\boldsymbol{Q}_r(x_0) & 0 & \ldots & 0\\ 0 & \boldsymbol{Q}_r(x_1) & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \boldsymbol{Q}_r(x_N)\end{bmatrix}_{(N+1)^2\times(N+1)^2}\ \text{and}\ \overline{\boldsymbol{S}}\left(\mu_r,\gamma_r\right)=\begin{bmatrix}\boldsymbol{S}(\mu_r,\gamma_r)\\ \boldsymbol{S}(\mu_r,\gamma_r)\\ \vdots\\ \boldsymbol{S}(\mu_r,\gamma_r)\end{bmatrix}_{(N+1)^2\times(N+1)}.$$

3.2. Method of Solution

We can now construct the matrix form of Eq. (1). By substituting the matrix relations (9) and (12) into Eq. (1), we have

$$\underbrace{\left\{\sum_{k=0}^{m_1}\boldsymbol{P}_k\boldsymbol{X}\boldsymbol{S}\left(\delta_k,\tau_k\right)-\sum_{r=0}^{m_2}\lambda_r\overline{\boldsymbol{X}}\ \overline{\boldsymbol{K}_r}\ \overline{\boldsymbol{Q}_r}\ \overline{\boldsymbol{S}}\left(\mu_r,\gamma_r\right)\right\}\boldsymbol{K}(\beta)}_{\boldsymbol{W}}\boldsymbol{Y}=\boldsymbol{G}\ , \tag{13}$$

where

$$\boldsymbol{G}=\left[g(x_0)\quad g(x_1)\quad\cdots\quad g(x_N)\right]^T.$$

For brevity, we can write the form (13) as

$$\boldsymbol{W}\boldsymbol{Y}=\boldsymbol{G}\ \text{or}\ \left[\boldsymbol{W}\ ;\ \boldsymbol{G}\right], \tag{14}$$

where

$$\left[\boldsymbol{W}\ ;\ \boldsymbol{G}\right]=\begin{bmatrix}w_{00} & w_{01} & \cdots & w_{0N} & ; & g(x_0)\\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g(x_1)\\ \vdots & \vdots & \vdots & \vdots & ; & \vdots\\ w_{N0} & w_{N1} & \cdots & w_{NN} & ; & g(x_N)\end{bmatrix}.$$

In the form (14), if $\operatorname{rank}\boldsymbol{W}=\operatorname{rank}\left[\boldsymbol{W}\ ;\ \boldsymbol{G}\right]=N+1$, then we can write $\boldsymbol{Y}=\left(\boldsymbol{W}\right)^{-1}\boldsymbol{G}$. Therefore, the unknown coefficients in the matrix $\boldsymbol{Y}$ are uniquely determined. Thereby, we get the generalized Mott polynomial solution in the form (2).

4. RESIDUAL ERROR ANALYSIS

The residual correction was used in [13]. In this section, an error analysis based on the residual function is constituted to improve the Mott polynomial solution $y_N(x)$ of Eq. (1). The residual function is obtained by substituting $y_N(x)$ into Eq. (1) under a linear operator L as follows:

$$R_N(x) = L[y_N(x)] - g(x), \tag{15}$$

where

$$L[y_N(x)] = D(x) - I(x).$$

Also, an error function $e_N(x)$ is defined to be

$$e_N(x) = y(x) - y_N(x), \tag{16}$$

where $y(x)$ is the exact solution of Eq. (1). By Eqs. (15) and (16), we obtain the error equation

$$L[e_N(x)] = L[y(x)] - L[y_N(x)] = -R_N(x). \tag{17}$$

The error equation (17) can be solved by using the same procedure described in Section 3.2. Thereby, we get

$$e_{N,M}(x) = \sum_{n=0}^{M} y_n^* M_n(x,\beta),\ (M > N),$$

where $e_{N,M}(x)$ is the Mott polynomial solution of the error equation (17). Thus, the corrected Mott polynomial solution and the corrected error function are $y_{N,M}(x) = y_N(x) + e_{N,M}(x)$ and $E_{N,M}(x) = y(x) - y_{N,M}(x),$ respectively.

