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I-WEIGHTED LACUNARY STATISTICAL τ-CONVERGENCE IN LOCALLY SOLID RIESZ SPACES

ŞÜKRAN KONCA AND ERGIN GENÇ

Abstract. An ideal I is a family of subsets of positive integers $\mathbb{N}$ which is closed under taking finite unions and subsets of its elements. In this paper, we introduce the notions of ideal versions of weighted lacunary statistical τ-convergence, statistical τ-Cauchy, weighted lacunary τ-boundedness of sequences in locally solid Riesz spaces endowed with the topology τ. We also prove some topological results related to these concepts in locally solid Riesz space.

1. Introduction

A Riesz space is an ordered vector space which is lattice at the same time. A locally solid Riesz space is a Riesz space equipped with a linear topology that has a base consisting of solid sets. The Riesz space was first introduced by F. Riesz in 1928, at the International Mathematical Congress in Bologna [1]. Soon after, in the mid-thirties, H. Freudental [2] and L. V. Kantrovich [3] independently set up the axiomatic foundation and derived a number of properties dealing with the lattice structure of Riesz space. Riesz space have many applications in measure theory, operator theory and optimization. They also have some applications in economics [4]. For further results we may refer to [5, 6, 7].

The notion of statistical convergence was introduced by Fast [8] and Steinhaus [9] independently in the same year 1951. Recently, a new concept has been introduced by Başarır and Konca [10] for sequences which is called weighted lacunary statistical convergence (also for the concept weighted statistical convergence see [11, 12]). Then they extended this concept to locally solid Riesz space in [13]. The concept of I-convergence (I denotes the ideal of subsets of $\mathbb{N}$) was initially introduced by Kostyrko et al. [14] as a generalization of statistical convergence. More applications of ideals can be seen in [15, 16, 17, 18].

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In this paper, we define I-weighted lacunary statistical τ-convergence, I-weighted lacunary statistical τ-boundedness and examine some inclusion relations in locally solid Riesz space.

2. Definitions and Preliminaries

Let X be a real vector space and "$\leq$" be a partial order on this space. Then X is said to be an ordered vector space if it satisfies the following properties:

(1) $\forall x, y \in X$ and $y \leq x$ imply $y + z \leq x + z$ for each $z \in X$,
(2) $\forall x, y \in X$ and $y \leq x$ imply $\alpha y \leq \alpha x$ for each $\alpha \geq 0$.

In addition, if X is a lattice with respect to the partial order, then X is said to be a Riesz space (or a vector lattice) [7].

A subset S of a Riesz space X is said to be solid if $y \in S$ and $|x| \leq |y|$ implies $x \in X$. A topological vector space (X, τ) is a vector space X which has a linear topology τ such that the algebraic operations of additions and scalar multiplication in X are continuous. Every linear topology τ on a vector space X has a base $\mathcal{N}_{sol}$ for the neighborhoods of zero satisfying the following properties:

(1) Each $Y \in \mathcal{N}_{sol}$ is a balanced set, that is, $\alpha x \in Y$ holds for all $x \in Y$ and every $\alpha \in \mathbb{R}$ with $|\alpha| \leq 1$.
(2) Each $Y \in \mathcal{N}_{sol}$ is an absorbing set, that is, for every $x \in X$ there exists $\alpha > 0$ such that $\alpha x \in Y$.
(3) For each $Y \in \mathcal{N}_{sol}$, there exists some $W \in Y$ with $W + W \subseteq Y$.

A linear topology τ on a Riesz space X is said to be locally solid Riesz space if τ has a base at zero consisting of solid sets. A locally solid Riesz space (X, τ) is a Riesz space equipped with a locally solid topology τ [7].

Throughout the paper, the symbol $\mathcal{N}_{sol}$ will denote any base at zero consisting of solid sets and satisfying the above conditions (1), (2), (3) in a locally solid Riesz topology τ. For our convenience, here and in where follows, we shall write a word "LSRS" instead of a locally solid Riesz space and we mean $\lim_{k\to\infty} x_k$ by $\lim x$ for brevity.

