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## A MULTIPLE SCALES METHOD FOR SOLVING NONLINEAR KdV7 EQUATION

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### ABSTRACT

In this report, a method of multiple scales is presented for the analysis of the (1+1)-dimensional seventh-order Korteweg-de Vries (KdV7) equation and we derive nonlinear Schrödinger (NLS) type equation. Also we found the exact solutions for (1+1)-dimensional KdV7 equation by using the  $\left(\frac{G'}{G}\right)$ -expansion method. These methods are very simple and effective for getting integrability and exact solutions of KdV type equations.

**Keywords:** Multiple scales method, KdV7 equation,  $\left(\frac{G'}{G}\right)$ -expansion method, Exact solutions

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### 1. INTRODUCTION

The Korteweg-de Vries (KdV) equation was derived in various physical contexts. The Korteweg-de Vries (KdV) equation can also be stated as modelling the one-sided breeding of small amplitude long wavelength gravity waves in a shallow channel. The Korteweg-de Vries (KdV) equation is attained at definite approximation degree in all cases and so it can't be accepted that it represents physical reality with perfect accuracy. Thus, one important question that emerges is what happens to the solutions of Korteweg-de Vries (KdV) when perturbations, resulting from the terms neglected in its derivation, are operated. For example, it can be asked whether the perturbed Korteweg-de Vries (KdV) equation has a solitary-wave-type solution. The answer would be related to physical origin and nature of the perturbation.

That multiple scales analysis of the Korteweg-de Vries (KdV) equation cause to the Nonlinear Schrödinger (NLS) equation for the regulated amplitude is widely known. [1-5]. In [1] Zakharov and Kuznetsov indicated a much deeper correspondence between these integrable equations not only at the level of the equation, but also at the level of the linear spectral problem by showing that a multiple scales analysis of the Schrödinger spectral problem leads to the Zakharov-Shabat problem for the Nonlinear Schrödinger (NLS) equation. The same relationship between integrable fifth-order nonlinear evolution equations and NLS equation was also indicated by Dağ and Özer [6].

Traveling waves emerge in many physical structures in solitary wave theory such as solitons, cuspons, compactons, peakons, kinks and others. To find a certain travelling wave solutions of nonlinear evolution equations (NEEs), too many methods have been developed recently. To construct the travelling wave solutions of NEEs, the  $\left(\frac{G'}{G}\right)$ -expansion method is simple, clear and proper among these methods. The method is based on the explicit linearization of nonlinear evolution equations for travelling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. Besides, it transforms a nonlinear equation to a simplest algebraic computation [7]. The  $\left(\frac{G'}{G}\right)$ -expansion method was used by Zhang et al. [8] and Aslan [9] to address some physically important nonlinear differential difference equations. Generalized  $\left(\frac{G'}{G}\right)$ -expansion method was recommended by

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Zhang et al. [10] to extend and develop the works of Wang et al. [11] and Tang et al. [12] in order to solve high-dimensional equations and variable-coefficient equations. To get new certain travelling wave solutions in the sense of modified Riemann Liouville derivative, the  $\left(\frac{G'}{G}\right)$ -expansion method was applied to fractional differential difference equation [13].

The research was organized as follows. In Section 2 and in section 3, the method of multiple scales was expressed shortly and  $\left(\frac{G'}{G}\right)$ -expansion method in turn. In Section 4, the family of seventh-order KdV equations were introduced. In section 5, these methods were implemented to  $(1+1)$ -dimensional seventh order Korteweg-de Vries Equation (KdV7) [14]. Conclusions were discussed in the last section.

## 2. MULTIPLE SCALES METHOD

In this section, we consider the application of the multiple scales method to NEEs. Using the technique of Zakharov and Kuznetsov [1], getting the NLS type equations from KdV type equations was shown step by step.

Let consider the general evolution equation in the following form.

$$u_t = K(u, u_x, u_y, \dots) \tag{2.1}$$

Where  $K[u]$  is a function of  $u$  and its derivatives with respect to the  $x$ -spatial variables. The well-known of this type equations are KdV equation.

