ESKİŞEHİR TEKNİK ÜNİVERSİTESİ BİLİM VE TEKNOLOJİ DERGİSİ B- TEORİK BİLİMLER

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ANALYTICAL EVALUATION OF THE EINSTEIN INTEGRATE USING BINOMIAL EXPANSION THEOREM AND POWER SERIES

Hüseyin KOÇ *

Department of Electrical and Electronics Engineering, Faculty of Engineering, Muş Alparslan University, Muş, Turkey

ABSTRACT

The objective of this paper is to provide an efficient and reliable analytical expression for the Einstein integrals using the binomial expansion theorem and power series. The obtained analytical expressions are valid for all values of their parameters. The algorithm can be used in the software and simulation programs. By comparing the literature results, the obtained results show that our evaluation method gives reliable computational efficiency.

Keyword: Einstein integrals, Binomial expansion theorem, Bed load, Sediment transport

BİNOM GENİŞLEME TEOREMİ VE KUVVET SERİLERİ KULLANILARAK EİNSTEİN İNTEGRALİNİN ANALİTİK DEĞERLENDİRMESİ

ÖZET

Bu çalışmanın amacı, binom genişleme teoremi ve kuvvet serileri kullanılarak Einstein integralleri için etkili ve güvenilir analitik ifadeler elde etmektir. Elde edilen analitik ifadeler parametrelerin tüm değerleri için geçerlidir. Algoritma, yazılım ve simülasyon programlarında kullanılabilir. Literatür sonuçları ile karşılaştırıldığında elde edilen sonuçlar, değerlendirme yöntemimizin güvenilir hesaplama verimliliği sağladığını göstermektedir.

Anahtar Kelimeler: Einstein integralleri, Binom genişleme teoremi, Yatak yükü, Tortu taşımacılığı

1. INTRODUCTION

In order to continue the economic and cultural developments along a river, it is essential to understand principles of sediment transport for application to the solution of some engineering and environmental problems [1, 3]. For this reason, many sediment transport equations developed based on different approximations to predict bed load transport rates. The Einstein bed load function is one of the developed equations [1, 2, 4, 8, 12, 14].

The computation of Einstein bed load function requires the calculation of integrals J_1 and J_2 [8, 13]. Various methods have been used for the solution of these integrals [3, 5, 6, 8, 13]. Einstein [3] used the numerical integration technique to introduce a graphical solution for the calculation of these integrals. Guo and Hui [5] and Guo and Wood [6] obtained an analytical expression for solution of I_1 integrate using the Beta function. On the other hand, Guo [7, 8] used the binomial theorem and psi function to solve J_1 and J_2 integrals.

The aim of this work is to provide an accurate analytical method to estimate the Einstein integrals that the method will be simply applied in software programs.

*Corresponding Author: huseyinnkoc@yahoo.com
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2. THE EINSTEIN INTEGRALS AND METHOD

The Einstein integrals examined in the present work is defined [3]

$$J_1(z) = \int_E^1 \left(\frac{1-x}{x}\right)^z dx \tag{1}$$

and

$$J_2(z) = \int_E^1 \left(\frac{1-x}{x}\right)^z \ln x \, dx \tag{2}$$

in which Rouse number z express the ratio of the sediment properties to the hydraulic characteristic of the flow, E is relative layer thickness to water depth [11, 13].

In order to obtain the integration of $\mathrm{J}_1(\mathrm{z})$, Eq.(1), it is used the following binomial expansion theorems for an arbitrary real or complex n series and $|\mathrm{x}| > |\mathrm{y}|$ (see Refs. [9, 10, 15-17]).

$$(x \pm y)^n = \sum_{m=0}^{\infty} (\pm 1)^m F_m(n) x^{n-y} y^m \tag{3}$$

where $F_0(n) = 1$ and

$$F_m(n) = \begin{cases} \frac{n!}{[m!(n-m)!]} \; for\ integer\ n, \\ \frac{(-1)^m \Gamma(m-n)}{m! \Gamma(-n)} \; for\ integer\ n. \end{cases} \tag{4}$$