5. NUMERICAL EXAMPLES

In this section, we consider some stiff examples. To solve these examples, a general computer program has been designed on Mathematica. Thus, the numerical results can be sensitively monitored in terms of different values of β along with computation limit N. The corrected absolute and estimated absolute errors are calculated respectively as

$$|E_{N,M}(x)| = |y(x) - y_{N,M}(x)| \text{ and } |e_{N,M}(x)| = |y_{N,M}(x) - y_N(x)|.$$

In numerical results, we also use L_∞ error norm, which is defined as

$$L_\infty = \underset{x\in[a,b]}{\text{maks}} \{|y(x) - y_N(x)|\}.$$

Thus, as N is increased, we can prescribe

$$\|y(x) - y_N(x)\|_\infty < \varepsilon,$$

which means that ε determines the precision of the method.

Example 5.1 Consider the functional delay integral equation with variable bounds

$$(x-1)y(x+1)-2xy(2x-1)=g(x)+2\int_{x}^{x^2+1}(x^2t+xt^2)y(2t)dt,\ 0\le x\le 1,$$

where

$$g(x)=-2-94x/15+6x^2-17x^3-25x^4/3-18x^5-47x^6/5-50x^7/3-8x^8$$
$$-8x^9-2x^{10}-8x^{11}/5.$$

In this equation, $P_0(x)=x-1$, $\{\delta_0=\beta_0=1\}$, $P_1(x)=-2x$, $\{\delta_1=2,\beta_1=-1\}$, $\lambda_0=2$, $K_0(x,t)=x^2t+xt^2$, $\{\mu_0=2,\gamma_0=0\}$, $v_r(x)=x^2+1$ and $u_r(x)=x$. The exact solution is $y(x)=x^2+1$. We deal with the equation for N=2 and so the Mott polynomial solution is of the form

$$y_N(x)=\sum_{n=0}^{2}y_nM_n(x,\beta).$$

Also, the collocation points (3) are obtained as

$$\{x_0=0,x_1=1/2,x_2=1\}.$$

By considering the form (13), we construct the fundamental matrix equation:

$$\left\{\boldsymbol{P}_0\boldsymbol{XS}(\delta_0,\tau_0)+\boldsymbol{P}_1\boldsymbol{XS}(\delta_1,\tau_1)-\lambda_0\left(\overline{\boldsymbol{X}}\right)\left(\overline{\boldsymbol{K}_0}\right)\left(\overline{\boldsymbol{Q}_0}\right)\overline{\boldsymbol{S}}(\mu_0,\gamma_0)\right\}\boldsymbol{K}(\beta)\boldsymbol{Y}=\boldsymbol{G},$$

where

$$\boldsymbol{P}_0=\begin{bmatrix}-1&0&0\\0&-1/2&0\\0&0&0\end{bmatrix},\ \boldsymbol{P}_1=\begin{bmatrix}0&0&0\\0&-1&0\\0&0&-2\end{bmatrix},\ \boldsymbol{X}=\begin{bmatrix}1&0&0\\1&1/2&1/4\\1&1&1\end{bmatrix},\ \boldsymbol{S}(1,1)=\begin{bmatrix}1&1&1\\0&1&2\\0&0&1\end{bmatrix},$$

$$\boldsymbol{S}(2,-1)=\begin{bmatrix}1&-1&1\\0&2&-4\\0&0&4\end{bmatrix},\ \boldsymbol{K}(\beta)=\begin{bmatrix}1&0&0\\0&-\beta&0\\0&0&\beta^2\end{bmatrix},\ \overline{\boldsymbol{X}}=\begin{bmatrix}1&0&0&0&0&0&0&0&0\\0&0&0&1&1/2&1/4&0&0&0\\0&0&0&0&0&0&1&1&1\end{bmatrix},$$

$$\overline{\boldsymbol{K}_0}=\begin{bmatrix}0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1&0\end{bmatrix},\ \overline{\boldsymbol{S}}(2,0)=\begin{bmatrix}1&0&0\\0&2&0\\0&0&4\\1&0&0\\0&2&0\\0&0&4\\1&0&0\\0&2&0\\0&0&4\end{bmatrix},\ \boldsymbol{G}=\begin{bmatrix}-2\\-18351/2560\\-1369/15\end{bmatrix},$$

$$\overline{\boldsymbol{Q}_0}=\begin{bmatrix} 1 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/3 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/4 & 1/5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3/4 & 21/32 & 39/64 & 0 & 0 & 0 \\ 0 & 0 & 0 & 21/32 & 39/64 & 609/1024 & 0 & 0 & 0 \\ 0 & 0 & 0 & 39/64 & 609/1024 & 3093/5120 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3/2 & 7/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3/2 & 7/3 & 15/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7/3 & 15/4 & 31/5 \end{bmatrix}.$$