Let $E \subseteq \mathbb{N}$. Then the natural density of E is denoted by $\delta(E)$ and defined by

$$\delta(E) = \lim_{x\to\infty} |\{k \leq n : k \in E\}|,$$

where the vertical bars denote the cardinality of the respective set [19].

Let (X, τ) be a locally solid Riesz space. A sequence $x = (x_k)$ in X is said to be $S(\tau)$-convergent to an element x_0 in X if for each τ- neighborhood V of zero

$$\delta(\{k \in \mathbb{N} : x_k - x_0 \notin V\}) = 0$$

i.e.,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : x_k - x_0 \notin V\}| = 0.$$

In this case, we write $S(\tau)$-$\lim x = x_0$ or $(x_k) \overset{S(\tau)}{\to} x_0$ [20].

By a lacunary sequence we mean an increasing sequence of integers $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout the paper, the intervals determined by θ will be denoted by $I_r := (k_{r-1}, k_r]$ and the ratio k_r/k_{r-1} will be abbreviated by q_r.

Let (p_k) be a sequence of positive real numbers and $P_n = p_1+p_2+...+p_n$ for $n \in \mathbb{N}$. Then the Riesz transformation of $x = (x_k)$ is defined as $t_n = 1/P_n \sum_{k=1}^{n} p_k x_k$. If the transformation sequence (t_n) has a finite limit x_0 then the sequence $x = (x_k)$ is said to be Riesz convergent to x_0. Let us note that if $P_n \to \infty$ as $n \to \infty$ then Riesz mean is a regular summability method. Throughout the paper, let $P_n \to \infty$ as $n \to \infty$ and $P_n = p_0 = 0$ [11].

Let θ be a lacunary sequence of positive real numbers such that $H_r := \sum_{k \in I_r} p_k$, $P_{k_r} := \sum_{k \in (0,k_r]} p_k$, $P_{k_{r-1}} := \sum_{k \in (0,k_{r-1}]} p_k$, $Q_r := \frac{P_{k_r}}{P_{k_{r-1}}}$, $P_0 = 0$ and the intervals determined by θ and p_k are denoted by $I_r^{'} = \left(P_{k_{r-1}}, P_{k_r}\right]$. It is easy to see that $H_r = P_{k_r} - P_{k_{r-1}}$. If we take $p_k = 1$ for all $k \in \mathbb{N}$, then $H_r, P_{k_r}, P_{k_{r-1}}, Q_r$ and $I_r^{'}$ reduce to h_r, k_r, k_{r-1}, q_r and I_r respectively. Throughout the paper we assume that $P_n \to \infty$ as $n \to \infty$ such that $H_r \to \infty$ as $n \to \infty$ [10].

A family of sets $I \subseteq P(\mathbb{N})$ (power sets of $\mathbb{N}$) is said to be an ideal if;

(1) $\emptyset \in I$,
(2) $A \cup B \in I$, for each $A, B \in I$,
(3) $A \in I$ and $B \subseteq A$ imply $B \in I$.

A non-empty family of sets $F \subseteq P(\mathbb{N})$ is a filter on $\mathbb{N}$ if and only if:

(1) $\emptyset \notin F$,
(2) $A \cap B \in F$, for each $A, B \in F$,
(3) $A \in F$ and $B \supset A$ imply $B \in F$.

I is called non-trivial if $I \neq \emptyset$ and $\mathbb{N} \notin I$. It is called admissible ideal when it contains all singletons $\{\{n\} : n \in \mathbb{N}\}$ [14]. We assume that I is non-trivial admissible ideal of $\mathbb{N}$ throughout the paper.

If we take $I = I_f = \{A \subseteq \mathbb{N} :$ A is finite subset$\}$. Then I_f is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincide with the usual convergence.

A sequence $x = (x_k)$ in a topological space X is said to be I-convergent to $x_0 \in X$ if for every neighborhood V of x_0

$$\{k \in \mathbb{N} : x_k \notin V\} \in I.$$

In this case, we write $I - \lim x = x_0$ or $x_k \overset{I}{\to} x_0$ and I denotes the set of all ideal convergent sequences [16].