Let  $L[\partial_x, \partial_y]u$  are the linear part of  $K[u]$ . So, using  $K[u]$  we can reach the dispersion relation for the Equation (2.1). Substituting the wave solution space

$$\begin{aligned} u_k &= Ae^{i(kx+ry-\omega(k,r)t)} \\ &\equiv Ae^{i\theta} \end{aligned} \tag{2.2}$$

into the linear part of Equation (2.1)

$$u_t = L[\partial_x, \partial_y]u \tag{2.3}$$

we get the dispersion relation

$$\omega(k, r) = iL[ik, ir] \tag{2.4}$$

Then, dispersion relation (2.4) is substituted in Equation (2.1). We assume the following series expansions for the solution of the Equation (2.1):

$$u(x, y, t) = \sum_{n=1}^{\infty} \varepsilon^n U_n(x, y, t, \xi, \tau) \tag{2.5}$$

Based on this solution, we also define slow space  $\xi$  and multiple time variable  $\tau$  with respect to the scaling parameter  $\varepsilon > 0$  respectively as follows.

$$\begin{aligned} \xi &= \varepsilon \left( x - \frac{d\omega(k, r)}{dk} t \right) \\ \tau &= -\frac{1}{2} \varepsilon^2 \left( \frac{d^2 \omega(k, r)}{dk^2} \right) t \end{aligned} \tag{2.6}$$

A nonlinear equation modulates the amplitude of this plane wave solution in such a way that may consider it depend upon the slow variables. If we choose the slow variables different forms, we can derive higher order NLS equations. The multiple scales analysis starts with the assumption:

$$u(x, y, t) = U(x, y, t, \xi, \tau) \tag{2.7}$$

and solution of  $U$  is in the form

$$U(x, y, t, \xi, \tau) = (\varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \dots) \tag{2.8}$$

Then, considering transformation (2.7) and solution (2.8), using dispersion relation (2.4) and slow variables (2.6), we get  $u$  and its derivatives with respect to  $\varepsilon$  in Equation (2.1). And we substitute these terms with (2.7) and (2.8) in the Equation (2.1). Collecting all terms with the same order of  $\varepsilon$  together, the left hand side of Equation (2.1) is converted into a polynomial in  $\varepsilon$ . Then setting each coefficient of this polynomial to zero, we get a set of algebraic equations. Using wave solution space (2.2) and dispersion relation (2.4), these equations can be solved by iteration and by use of Maple. So we can get NLS type equations from Equation (2.1). Also, from this procedure we can reach numerical solutions of KdV type equations.

### 3. $\left(\frac{G'}{G}\right)$ -EXPANSION METHOD

We can describe the  $\left(\frac{G'}{G}\right)$ -expansion method step by step as follows [11, 15, 16]:

Let consider nonlinear partial differential equation

$$P(u, u_t, u_x, u_{tt}, u_{xx}, \dots) = 0, \tag{3.1}$$

in the form where  $P$  is a polynomial of  $u(x, t)$  and its derivatives.

Step 1: The traveling wave transformation

$$u(x, t) = U(\xi), \quad \xi = kx - ct \tag{3.2}$$

where  $c$  is the wave speed,  $k$  is the wave number. The travelling wave transformation for the travelling wave solutions of Equation (3.1). With this wave transformation Equation (3.1) can be reduced to

$$Q(U, -cU', kU', c^2U'', k^2U'', \dots) = 0, \tag{3.3}$$

ordinary differential equation where  $U = U(\xi)$  and prime denotes derivatives with respect to  $\xi$ .

Step 2: We predict the solution of (3.3) equation in the finite series form

$$U(\xi) = \sum_{l=0}^m \alpha_l \left( \frac{G'(\xi)}{G(\xi)} \right)^l, \quad \alpha_m \neq 0, \tag{3.4}$$

where  $m$  is a positive integer to be determined and  $\alpha_l$ 's are constants to be determined later,  $G(\xi)$  satisfies a second order linear ordinary differential equation:

$$\frac{d^2G(\xi)}{d\xi^2} + \lambda \frac{dG(\xi)}{d\xi} + \mu G(\xi) = 0, \tag{3.5}$$

where  $\lambda$  and  $\mu$  are arbitrary constants. Using the general solutions of Equation (3.5), we get following cases:

$$\frac{G'(\xi)}{G(\xi)} = \begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2} \left( \frac{C_1 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + C_2 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{C_1 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + C_2 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0 \\ \frac{\sqrt{4\mu-\lambda^2}}{2} \left( \frac{-C_1 \sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi) + C_2 \cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}{C_1 \cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi) + C_2 \sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)} \right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu < 0 \end{cases} \tag{3.6}$$

where  $C_1$  and  $C_2$  are arbitrary constants.

Step 3: We can easily determine the degree  $m$  of Equation (3.4) by using the homogeneous balance principle for the highest order nonlinear term(s) and the highest order partial derivative of  $U(\xi)$  in Equation (3.3).