After making the required calculations, we get the matrix form

$$[\boldsymbol{W}\ ;\ \boldsymbol{G}]=\begin{bmatrix} -1 & \beta & -\beta^2 & ; & -2 \\ -2.4375 & 2.5488\beta & -4.7309\beta^2 & ; & -7.1684 \\ -9.6667 & 26.3333\beta & -81.6\beta^2 & ; & -91.2667 \end{bmatrix}.$$

When this form is solved the Mott coefficient matrix is obtained as follows:

$$\boldsymbol{Y}=\begin{bmatrix}1 & 0 & 1/\beta^2\end{bmatrix}^T.$$

Thus, we get the solution

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{2} y_n M_n(x,\beta) = y_0 M_0(x,\beta) + y_1 M_1(x,\beta) + y_2 M_2(x,\beta) \\ &= 1\cdot 1 + 0\cdot(-\beta x) + \frac{1}{\beta^2}\cdot(\beta^2 x^2) \\ &= x^2+1, \end{aligned}$$

which is the exact solution. Similarly, the same solution can be obtained for $N>2$.

Example 5.2 [16,18,19] Consider the Volterra integral equation

$$y(x)=1-x-\frac{x^2}{2}+\int_0^x (x-t)y(t)dt\ ,\ 0\le x\le 1.$$

The exact solution of the equation is $y(x)=1-\sinh(x)$. By considering the form (13), we give the fundamental matrix equation

$$\left\{\boldsymbol{P}_0\boldsymbol{X}\boldsymbol{S}(\delta_0,\tau_0)-\lambda_0\left(\overline{\boldsymbol{X}}\right)\left(\overline{\boldsymbol{K}_0}\right)\left(\overline{\boldsymbol{Q}_0}\right)\overline{\boldsymbol{S}}(\mu_0,\gamma_0)\right\}\boldsymbol{K}(\beta)\boldsymbol{Y}=\boldsymbol{G}.$$

When this equation is solved, we obtain the generalized Mott polynomial solutions for different N. As seen from Table 1, the absolute, estimated absolute and corrected absolute errors are far better than those obtained by the collocation and Adomian methods [16,18,19]. Figure 1 indicates that L_∞ errors dramatically decrease in logarithmic scale. The logarithmic plot of L_∞ errors are also compared for N=10 and different β in Figure 2. It is clearly observed from Figure 2 that the least error value is obtained when $\beta = 0.6$. In addition, the parameter-β can be taken as either $\beta = -1$ or $0.6 \le \beta \le 1$. However, when $\beta = 0$, we obtain the matrix system, which has no solution. For the best approximation of the generalized Mott polynomial solution to the exact solution, the parameter-β is chosen as $\beta = 0.6$. This consistence is also seen in Figure 3.

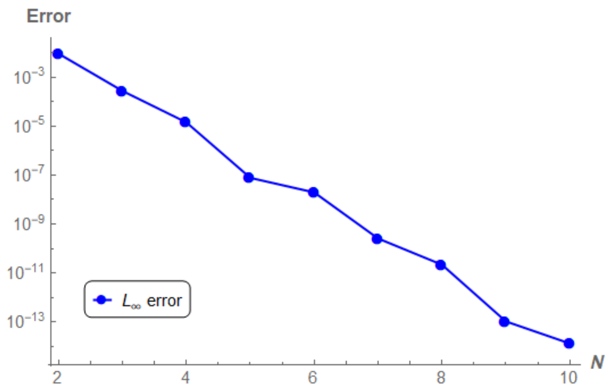

Figure 1. Logarithmic plot of L_∞ error with respect to N for Example 5.2.

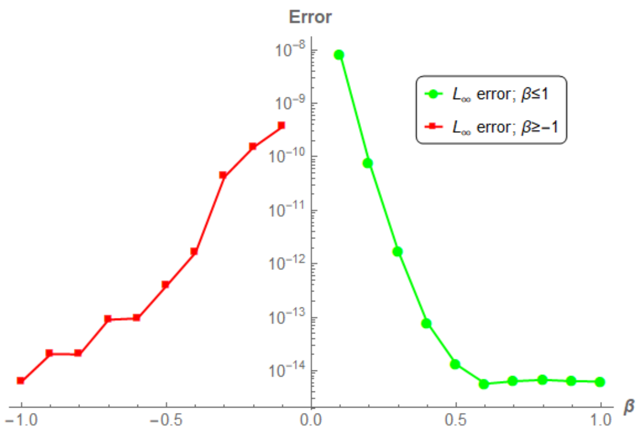

Figure 2. Logarithmic plot of L_∞ errors in terms of β for Example 5.2.