Let (X, τ) be a locally solid Riesz space. A sequence $x = (x_k)$ in X is said to be $I(\tau)$-convergent to an element x_0 in X if for each τ-neighborhood V of zero

$$\{k \in \mathbb{N} : x_k - x_0 \notin V\} \in I$$

i.e.,

$$\{k \in \mathbb{N} : x_k - x_0 \in V\} \in F.$$

In this case, we write $I(\tau)\text{-}\lim x = x_0$ or $x_k \overset{I(\tau)}{\to} x_0$ [15].

3. Main Results

Definition 3.1. Let (X,τ) be a LSRS and θ be a lacunary sequence. We say that $x=(x_k)$ in X is said to be $S^{I}_{(\bar{N},\theta)}(\tau)$-convergent to $x_0\in X$ if for every neighborhood V of zero and $\delta>0$,

$$\left\{r\in\mathbb{N}:\frac{1}{H_r}\left|\left\{k\in I_r^{'}:p_k\left(x_k-x_0\right)\notin V\right\}\right|\geq\delta\right\}\in I, \tag{3.1}$$

that is;

$$\left\{r\in\mathbb{N}:\frac{1}{H_r}\left|\left\{k\in I_r^{'}:p_k\left(x_k-x_0\right)\notin V\right\}\right|<\delta\right\}\in F\left(I\right).$$

In this case, we write $S^{I}_{(\bar{N},\theta)}(\tau)$ or $x_k\overset{S^{I}_{(\bar{N},\theta)}(\tau)}{\rightarrow}x_0$.

Remark 3.2. (1) If we take $p_k=1$ for all $k\in\mathbb{N}$ in (3.1) then we obtain the definition of I-lacunary statistical τ-convergence which can be given as follows:

Let (X,τ) be a LSRS and $\theta=(k_r)$ a lacunary sequence. Then a sequence $x=(x_k)$ in X is said to be $S^{I}_{\theta}(\tau)$-convergent to x_0 if for every τ-neighborhood V of zero and for any $\delta>0$,

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\left|\left\{x\in I_r:x_k-x_0\notin V\right\}\right|\geq\delta\right\}\in I.$$

In this case, we write $S^{I}_{\theta}(\tau)-\lim_{k\to\infty}x=x_0$ or $x_k\overset{S^{I}_{\theta}(\tau)}{\rightarrow}x_0$.

(2) For $I=I_{fin}$ and $p_k=1$ for all $k\in\mathbb{N}$, $S^{I}_{(\bar{N},\theta)}(\tau)$-convergence reduces to $x_k\overset{S_{\theta}(\tau)}{\rightarrow}x_0$ in [21].

Now, we give the definition of I-weighted statistical τ-convergence in LSRS.

Definition 3.3. Let (X,τ) be a LSRS and θ be a lacunary sequence. We say that $x=(x_k)$ in X is said to be $S^{I}_{\bar{N}}(\tau)$-convergent to $x_0\in X$ if for every neighborhood V of zero and $\delta>0$, $\left\{r\in\mathbb{N}:\frac{1}{P_n}\left|\left\{k\leq P_n:p_k\left(x_k-x_0\right)\notin V\right\}\right|\geq\delta\right\}\in I$. In this case, we write $S^{I}_{\bar{N}}(\tau)-\lim_{k\to\infty}x=x_0$ or $x_k\overset{S^{I}_{\bar{N}}(\tau)}{\rightarrow}x_0$.

Definition 3.4. Let (X,τ) be a LSRS. We say that $x=(x_k)$ in X is said to be $S^{I}_{(\bar{N},\theta)}(\tau)$-bounded if for every neighborhood V of zero and there exists $\alpha>0$ such that, for any $\delta>0$,

$$\left\{r\in\mathbb{N}:\frac{1}{H_r}\left|\left\{k\in I_r^{'}:\alpha p_kx_k\notin V\right\}\right|\geq\delta\right\}\in I.$$

Theorem 3.5. Let (X,τ) be a Hausdorff LSRS and $x=(x_k)$, $y=(y_k)$ be two sequences in X. Then the followings hold:

(1) If $S^{I}_{(\bar{N},\theta)}(\tau)-\lim x=x_0$ and $S^{I}_{(\bar{N},\theta)}(\tau)-\lim x=y_0$, then $x_0=y_0$.
(2) If $S^{I}_{(\bar{N},\theta)}(\tau)-\lim x=x_0$, then $S^{I}_{(\bar{N},\theta)}(\tau)-\lim \alpha x=\alpha x_0$, for every $\alpha\in\mathbb{R}$,
(3) If $S^{I}_{(\bar{N},\theta)}(\tau)-\lim x=x_0$ and $S^{I}_{(\bar{N},\theta)}(\tau)-\lim y=y_0$, then $S^{I}_{(\bar{N},\theta)}(\tau)-\lim x+y=x_0+y_0$.

Proof. (1) Suppose that $S^{I}_{(\bar{N},\theta)}(\tau)-\lim x = x_0$ and $S^{I}_{(\bar{N},\theta)}(\tau)-\lim x = y_0$. Let V be any τ-neighborhood of zero. Then there exists a $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq V$. Choose $W \in \mathcal{N}_{sol}$ such that $W+W \subseteq Y$. Since $S^{I}_{(\bar{N},\theta)}(\tau)-\lim x = x_0$ and $S^{I}_{(\bar{N},\theta)}(\tau)-\lim x = y_0$, we can write for $\delta > 0$,

$$A_1 = \left\{ r \in \mathbb{N} : \frac{1}{H_r} \left| \left\{ k \in I_r' : p_k (x_k - x_0) \notin V \right\} \right| < \delta \right\} \in F(I),$$

and

$$A_2 = \left\{ r \in \mathbb{N} : \frac{1}{H_r} \left| \left\{ k \in I_r' : p_k (x_k - y_0) \notin V \right\} \right| < \delta \right\} \in F(I).$$

Since $S^{I}_{(\bar{N},\theta)}(\tau)-\lim x = x_0$ and $S^{I}_{(\bar{N},\theta)}(\tau)-\lim x = y_0$, we get $A_1, A_2 \in F(I)$ and for every $r \in A_1$ and A_2

$$\frac{1}{H_r} \left| \left\{ k \in I_r' : p_k (x_k - x_0) \notin W \right\} \right| < \frac{\delta}{2},$$

similarly

$$\frac{1}{H_r} \left| \left\{ k \in I_r' : p_k (x_k - y_0) \notin W \right\} \right| < \frac{\delta}{2}.$$

Now, let $A_1 \cap A_2 = A$. Then we have

$$p_k (x_0 - y_0) = p_k (x_0 - x_k + x_k - y_0) = p_k (x_k - x_0) + p_k (x_k - y_0) \in W+W \subseteq Y \subseteq V.$$

Thus

$$\frac{1}{H_r} \left| \left\{ k \in I_r' : p_k (x_k - x_0) \notin W \right\} \right| \leq \frac{1}{H_r} \left| \left\{ k \in I_r' : p_k (x_k - x_0) \notin W \right\} \right| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Consequently, we obtain

$$\left\{ r \in \mathbb{N} : \frac{1}{H_r} \left| \left\{ k \in I_r' : p_k (x_k - y_0) \notin V \right\} \right| < \delta \right\} \in F(I).$$

Finally, for every τ-neighborhood V of zero, we get $p_k(x_0 - y_0) \in V$. Since (p_k) is a sequence of positive real numbers and (X, τ) is Hausdorff, the intersection of all τ-neighborhood V of zero is the singleton set $\{\theta\}$. Thus we get $x_0 - y_0 = \theta$, i.e., $x_0 = y_0$.