Step 4: As a final step, substituting (3.4) together with (3.5) into Equation (3.3) and collecting all terms with the same order of  $\left(\frac{G'(\xi)}{G(\xi)}\right)$ , the left hand side of Equation (3) is converted into a polynomial in  $\left(\frac{G'(\xi)}{G(\xi)}\right)$ . For example, for  $m = 1$  in Equation (3.5) is in the form

$$\frac{d^2G(\xi)}{d\xi^2} + \lambda \frac{dG(\xi)}{d\xi} + \mu G(\xi) = 0 \Rightarrow G''(\xi) = -\lambda G'(\xi) - \mu G(\xi), \tag{3.7}$$

where

$$U(\xi) = a_0 + a_1 \left( \frac{G'(\xi)}{G(\xi)} \right), \tag{3.8}$$

so we get

$$U' = a_1 \left( -\lambda \frac{G'}{G} - \mu - \left(\frac{G'}{G}\right)^2 \right) \tag{3.9}$$

and

$$\begin{aligned} U'' &= a_1 \left( \lambda\mu + (\lambda^2 + 2\mu)\left(\frac{G'}{G}\right) + 3\lambda\left(\frac{G'}{G}\right)^2 + 2\left(\frac{G'}{G}\right)^3 \right) \\ &\vdots \end{aligned} \tag{3.10}$$

as polynomials of  $\left(\frac{G'}{G}\right)$ . As a final step, substituting Equations. (3.4) and (3.8 - 3.10) together with (3.5) into Equation (3.3) and collecting all terms with the same order of  $\left(\frac{G'}{G}\right)$  together, the left hand side of Equation (3.3) is converted into a polynomial in  $\left(\frac{G'}{G}\right)$ . Equating each coefficient of  $\left(\frac{G'}{G}\right)^l$  ( $l = 0, 1, 2, \dots$ ) to zero yields a set of algebraic equations for “ $\alpha_l$  ( $l = 0, 1, 2, \dots, N$ ),  $k$ ,  $c$ ”. Solving these algebraic

equations system, we can define “ $\alpha_l$  ( $l = 0, 1, 2, \dots, N$ ),  $k$ ,  $c$ ”. Finally, we substitute these values into expression (3.4) and obtain various kinds of exact solutions to Equation (3.1) by use of Maple.

#### 4. (1+1)-DIMENSIONAL KdV7 EQUATION

Take into consideration the family of seventh-order KdV equations [17],

$$u_t = u_{xxxxxx} + auu_{xxxx} + bu_x u_{xxx} + cu_{xx} u_{xx} + du^2 u_{xxx} + euu_x u_{xx} + fu_x^3 + gu^3 u_x \tag{4.1}$$

where  $a, b, c, d, e, f$  and  $g$  are non-zero parameters. Special cases of (4.1) are known in the literature. For  $a = 21, b = 42, c = 63, d = 126, e = 378, f = 63, g = 252$

$$u_t = u_{xxxxxx} + 21uu_{xxxx} + 42u_x u_{xxx} + 63u_{xx} u_{xx} + 126u^2 u_{xxx} + 378uu_x u_{xx} + 63u_x^3 + 252u^3 u_x \tag{4.2}$$

the equation (4.1) degrades to the SK-Ito equation, because of seventh-order Sawada-Kotera-Ito equation (4.2). For  $a = 42, b = 147, c = 252, d = 504, e = 2268, f = 630, g = 2016$

$$u_t = u_{xxxxxx} + 42uu_{xxxx} + 147u_x u_{xxx} + 252u_{xx} u_{xx} + 504u^2 u_{xxx} + 2268uu_x u_{xx} + 630u_x^3 + 2016u^3 u_x \tag{4.3}$$

the equation (4.1) pertains to the Kaup-Kupershmidt hierarchy (4.3) [18, 19]. For  $a = 14, b = 42, c = 70, d = 70, e = 280, f = 70, g = 140$

$$u_t = u_{xxxxxx} + 14uu_{xxxx} + 42u_x u_{xxx} + 70u_{xx} u_{xx} + 70u^2 u_{xxx} + 280uu_x u_{xx} + 70u_x^3 + 140u^3 u_x \tag{4.4}$$

the equation (4.1) pertains to the KdV hierarchy studied by Lax (4.4) [20].