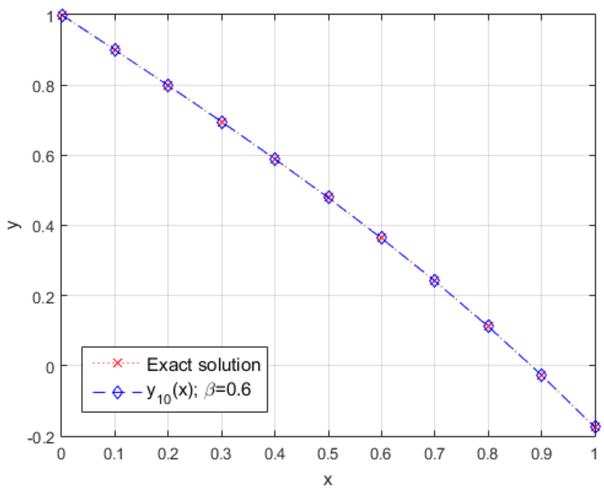

Figure 3. Comparison of the generalized Mott polynomial and exact solutions according to $\beta = 0.6$ for Example 5.2.

Table 1. Comparison of the absolute, estimated absolute and corrected absolute errors for Example 5.2.

x_i	$\lvert e_{10}(x_i)\rvert$; $\beta=1/2$	$\lvert e_{10,12}(x_i)\rvert$; $\beta=1/2$	$\lvert E_{10,12}(x_i)\rvert$; $\beta=1/2$	Collocation meth. *N*=10 [18]	Collocation meth. [19]	Adomian meth. [16]
0	0	0	0	$1.11e-16$	$1.98e-14$	0
0.2	$9.99201e-16$	$2.63348e-15$	$3.55271e-15$	$9.45e-10$	$2.35e-07$	$2.20e-05$
0.4	$2.22045e-15$	$2.18811e-15$	0	$1.03e-09$	$4.77e-07$	$8.33e-05$
0.6	$3.55271e-15$	$3.39153e-15$	$1.11022e-16$	$1.81e-09$	$7.39e-07$	$1.77e-04$
0.8	$4.87110e-15$	$6.09497e-15$	$1.17961e-15$	$3.94e-08$	$1.03e-06$	$2.97e-04$
1.0	$6.30052e-15$	$1.27857e-14$	$6.49480e-15$	$6.33e-07$	$1.36e-06$	$4.38e-04$

Example 5.3 [20, 21] Consider the Volterra integral equation with functional kernel

$$y(x)=\cos(x)-\int_0^x (x-t)\cos(x-t)\,y(t)\,dt,\ 0\le x\le 1.$$

The exact solution of the equation is $y(x)=1/3\left(2\cos\sqrt{3}x+1\right)$ [20]. Let us solve this equation by taking *N*=4, 6 and *M*=9. In Figure 4, the Mott polynomial solutions are illustrated along with the exact solution. It is obviously seen that the Mott polynomial solutions coincide well with the exact solution. Also, the absolute errors in Figure 5 are decreased thanks to both *N* and the residual error analysis. Table 2 shows that the better results for *N*=6 and $\beta=0.5$ are obtained in comparison with Coiflet-Galerkin method (*n*=6) [21].

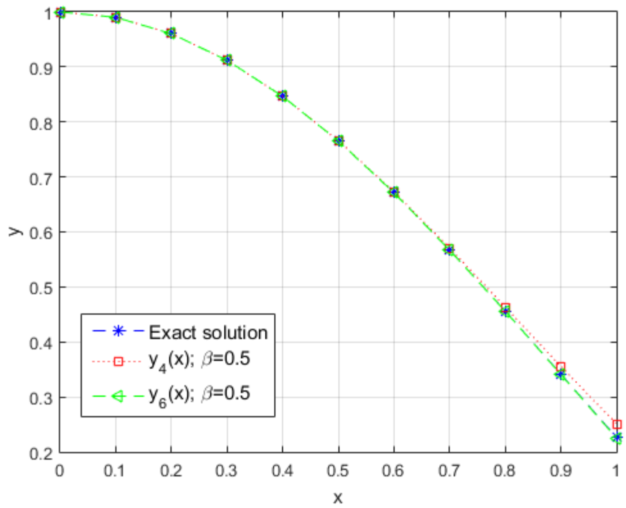

Figure 4. Comparison of the exact and Mott polynomial solutions with respect to *N* for Example 5.3.