(2) We assume that $S^{I}_{(\bar{N},\theta)}(\tau)-\lim x = x_0$ and let V be an arbitrary $\tau-$neighbourhood of zero. Then there exists $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq V$. Since $S^{I}_{(\bar{N},\theta)}(\tau) - \lim x = x_0$, we can write for any $\delta > 0$

$$B = \left\{ r \in \mathbb{N} : \frac{1}{H_r} \left| \left\{ k \in I_r' : p_k (x_k - x_0) \notin Y \right\} \right| < \delta \right\} \in F(I).$$

Since Y is balanced, $p_k (x_k - x_0) \in Y$ implies that $\alpha p_k (x_k - x_0) \in Y$ for every $\alpha \in \mathbb{R}$ with $|\alpha| < 1$. Therefore

$$\begin{aligned} \left\{ k \in I_r' : p_k (x_k - x_0) \in Y \right\} &\subseteq \left\{ k \in I_r' : \alpha p_k (x_k - x_0) \in Y \right\} \\ &\subseteq \left\{ k \in I_r' : \alpha p_k (x_k - x_0) \in V \right\}. \end{aligned}$$

Thus we obtain

$$\left\{ r \in \mathbb{N} : \frac{1}{H_r} \left| \left\{ k \in I_r' : \alpha p_k (x_k - x_0) \notin V \right\} \right| < \delta \right\} \in F(I)$$

for each τ-neighborhood V of zero.

Now let $|\alpha| > 1$ and $[\![\alpha]\!]$ be the smallest integer greater than or equal to $|\alpha|$. Then there exists $W \in N_{sol}$ such that $[\![\alpha]\!]\, W \subseteq Y$. Since $S^{I}_{(\bar{N},\theta)}(\tau)-\lim x = x_0$, we have

$$B = \left\{ r \in \mathbb{N} : \frac{1}{H_r} \left| \left\{ k \in I_r^{'} : p_k\left(x_k - x_0\right) \notin Y \right\} \right| < \delta \right\} \in F\left(I\right).$$

We also have,

$$\left|\alpha p_k\left(x_k - x_0\right)\right| = |\alpha|\left|p_k\left(x_k - x_0\right)\right| \leq [\![\alpha]\!]\left|p_k\left(x_k - x_0\right)\right| \in [\![\alpha]\!]\, W \subseteq Y \subseteq V.$$

Since Y is solid, we have $\alpha p_k\left(x_k - x_0\right) \in Y$ this implies that $\alpha p_k\left(x_k - x_0\right) \in V$. Consequently,

$$\left\{ r \in \mathbb{N} : \frac{1}{H_r} \left| \left\{ k \in I_r^{'} : \alpha p_k\left(x_k - x_0\right) \notin V \right\} \right| < \delta \right\} \in F\left(I\right),$$

for each τ-neighborhood V of zero. Hence $S^{I}_{(\bar{N},\theta)}(\tau) - \lim \alpha x = \alpha x_0$, for every $\alpha \in \mathbb{R}$.

(3) Let V be an arbitrary $\tau-$neighbourhood of zero. Then there exists $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq V$. We choose $W \in \mathcal{N}_{sol}$ such that $W + W \subseteq Y$. Since $S^{I}_{(\bar{N},\theta)}(\tau) - \lim x = x_0$ and $S^{I}_{(\bar{N},\theta)}(\tau) - \lim y = y_0$, we have

$$C_1 = \left\{ r \in \mathbb{N} : \frac{1}{H_r} \left| \left\{ k \in I_r^{'} : p_k\left(x_k - x_0\right) \notin V \right\} \right| < \frac{\delta}{2} \right\} \in F\left(I\right),$$

and

$$C_2 = \left\{ r \in \mathbb{N} : \frac{1}{H_r} \left| \left\{ k \in I_r^{'} : p_k\left(y_k - y_0\right) \notin V \right\} \right| < \frac{\delta}{2} \right\} \in F\left(I\right).$$

Let $C_1 \cap C_2 = C$. Hence we have also,

$$p_k\left(\left(x_k + y_k\right) - \left(x_0 + y_0\right)\right) = p_k\left(x_k - x_0\right) + p_k\left(y_k - y_0\right) \in W + W \subseteq Y \subseteq V,$$

for $r \in C_1 \cap C_2$ we get

$$\begin{aligned} \tfrac{1}{H_r} \left| \left\{ k \in I_r^{'} : p_k\left(\left(x_k + y_k\right) - \left(x_0 + y_0\right)\right) \notin W \right\} \right| &\leq \tfrac{1}{H_r} \left| \left\{ k \in I_r^{'} : p_k\left(x_k - x_0\right) \notin W \right\} \right| \\ &+ \tfrac{1}{H_r} \left| \left\{ k \in I_r^{'} : p_k\left(y_k - y_0\right) \notin W \right\} \right| < \tfrac{\delta}{2} + \tfrac{\delta}{2} = \delta. \end{aligned}$$