#### 5. APPLICATIONS

In this section, we apply multiple scales method and  $\left(\frac{G'}{G}\right)$ -expansion method summarized in section 2 and section 3 to (1+1) dimensional KdV7 Equation (4.4). To find dispersion relation for (4.4), we consider the linear part of (4.4) in the form

$$u_t = u_{xxxxxx} \tag{5.1}$$

and linear differential Equation (5.1) satisfies the solution

$$u(x, t) = e^{i\theta}, \theta = kx - w(k)t \quad (5.2)$$

Substituting the solution (5.2) into the linear differential Equation (5.1), we get

$$we^{i(kx-wt)} = k^7 e^{i(kx-wt)} \quad (5.3)$$

and from this we reach the

$$w(k) = k^7 \quad (5.4)$$

dispersion relation. Thus the solution of linear differential Equation (5.1) is as follows:

$$u(x, t) = e^{i(kx-k^7t)} \quad (5.5)$$

Let the solution of Equation (4.4) is in the form

$$u(x, t) = U(x, t, \xi, \tau) \quad , \quad U(x, t, \xi, \tau) = \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \dots \quad (5.6)$$

and slow variables are in the form

$$\begin{aligned} \xi &= \varepsilon \left( x - \frac{dw(k)}{dk} t \right) \\ &= \varepsilon (x - 7k^6 t) \\ \tau &= -\frac{1}{2} \varepsilon^2 \left( \frac{d^2 w(k)}{dk^2} t \right) \\ &= -21\varepsilon^2 k^5 t \end{aligned} \quad (5.7)$$

Then using (5.4-5.7), the terms included derivative in (4.4) are obtained as follows:

$$\begin{aligned}
 D_x &= \partial_x + \varepsilon \partial_\xi \\
 D_t &= \partial_t - 7\varepsilon k^6 \partial_\xi - 21\varepsilon^2 k^5 \partial_\tau \\
 D_{xx} &= \partial_{xx} + 2\varepsilon \partial_{x\xi} + \varepsilon^2 \partial_{\xi\xi} \\
 D_{xxx} &= \partial_{xxx} + 3\varepsilon \partial_{xx\xi} + 3\varepsilon^2 \partial_{x\xi\xi} + \varepsilon^3 \partial_{\xi\xi\xi} \\
 D_{xxxx} &= \partial_{xxxx} + 4\varepsilon \partial_{xxx\xi} + 6\varepsilon^2 \partial_{xx\xi\xi} + 4\varepsilon^3 \partial_{x\xi\xi\xi} + \varepsilon^4 \partial_{\xi\xi\xi\xi} \\
 D_{xxxxx} &= \partial_{xxxxx} + 5\varepsilon \partial_{xxx\xi\xi} + 10\varepsilon^2 \partial_{xx\xi\xi\xi} + 10\varepsilon^3 \partial_{x\xi\xi\xi\xi} \\
 &\quad + 5\varepsilon^4 \partial_{\xi\xi\xi\xi\xi} + \varepsilon^5 \partial_{\xi\xi\xi\xi\xi\xi} \\
 D_{xxxxxx} &= \partial_{xxxxxx} + 7\varepsilon \partial_{xxx\xi\xi\xi} + 21\varepsilon^2 \partial_{xx\xi\xi\xi\xi} + 35\varepsilon^3 \partial_{x\xi\xi\xi\xi\xi} \\
 &\quad + 35\varepsilon^4 \partial_{\xi\xi\xi\xi\xi\xi} + 21\varepsilon^5 \partial_{\xi\xi\xi\xi\xi\xi\xi} + 7\varepsilon^6 \partial_{\xi\xi\xi\xi\xi\xi\xi\xi} + \varepsilon^7 \partial_{\xi\xi\xi\xi\xi\xi\xi\xi\xi}
 \end{aligned} \tag{5.8}$$

Substituting (5.6) and (5.8) into the (4.4), we get a polynomial in  $\varepsilon$ . Equating each coefficient of this polynomial to zero, we find

$$\varepsilon = u_{1t} - u_{1xxxxxx} = 0 \tag{5.9}$$

$$\begin{aligned}
 \varepsilon^2 &= u_{2t} - 7k^6 u_{1\xi} - u_{2xxxxxx} - 7u_{1xxxx\xi} - 14u_1 u_{1xxxx} \\
 &\quad - 42u_{1x} u_{1xxx} - 70u_{1xx} u_{1xxx} = 0
 \end{aligned} \tag{5.10}$$

$$\begin{aligned}
 \varepsilon^3 &= u_{3t} - 7k^6 u_{2\xi} - 21k^5 u_{1\tau} - u_{3xxxxxx} - 7u_{2xxxx\xi} \\
 &\quad - 21u_{1xxxx\xi\xi} - 14u_1 (u_{2xxxx} + 5u_{1xxx\xi}) - 14u_2 u_{1xxxx} \\
 &\quad - 42u_{1x} (u_{2xxx} + 4u_{1xxx\xi}) - 42(u_{2xx} + u_{1x\xi}) u_{1xxx} \\
 &\quad - 70u_{1xx} (u_{2xxx} + 3u_{1xx\xi}) - 70(u_{2xx} + 2u_{1x\xi}) u_{1xxx} \\
 &\quad - 70u_1^2 u_{1xxx} - 280u_1 u_{1x} u_{1xx} - 70u_{1x}^3 = 0
 \end{aligned} \tag{5.11}$$