It is noticed that for $m < 0$ the binomial coefficient $F_m(n)$ in Eq.(4) is zero and the positive integer *n* terms with negative factorials do not contribute to the summation. $\Gamma(\alpha)$ is defined by well-known gamma functions [9]

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt. \tag{5}$$

$J_1(z)$ for non-integer *z* can rewrite in the following form:

$$J_1(z) = \int_E^1 \left(\frac{1-x}{x}\right)^z dx = \int_E^1 (1-x)^z x^{-z} dx \tag{6}$$

Taking into account Eq.(5) in Eq.(6), it is obtained the following formulas for the $J_1(z)$ integrate:

$$J_1(z) = \int_E^1 \left(\frac{1-x}{x}\right)^z dx = \sum_{n=0}^{\infty} (-1)^n f_n(z) \int_E^1 x^{n-z} dx \tag{7}$$

$$= \sum_{n=0}^{\infty} (-1)^n f_n(z) I_n(z) \tag{8}$$

Where

$$I_n(z) = \left(\frac{1-E^{n-z+1}}{n-z+1}\right). \tag{9}$$

For any integer *z* of $J_1(z)$ integral, one can obtain the following formulas by a simple transformation

$$J_1(z) = \int_E^1 \left(\frac{1-x}{x}\right)^z dx = \sum_{k=0}^{z} \frac{(-1)^k}{k!} \prod_{j=0}^{k-1} (z-j) \int_E^1 x^{k-z} dx \tag{10}$$

$$= \sum_{k=0}^{z-2 \geq 0} \frac{(-1)^k}{k!} I_1(z) \prod_{i=0}^{k-1} (z-i) + g_1(z). \tag{11}$$

where

$$I_1(z) = \int_E^1 x^{k-z} dx = \left(\frac{E^{k-z+1}}{z-k-1}\right). \tag{12}$$

and

$$g_1(z) = (-1)^z\{z\,\text{In}E - E + 1\}. \tag{13}$$

The $J_2(z)$ integral in Eq.(2) can be handled in similar way. In this context, Eq.(2) can rewrite in the following form:

$$J_2(z) = \int_E^1 \left(\frac{1-x}{x}\right)^z \text{In}\,x dx = \int_E^1 (1-x)^z x^{-z} \text{In}\,x dx \tag{14}$$

For non-integer z value of $J_2(z)$ integral, it can be obtained the following formula for by applying the binomial expansion theorem given by Eq.(5) in Eq.(14),

$$J_2(z) = \sum_{m=0}^{\infty} (-1)^m f_m(z) \int_E^1 x^{m-z} \text{In}\,x dx \tag{15}$$

$$= \sum_{m=0}^{\infty} (-1)^m f_m(z)\, I_m(z) \tag{16}$$

where

$$I_m(z) = \frac{1}{(m-z+1)^2}\{E^{m-z+1}[1-(m-z+1)\text{In}(E)] - 1\}. \tag{17}$$

For integer z of $J_2(z)$ integral, one can show that the following solution exists by applying a simple transformation in Eq.(2),

$$J_2(z) = \int_E^1 \left(\frac{1-x}{x}\right)^z \text{In}\,x dx = \sum_{k=0}^{z} \frac{(-1)^k}{k!} \prod_{j=0}^{k-1}(z-j) \int_E^1 x^{k-z} \text{In}\,x dx \tag{18}$$

$$= \sum_{k=0}^{z-2\geq 0} \frac{(-1)^k}{k!} I_2(z) \prod_{j=0}^{k-1}(z-j) + g_2(z) \tag{19}$$

in which

$$I_2(z) = \left\{-\frac{1}{(k-z+1)^2}\left[E^{k-z+1}\{(k-z+1)\text{In}E - 1\} + 1\right]\right\} \tag{20}$$

and

$$g_1(z) = (-1)^z \left\{\frac{z}{2}\,\text{In}^2 E - E\,\text{In}E + E - 1\right\}. \tag{21}$$

Several efficient methods for the calculation of J_1 and J_2 integrals given by Eqs.(1)-(2) have been proposed by various authors [8, 13].