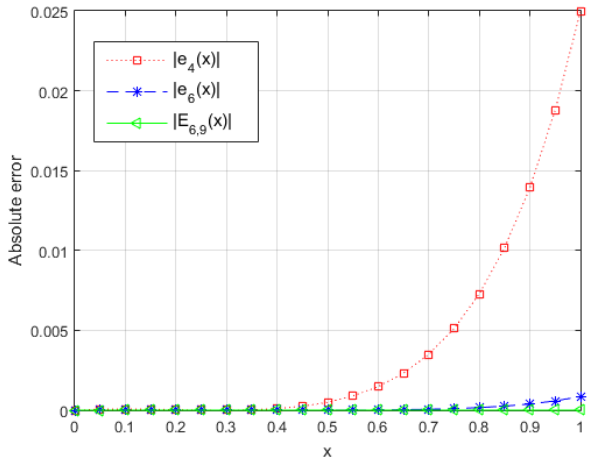

Figure 5. Comparison of the absolute errors with respect to N and $\beta = 0.5$ for Example 5.3.

Table 2. Comparison of the corrected absolute errors in terms of β for Example 5.3.

x_i	$\lvert E_{6,9}(x_i)\rvert$; $\beta = 0.1$	$\lvert E_{6,9}(x_i)\rvert$; $\beta = 0.5$	Coiflet-Galerkin meth. n=6 [21]
0	$7.90e-09$	$9.08e-14$	$5.7e-10$
0.2	$1.37e-08$	$2.43e-11$	$4.7e-08$
0.4	$1.51e-10$	$1.52e-11$	$1.0e-07$
0.6	$1.11e-09$	$1.72e-11$	$1.5e-07$
0.8	$9.97e-09$	$3.00e-11$	$1.8e-07$
1.0	$2.00e-09$	$1.17e-10$	$1.6e-07$

Example 5.4 Finally, consider the delay integral equation with variable bounds

$$x^2 y(x+1) + \sin(x) y(0.5x) = g(x) + \int_0^{x^2} (x+t)^3 y(t+1)\,dt - \int_x^{e^x} (x^2+t^2) y(0.2t-1)\,dt, \quad 0 \le x \le 1,$$

where

$$\begin{aligned} g(x) = &-5.0488x^3 + \left(135.076 + e^x\left(-42.07 - 2.7e^x\right) - 2.7x^2\right)\cos\left(0.2e^x\right) \\ &+ \left(-135.076 + x\left(42.074 + 5.403x\right)\right)\cos\left(0.2e^x\right) + \left(210.368 + e^x\left(27.015 - 4.207e^x\right)\right)\sin\left(0.2e^x\right) \\ &- 210.368\sin(0.2x) + \sin(0.5x)\sin(x) + x^5\left(-6. + x^4\right)\cos\left(1+x^2\right) - 27.015x\sin(0.2x) \\ &+ x^2\left(-4.207\sin\left(0.2e^x\right) + 8.415\sin(0.2x) + \sin(x+1) + x\left(6 - 3.x^4\right)\sin\left(1+x^2\right)\right). \end{aligned}$$

The exact solution of this stiff equation is $y(x) = \sin x$. By the form (13), we give the fundamental matrix equation

$$\left\{ \begin{array}{l} \mathbf{P}_0 \mathbf{X}\mathbf{S}(\delta_0,\tau_0) + \mathbf{P}_1 \mathbf{X}\mathbf{S}(\delta_1,\tau_1) - \lambda_0 \overline{\mathbf{X}}\ \overline{\mathbf{K}}_0\ \overline{\mathbf{Q}}_0\ \overline{\mathbf{S}}(\mu_0,\gamma_0) \\ -\lambda_1 \overline{\mathbf{X}}\ \overline{\mathbf{K}}_1\ \overline{\mathbf{Q}}_1\ \overline{\mathbf{S}}(\mu_1,\gamma_1) \end{array} \right\} \mathbf{K}(\beta)\mathbf{Y} = \mathbf{G}.$$