Then, we can find the solution of (5.9) as follows

$$u_1(x, t, \xi, \tau) = v_1(\xi, \tau) e^{i(kx - k^7 t)} + v_{-1}(\xi, \tau) e^{-i(kx - k^7 t)} \tag{5.12}$$

where  $v_{-1}$  is complex conjugate of  $v_1$ . Substituting the solution (5.12) into (5.10), the solution of (5.10) is in the form,

$$u_2(x, t, \xi, \tau) = v_2(\xi, \tau)e^{2i(kx-k^7t)} + v_{-2}(\xi, \tau)e^{-2i(kx-k^7t)} + f_1(\xi, \tau) \quad (5.13)$$

where  $f_1(\xi, \tau)$  is integration constant. Thus we get

$$v_2(\xi, \tau) = \frac{v_1^2(\xi, \tau)}{k^2}, \quad v_{-2}(\xi, \tau) = \frac{v_{-1}^2(\xi, \tau)}{k^2} \quad (5.14)$$

where  $v_{-1}$  is the complex conjugate of  $v_1$  and  $v_{-2}$  is the complex conjugate of  $v_2$ . Substituting solutions (5.11), (5.13) and (5.14) into the (5.11), we find the solution of (5.11) in the form

$$u_3(x, t, \xi, \tau) = v_3(\xi, \tau)e^{3i(kx-k^7t)} + v_{-3}(\xi, \tau)e^{-3i(kx-k^7t)} + f_2(\xi, \tau)e^{2i(kx-k^7t)} + f_3(\xi, \tau)e^{-2i(kx-k^7t)} \quad (5.15)$$

where  $f_2(\xi, \tau)$  and  $f_3(\xi, \tau)$  is integration constant. Then we get

$$v_3(\xi, \tau) = \frac{3v_1^3(\xi, \tau)}{4k^4}, \quad v_{-3}(\xi, \tau) = \frac{3v_{-1}^3(\xi, \tau)}{4k^4} \quad (5.16)$$

and

$$\begin{aligned} f_1(\xi, \tau) &= \frac{-2v_1v_{-1}}{k^2} \\ f_2(\xi, \tau) &= \frac{-2iv_{-1}v_{-1}\xi}{k^3} \\ f_3(\xi, \tau) &= \frac{2iv_1v_1\xi}{k^3} \end{aligned} \quad (5.17)$$

where  $v_{-3}$  and  $f_3$  are the complex conjugates of  $v_3$  and  $f_2$  respectively. Thus, the solutions of (5.9) - (5.10) are obtained as

$$\begin{aligned} u_1 &= v_1(\xi, \tau)e^{i\theta} + v_{-1}(\xi, \tau)e^{-i\theta} \\ u_2 &= k^{-2}(-2v_1(\xi, \tau)v_{-1}(\xi, \tau) + v_1^2(\xi, \tau)e^{2i\theta} + v_{-1}^2(\xi, \tau)e^{-2i\theta}) \\ u_3 &= k^{-3}(-2iv_{-1}(\xi, \tau)v_{-1\xi}(\xi, \tau)e^{-2i\theta} + 2iv_1(\xi, \tau)v_{1\xi}(\xi, \tau)e^{2i\theta}) \\ &\quad + \frac{3}{4}k^{-4}(v_1^3(\xi, \tau)e^{3i\theta} + v_{-1}^3(\xi, \tau)e^{-3i\theta}) \end{aligned} \quad (5.18)$$

where



$$\theta = (kx - k^7 t) \tag{5.19}$$

Finally, substituting the solutions (5.18) into (5.11), we get

$$i v_{1\tau} = v_{1\xi\xi} - \frac{2}{k^2} v_1 v_{-1} \tag{5.20}$$

$$i v_{1\tau} = v_{1\xi\xi} - \frac{2}{k^2} v_1 |v_1|^2$$

Describing as  $q = \frac{v_1}{k}$  and  $q_{-1} = \frac{v_{-1}}{k}$  (5.20) equation we get the (1 + 1)-dimensional NLS type equations in the form