3. RESULTS AND DISCUSSION

In this paper we present an efficient and simple approximation to Einstein integrals J_1 and J_2. This approximation is valid over the entire values (integer and non-integer) of z and the bed-layer thickness E. The results of the calculations for J_1 and J_2 integrals are shown in Tables 1 and 2. The results obtained from literature [3, 13] with Mathematica are also shown in Tables 1 and 2. Table 1 shows the calculations for J_1 and J_2 integrals with the literature results for non-integer values z [3, 13]. Like Table 1, Table 2 presents the calculations for J_1 and J_2 integrals with the literature results for integer values z. As can be seen from Tables 1 and 2, the obtained results are in excellent agreement with the literature results.

Table 1. The comparative values of Einstein's integrals $J_1(z)$ and of $J_2(z)$ for noninteger z values.

z	e	Einstein (1950) Table-2		Guo and Julien (2004) Eqs. (9) and (11)		Mathematica		This Study	
		J1(Z)	J2(Z)	J1(Z)	J2(Z)	J1(Z)	J2(Z)	J1(Z)	J2(Z)
0.2	1	0	0	-	-	0	-	0	0
	0.9	0.047736	0.003029	-	-	0.0531785	-0.00301493	0.0532254	-0.00301797
	0.6	0.28678	0.076114	-	-	0.292234	-0.076098	0.292281	-0.076101
	0.5	0.38286	0.13380	0.388439	-0.139753	0.388314	-0.133788	0.388361	-0.133791
	0.2	0.72505	0.51118	0.730506	-0.514015	0.730506	-0.511136	0.730553	-0.511139
	0.12	0.83681	0.71775	0.842263	-0.72014	0.842263	-0.717697	0.84231	-0.717701
	0.09	0.88290	0.82186	0.888354		0.888354		0.888401	
	0.04	0.96866	1.0595	0.974119		0.974119		0.974166	
	0.016	1.0178	1.2376	1.02329		1.02329		1.02334	
	0.008	1.0372	1.03240	1.04271		1.04271		1.04276	
	0.0007	1.0597	1.4517	1.06522		1.06522		1.06527	
	0.0003	1.0616	1.4658	1.06706		1.06706		1.06711	
	0.00009	1.0628	1.4759	1.06823		1.06823		1.06828	
	0.00002	1.0631	1.4809	1.06874		1.06874		1.06879	
	0.00001	1.0634	1.4820	1.06883		1.06883		1.06888	
0.4	1	0	0	-		0		0	
	0.9	0.027451	0.0017824	-		0.0291347		0.0291512	
	0.6	0.21975	0.062683	-		0.221441		0.221457	
	0.5	0.31211	0.11826	3.14053		3.13803		3.1382	
	0.2	0.70487	0.55950	0.706555		0.706555		0.706571	
	0.12	0.86118	0.84921	0.862865		0.862865		0.862881	
	0.09	0.93201	1.0093	0.933702		0.933702		0.933718	
	0.04	1.0795	1.4196	1.08117		1.08117		1.08119	
	0.016	1.1805	1.7870	1.18222		1.18222		1.18224	
	0.008	1.2277	1.9975	1.22943		1.22943		1.22945	
	0.0007	1.2983	2.4038	1.29998		1.29998		1.30000	
	0.0003	1.3068	2.4689	1.30848		1.30848		1.3085	
	0.00009	1.3134	2.5259	1.31508		1.31508		1.31509	
	0.00002	1.3171	2.5627	1.31878		1.31878		1.3188	
	0.00001	1.3179	2.5723	1.31964		1.31964		1.31966	
0.6	1	0	0	-		0		0	