We apply the present method to solve this equation by taking N from 4 to 12. Table 3 shows that the accuracy of the method is increased when N is increased. The logarithmic plot of L_∞ errors obtained for N=12 are illustrated in terms of different β in Figure 6. It can be noticed from Figure 6 that there is an important role of β in the results. That is, we encounter the following situations:

- The errors increase for $\beta \in [-0.6, 0.4] / \{0\}$.
- The errors decrease and stabilize for $\beta \in (-2, -1]$ or $\beta \in [0.7, 2]$.
- We have no solution for $\beta = 0$, so the matrix system (14) cannot produce any solution.

As seen from the above, β can be chosen on $(-2, -1]$ or on $[0.7, 2]$ to get the best approximation in Example 5.4. In addition, Figure 7 shows that the generalized Mott polynomial solution $y_{12}(x)$ is in very good agreement with the exact solution on wide interval [0,10].

Table 3. Comparison of L_∞ errors in terms of N for Example 5.4.

N	4	5	6	7	8	9	10
L_∞	$4.81e-02$	$4.14e-04$	$2.64e-04$	$8.21e-05$	$6.42e-06$	$4.74e-06$	$4.21e-07$

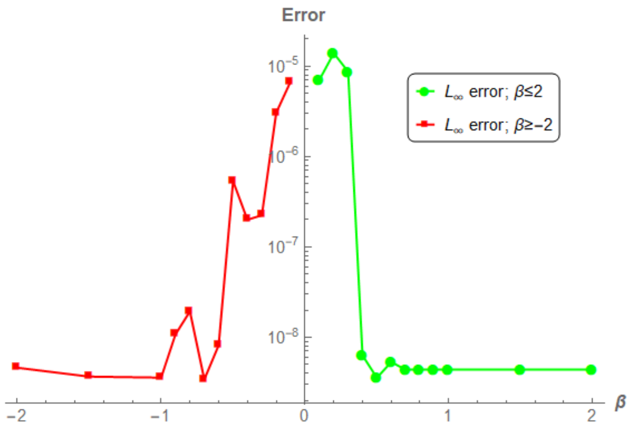

Figure 6. Logarithmic plot of L_∞ errors in terms of β for Example 5.4.

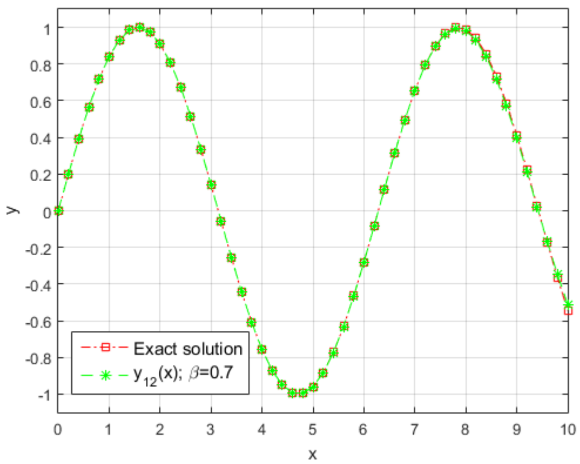

Figure 7. Oscillatory behavior of the generalized Mott polynomial solution and exact solution on [0,10] for Example 5.4.

6. CONCLUSIONS

In this study, we have introduced a new matrix-collocation method based on the generalized Mott polynomials to numerically solve the delay integral equations with variable bounds. Thanks to this method, the advanced algebraic properties of the generalized Mott polynomials could be determined in the future works. We have achieved good approximation to the exact solutions of stiff integral equations. In the considered examples, the better approximate solutions and results have been obtained by means of the generalized Mott polynomials with truncation limit N, β and residual error analysis technique, it is also monitored from the examples that the parameter-β has a different role in obtaining the approximate solutions. That is, the consistency of the approximate solution changes with respect to β, but we can determine $\beta \in [0,2]$ for the optimal approximation to the exact solutions as seen in Figures 2 and 6. It is clearly seen that the present method is accurate, efficient and simple. For future work, it would be interesting to apply the present method to other well-known problems, such as fractional and partial differential equations. However, some modifications are required.

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