$$i q_{\tau} = q_{\xi\xi} - 2q|q|^2 \tag{5.21}$$

Also, numerical solution of the (1 + 1)-dimensional KdV7 equation (4.4) is found as

$$\begin{aligned} u(x, t) = & \epsilon k (q(\xi, \tau) e^{i(kx - k^7 t)} + q_{-1}(\xi, \tau) e^{-i(kx - k^7 t)}) \\ & + \epsilon^2 (-2q(\xi, \tau) q_{-1}(\xi, \tau) + q^2(\xi, \tau) e^{2i(kx - k^7 t)} \\ & + q_{-1}^2(\xi, \tau) e^{-2i(kx - k^7 t)}) + k \epsilon^3 (2i q(\xi, \tau) q_{\xi}(\xi, \tau) e^{2i(kx - k^7 t)} \\ & - 2i q_{-1}(\xi, \tau) q_{-1\xi}(\xi, \tau) e^{-2i(kx - k^7 t)}) \\ & + \frac{3}{4} k \epsilon^3 (q_1^3(\xi, \tau) e^{3i(kx - k^7 t)} + q_{-1}^3(\xi, \tau) e^{-3i(kx - k^7 t)}) \end{aligned} \tag{5.22}$$

where  $q$  is solution of NLS equation.

Using the travelling wave transformation (5.23)

$$u(x, t) = U(\xi) \quad , \quad \xi = kx - ct \tag{5.23}$$

Equation (4.4) turns into

$$\begin{aligned} -cU' + 140kU^3U' + 70k^3(U')^3 + 280k^3UU'U'' + 70k^3U^2U''' + 70k^5U''U''' \\ + 42k^5U'U^{(4)} + 14k^5UU^{(5)} + k^7U^{(7)} = 0 \end{aligned} \tag{5.24}$$

ordinary differential equation. Suppose that the travelling wave solution of Equation (5.24) is in the form

$$U(\xi) = \sum_{l=0}^m a_l \left( \frac{G'(\xi)}{G(\xi)} \right)^l \quad , \quad a_m \neq 0 \tag{5.25}$$

According to the homogeneous balance procedure, in Equation (5.24) balancing the highest order derivative term  $U^{(7)}$  and the highest order nonlinear term  $U(U)^{(5)}$ , we get

$$U^{(7)} \sim UU^{(5)} \tag{5.26}$$

$$m + 7 = m + m + 5 \tag{5.27}$$

$$m = 2$$

Thus substituting (5.27) into (3.4), the travelling wave solution of Equation (5.24) is found as

$$U(\xi) = \alpha_0 + \alpha_1 \left( \frac{G'(\xi_n)}{G(\xi_n)} \right), \quad \alpha_1 \neq 0 \tag{5.28}$$

$$U = \alpha_0 + \alpha_1 \left( \frac{G'(\xi)}{G(\xi)} \right) + \alpha_2 \left( \frac{G'(\xi)}{G(\xi)} \right)^2, \quad \alpha_1, \alpha_2 \neq 0 \tag{5.29}$$

where  $\alpha_l$  ( $l = 0, 1$ ) are constants to be determined later and  $G(\xi)$  satisfies second order linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{5.30}$$

where  $\lambda$  and  $\mu$  are arbitrary constants. Using (5.30) and (5.28) the derivatives in Equation (5.24) are obtained as follows:

$$\begin{aligned} U' &= -\alpha_1 \left( \lambda \frac{G'}{G} + \mu + \left( \frac{G'}{G} \right)^2 \right) - 2\alpha_2 \left( \lambda \left( \frac{G'}{G} \right)^2 + \mu \left( \frac{G'}{G} \right) + \left( \frac{G'}{G} \right)^3 \right), \\ U'' &= \alpha_1 \left( \lambda \mu + (\lambda^2 + 2\mu) \left( \frac{G'}{G} \right) + 3\lambda \left( \frac{G'}{G} \right)^2 + 2 \left( \frac{G'}{G} \right)^3 \right) + 2\alpha_2 \left( 2\lambda^2 \left( \frac{G'}{G} \right)^2 \right. \\ &\quad \left. + 3\lambda \mu \left( \frac{G'}{G} \right) + 5\lambda \left( \frac{G'}{G} \right)^3 + \mu^2 + 4\mu \left( \frac{G'}{G} \right)^2 + 3 \left( \frac{G'}{G} \right)^4 \right), \\ U''' &= -\alpha_1 \left( \lambda^3 \frac{G'}{G} + \lambda^2 \mu + 8\lambda \mu \frac{G'}{G} + 2\mu^2 + 7 \left( \frac{G'}{G} \right)^2 \lambda^2 + 8 \left( \frac{G'}{G} \right)^2 \mu \right. \\ &\quad \left. + 12 \left( \frac{G'}{G} \right)^3 \lambda + 6 \left( \frac{G'}{G} \right)^4 \right) - 2\alpha_2 \left( 4\lambda^3 \left( \frac{G'}{G} \right)^2 + 7\lambda^2 \mu \left( \frac{G'}{G} \right) \right. \\ &\quad \left. + 19\lambda^2 \left( \frac{G'}{G} \right)^3 + 3\lambda \mu^2 + 26\lambda \mu \left( \frac{G'}{G} \right)^2 + 27\lambda \left( \frac{G'}{G} \right)^4 \right. \\ &\quad \left. + 8\mu^2 \left( \frac{G'}{G} \right) + 20\mu \left( \frac{G'}{G} \right)^3 + 12 \left( \frac{G'}{G} \right)^5 \right), \\ &\quad \vdots \end{aligned}$$