	0.9	0.015853	0.0010544	-		0.0163149		0.0163187	
	0.6	0.17195	0.052142	-		0.172417		0.172421	
	0.5	0.26079	0.10571	0.26163		0.261255		0.261259	
	0.2	0.71441	0.62474	0.714849		0.714849		0.714853	
	0.12	0.93330	1.0316	0.93374		0.93374		0.933744	
	0.09	1.0422	1.2779	1.04265		1.04265		1.04265	
	0.04	1.2964	1.9881	1.29685		1.29685		1.29685	
	0.016	1.5047	2.7483	1.5051		1.5051		1.5051	
	0.008	1.6196	3.2618	1.62007		1.62007		1.62007	
	0.0007	1.8448	4.5794	1.84521		1.84521		1.84521	
	0.0003	1.8841	4.8807	1.88451		1.88451		1.88452	
	0.00009	1.9213	5.2034	1.92175		1.92175		1.92176	
	0.00002	1.9485	5.4754	1.94897		1.94897		1.94898	
	0.00001	1.9565	5.5645	1.95696		1.95696		1.95696	
	1	0	0	-		0		0	
	0.9	0.0091963	0.0006271	-		0.00929023		0.0092908	
	0.6	0.13699	0.043745	-		0.137089		0.137089	
	0.5	0.22249	0.095397	0.223083		0.222583		0.222584	
	0.2	0.74963	0.70947	0.749694		0.749694		0.749695	
	0.12	1.0565	1.28151	1.0566		1.0566		1.0566	
	0.09	1.2240	1.6606	1.22409		1.22409		1.22409	
0.8	0.04	1.6633	2.8911	1.66335		1.66335		1.66335	
	0.016	2.0937	2.0937	2.09378		2.09378		2.09378	
	0.008	2.3742	5.7233	2.37421		2.37421		2.37422	
	0.0007	3.1064	10.080	3.10648		3.10648		3.10648	
	0.0003	3.2887	11.478	3.2887		3.2887		3.2887	
	0.00009	3.4998	13.314	3.49992		3.49992		3.49992	
	0.00002	3.7014	15.336	3.70149		3.70149		3.70149	
	0.00001	3.7758	16.166	3.77584		3.77584		3.77584	
	1	0	0	-		0		0	
	0.9	0.0031405	0.0002256	-		0.0031263		0.00312626	
	0.6	0.090824	0.031440	-		0.0908093		0.0908093	
	0.5	0.17014	0.079551	0.170896		0.170119		0.170119	
1.2	0.2	0.89522	0.95351	0.894997		0.894997		0.894997	
	0.12	1.5008	2.0884	1.50052		1.50052		1.50052	
	0.09	1.8974	2.9873	1.8971		1.8971		1.8971	
	0.04	3.2188	6.7252	3.21853		3.21853		3.21853	
	0.016	5.0743	13.580	5.07371		5.07371		5.07371	