$$\begin{aligned}
 U^{(7)} = & -\alpha_1(127\lambda^6(\frac{\dot{G}}{G})^2 + 240\lambda^5\mu(\frac{\dot{G}}{G}) + 1986\lambda^5(\frac{\dot{G}}{G})^3 + 20160\lambda(\frac{\dot{G}}{G})^7 \\
 & + 13440\mu(\frac{\dot{G}}{G})^6 + 5040(\frac{\dot{G}}{G})^8 + 4326\lambda^4\mu(\frac{\dot{G}}{G})^2 + 3072\lambda^3\mu^2(\frac{\dot{G}}{G}) + \dots
 \end{aligned}
 \tag{5.31}$$

If we solve this system by Maple, then we get

$$\alpha_0 = \alpha_0 \quad , \quad \alpha_1 = -2k^2\lambda \quad , \quad \alpha_2 = -2k^2 \quad , \quad k = k
 \tag{5.32}$$

$$\begin{aligned}
 c = & -k(70\lambda^2k^2\alpha_0^2 + 14\lambda^4k^4\alpha_0 + 560\alpha_0^2k^2\mu + 104\lambda^2k^6\mu^2 \\
 & + 16\lambda^4k^6\mu + 784\alpha_0k^4\mu^2 + 384k^6\mu^3 + \lambda^6k^6 \\
 & + 168\lambda^2k^4\alpha_0\mu + 140\alpha_0^3)
 \end{aligned}$$

Thus, hyperbolic and trigonometric solutions of Equation (5.24) are found as follows:

i) For  $\lambda^2 - 4\mu > 0$ ,

$$\frac{\dot{G}(\xi)}{G(\xi)} = \begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2} \left( \frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)} \right) - \frac{\lambda}{2} \quad , \quad \lambda^2 - 4\mu > 0 \\ \\ \frac{\sqrt{4\mu-\lambda^2}}{2} \left( \frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)} \right) - \frac{\lambda}{2} \quad , \quad \lambda^2 - 4\mu < 0 \\ \\ \frac{C_1}{C_1\xi + C_2} - \frac{\lambda}{2} \quad , \quad \lambda^2 - 4\mu = 0 \end{cases}
 \tag{5.33}$$

Substituting (5.32) and (3.6) into (5.28) we get hyperbolic function solutions of Equation (5.24)

$$\begin{aligned}
 U(\xi) = & \alpha_0 - 2k^2\lambda \left( \frac{\sqrt{\lambda^2-4\mu}}{2} \frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)} - \frac{\lambda}{2} \right) \\
 & - 2k^2 \left( \frac{\sqrt{\lambda^2-4\mu}}{2} \frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)} - \frac{\lambda}{2} \right)^2
 \end{aligned}
 \tag{5.34}$$

where  $C_1$  and  $C_2$  are arbitrary constants. From (5.34) using

$$\begin{aligned} \xi = & kx - (-k(70\lambda^2 k^2 \alpha_0^2 + 14\lambda^4 k^4 \alpha_0 + 560\alpha_0^2 k^2 \mu \\ & + 104\lambda^2 k^6 \mu^2 + 16\lambda^4 k^6 \mu + 784\alpha_0 k^4 \mu^2 + 384k^6 \mu^3 + \lambda^6 k^6 \\ & + 168\lambda^2 k^4 \alpha_0 \mu + 140\alpha_0^3)t) \end{aligned}$$

we obtain for  $\xi = \theta$ ,

$$u(x,t) = \alpha_0 - 2k^2 \lambda \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} v(x,t) - \frac{\lambda}{2} \right) - 2k^2 \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} v(x,t) - \frac{\lambda}{2} \right)^2 \quad (5.35)$$

where

$$v(x,t) = \frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \theta\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \theta\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \theta\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \theta\right)}$$

ii) For  $\lambda^2 - 4\mu < 0$ ,

Substituting (5.32) and (3.6) into (5.28) we find trigonometric function solutions of Equation (5.24)

$$\begin{aligned} U(\xi) = & \alpha_0 - 2k^2 \lambda \left( \frac{\sqrt{4\mu - \lambda^2}}{2} \frac{-C_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + C_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)}{C_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + C_2 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)} - \frac{\lambda}{2} \right) \\ & - 2k^2 \left( \frac{\sqrt{4\mu - \lambda^2}}{2} \frac{-C_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + C_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)}{C_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + C_2 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)} - \frac{\lambda}{2} \right)^2 \end{aligned} \quad (5.36)$$

where  $C_1$  and  $C_2$  are arbitrary constants. From (5.36) using

$$\begin{aligned} \xi = & kx - (-k(70\lambda^2 k^2 \alpha_0^2 + 14\lambda^4 k^4 \alpha_0 + 560\alpha_0^2 k^2 \mu \\ & + 104\lambda^2 k^6 \mu^2 + 16\lambda^4 k^6 \mu + 784\alpha_0 k^4 \mu^2 + 384k^6 \mu^3 + \lambda^6 k^6 \\ & + 168\lambda^2 k^4 \alpha_0 \mu + 140\alpha_0^3)t) \end{aligned}$$

we obtain for  $\xi = \theta$ ,

$$u(x,t) = \alpha_0 - 2k^2 \lambda \left( \frac{\sqrt{4\mu - \lambda^2}}{2} v(x,t) - \frac{\lambda}{2} \right) - 2k^2 \left( \frac{\sqrt{4\mu - \lambda^2}}{2} v(x,t) - \frac{\lambda}{2} \right)^2 \quad (5.37)$$

where

$$v(x,t) = \frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\theta\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\theta\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\theta\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\theta\right)}$$

iii) For  $\lambda^2 - 4\mu = 0$ ,

Substituting (5.32) and (3.6) into (5.28) we find trigonometric function solutions of Equation (5.24)

$$U(\xi) = \alpha_0 - 2k^2\lambda\left(\frac{C_1}{C_1\xi+C_2} - \frac{\lambda}{2}\right) - 2k^2\left(\frac{C_1}{C_1\xi+C_2} - \frac{\lambda}{2}\right)^2 \tag{5.38}$$

where  $C_1$  and  $C_2$  are arbitrary constants. From (5.36) using

$$\begin{aligned} \xi = & kx - (-k(70\lambda^2k^2\alpha_0^2 + 14\lambda^4k^4\alpha_0 + 560\alpha_0^2k^2\mu \\ & + 104\lambda^2k^6\mu^2 + 16\lambda^4k^6\mu + 784\alpha_0k^4\mu^2 + 384k^6\mu^3 + \lambda^6k^6 \\ & + 168\lambda^2k^4\alpha_0\mu + 140\alpha_0^3)t) \end{aligned}$$

we obtain for  $\xi = \theta$ ,

$$u(x,t) = \alpha_0 - 2k^2\lambda\left(\frac{C_1}{C_1\theta+C_2} - \frac{\lambda}{2}\right) - 2k^2\left(\frac{C_1}{C_1\theta+C_2} - \frac{\lambda}{2}\right)^2 \tag{5.39}$$

## 6. CONCLUSION

A robust multiple scales method was developed to solve nonlinear evolution equations. The relationship between NLS equation and KdV7 equation and the solutions of the KdV7 equation were also examined. In this study we only studied on deriving NLS type equations from KdV type equations and their solutions by use of multiple scales method. Application of the method to recursion operators of KdV7 equation and to the spectral problems can be investigated in future studies. Additionally, in the study, the  $\left(\frac{G'}{G}\right)$ -expansion method was used to find certain travelling wave solutions of  $(1+1)$ -dimensional KdV7 equation. It can be stated that the performance of the method is simple, reliable and these methods emerge new hyperbolic and trigonometric type certain solutions. The implementation of the methods is clear, simple and it can also be applied to other NEEs fractional differential equations and differential difference equations. Lastly, it is worth mentioning that the implementation of these proposed methods is very simple and straightforward, and it can also be applied to many other NEEs, differential difference equations and fractional differential equations. The details about these methods and their applications to other NEEs are given in [21].

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