	0.008	6.7511	21.108	6.75039		6.75039		6.75039	
	0.0007	14.969	71.621	14.9679		14.9679		14.9679	
	0.0003	18.915	102.00	18.9133		18.9133		18.9133	
	0.00009	25.806	162.24	25.8068		25.8068		25.8068	
	0.00002	37.115	276.53	37.114		37.114		37.114	
	0.00001	43.588	348.86	43.5864		43.5864		43.5864	
1.5	1	0	0	-		0		0	
	0.9	0.0014223	0.0001063	-		0.001415		0.00141499	
	0.6	0.68744	0.024911	-		0.0687335		0.0687335	
	0.5	0.14382	0.070593	0.144744		0.143806		0.143806	
	0.2	1.0791	1.2246	1.07855		1.07855		1.07855	
	0.12	2.0905	3.1279	2.08982		2.08982		2.08982	
	0.09	2.8481	4.8469	2.84747		2.84747		2.84747	
	0.04	5.8863	13.494	5.8856		5.8856		5.8856	
	0.016	11.480	34.276	11.478		11.478		11.478	
	0.008	17.919	63.267	17.9164		17.9164		17.9164	
	0.0007	70.970	396.60	70.9599		70.9599		70.9599	
	0.0003	110.82	704.16	110.81		110.81		110.81	
	0.00009	206.17	1541.1	206.135		206.135		206.135	
	0.00002	442.60	3943.2	442.515		442.515		442.515	
	0.00001	627.85	6016.0	627.753		627.753		627.753	
2.5	0.4	-	-	0.266253		0.266253		0.266253	
	0.09	-	-	14.7593		14.7593		14.7593	
	0.016	-	-	297.256		297.256		297.256	
	0.005	-	-	1822.5		1822.5		1822.5	
	0.00002	-	-	7452450		7452450		7452450	
3.01	0.0003	-	-	5984270		5984270		5984270	
	0.00009	-	-	67381800		67381800		67381800	
	0.00002	-	-	1385740000		1385740000		1385740000	
	0.00001	-	-	5581850000		5581850000		5581850000	
3.9	0.864	-	-	-		0.0000185799		0.0000185799	
	0.735	-	-	-		0.000819567		0.000819567	
	0.264	-	-	2.98829		2.98829		2.98829	
	0.125	-	-	66.8968		66.8968		66.8968	

Table 2. The comparative values of Einstein's integrals $J_1(z)$ and of $J_2(z)$ for integer z values

z	e	Einstein (1950) J2(Z) Tables-2,3 and 4	Guo and Julien (2004) Eqs. (9) and (11) J2(Z)	This Study J2(Z)
2	1	0	0	0
	0.9	-0.000032	-0.0000308262	-0.0000308262
	0.6	-0.01727	-0.0172712	-0.0172712
	0.5	-0.05927	-0.0592678	-0.0592678
	0.2	-1.9350	-1.93501	-1.93501
	0.001	-5862.0	-5862.03	-5862.03
	0.0001	-82024.0	-82020.6	-82020.6
3	1	0	0	0
	0.1	55.60	-62.8679	-62.8679
	0.01	19480	-19503.3	-19503.3
	0.001	3187×10^3	-3186.29×10^3	-3186.29×10^3
	0.0001	4353×10^5	-4352.71×10^5	-4352.71×10^5
	0.00001	5508×10^7	-5506.15×10^7	-5506.15×10^7
4	1	0	0	0
	0.1	363.2	-349.364	-349.364
	0.01	1343×10^3	-1343.89×10^3	-1343.89×10^3
	0.001	2177×10^6	-2178.69×10^6	-2178.69×10^6
	0.0001	2955×10^9	-2957.26×10^9	-2957.26×10^9
	0.00001	3723×10^{12}	-3726.31×10^{12}	-3726.31×10^{12}
5	1	0	0	0
	0.1	2602	-2662	-2662
	0.01	1019.8×10^5	-1019.61×10^5	-1019.61×10^5
	0.001	1653.5×10^9	-1653.51×10^9	-1653.51×10^9
	0.0001	2239×10^{13}	-2238.61×10^{13}	-2238.61×10^{13}
	0.00001	2816×10^{17}	-2815.55×10^{17}	-2815.55×10^{17}
6	1	0	0	0
	0.1	-	-19437.1	-19437.1
	0.01	-	-8178.02×10^6	-8178.02×10^6
	0.001	-	-1331.6×10^{12}	-1331.6×10^{12}
	0.0001	-	-1800.72×10^{17}	-1800.72×10^{17}
	0.00001	-	-2262.42×10^{22}	-2262.42×10^{22}
	0.000001	-	-2723.08×10^{27}	-2723.08×10^{27}

The numerical results show that the presented methods for solution of Einstein integrals are computationally efficient and can avoid computational and time overflow. Therefore, the method proposed in this work can be useful for the calculation of Einstein integrals for integer and non-integer values z. In particularly, this approximation may provide a simple and efficient way to incorporate Einstein bed load function into widely used hydraulic software.

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