Consequently,

$$\begin{aligned} &\left\{ r \in \mathbb{N} : \tfrac{1}{H_r} \left| \left\{ k \in I_r^{'} : p_k\left(\left(x_k + y_k\right) - \left(x_0 + y_0\right)\right) \notin V \right\} \right| < \delta \right\} \\ &\subseteq \left\{ r \in \mathbb{N} : \tfrac{1}{H_r} \left| \left\{ k \in I_r^{'} : p_k\left(y_k - y_0\right) \notin V \right\} \right| < \tfrac{\delta}{2} \right\} \\ &\cup \left\{ r \in \mathbb{N} : \tfrac{1}{H_r} \left| \left\{ k \in I_r^{'} : p_k\left(x_k - x_0\right) \notin V \right\} \right| < \tfrac{\delta}{2} \right\} \end{aligned}.$$

Since $S^{I}_{(\bar{N},\theta)}(\tau) - \lim x = x_0$ and $S^{I}_{(\bar{N},\theta)}(\tau) - \lim y = y_0$, it is obviously seen that the left side belongs to $F(I)$. This shows that $S^{I}_{(\bar{N},\theta)}(\tau) - \lim x + y = x_0 + y_0$.

□

Theorem 3.6. Let (X,τ) be a LSRS and $\theta=(k_r)$ be a lacunary sequence. If a sequence $x=(x_k)$ is $S^I_{(\bar{N},\theta)}(\tau)$-convergent and (p_k) is bounded, then $x=(x_k)$ is $S^I_{(\bar{N},\theta)}(\tau)$-bounded.

Proof. Suppose that $x=(x_k)$ is $S^I_{(\bar{N},\theta)}(\tau)-\lim x=x_0$ and (p_k) is bounded sequence. Let V be an arbitrary τ-neighborhood of zero. Then there exists $Y\in\mathcal{N}_{sol}$ such that $Y\subseteq V$. We choose another element $W\in\mathcal{N}_{sol}$ such that $W+W\subseteq Y$. Since $S^I_{(\bar{N},\theta)}(\tau)-\lim x=x_0$, we have;

$$D=\left\{r\in\mathbb{N}:\frac{1}{H_r}\left|\left\{k\in I_r^{'}:p_k\left(x_k-x_0\right)\notin W\right\}\right|\leq\delta\right\}\in I.$$

Since W is absorbing, there exists $a>0$ such that $ax_0\in W$. Let $b\leq 1$ and $b\leq a$. Since (p_k) is bounded then there exists a $M=\frac{a}{b}>0$ such that $p_k\leq M$ for all $k\in\mathbb{N}$. Then we can write $bp_k\leq a$ for all $k\in\mathbb{N}$. Since W is balanced $p_k(x_k-x_0)\in W$ implies that $bp_k(x_k-x_0)\in W$. Then we have $bp_kx_k=bp_k(x_k-x_0)+bp_kx_0\in W+W\subseteq Y\subseteq V$ for each $k\in\mathbb{N}\backslash D$. Thus,

$$\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\leq n:bx_k\notin V\}\right|\geq\delta\right\}\in I.$$

This shows that $x=(x_k)$ is $S^I(\tau)$-bounded. □

Theorem 3.7. Let (X,τ) be a LSRS, $\theta=(k_r)$ be a lacunary sequence and let $(x_k),(y_k),(z_k)$ be the sequences in X such that;

(1) $x_k\leq y_k\leq z_k$, for each $k\in\mathbb{N}$,
(2) $S^I_{(\bar{N},\theta)}(\tau)-\lim x=x_0=S^I_{(\bar{N},\theta)}(\tau)-\lim z=x_0$, then $S^I_{(\bar{N},\theta)}(\tau)-\lim y=x_0$.

Proof. Let V be an arbitrary $\tau-$neighbourhood of zero. Then there exists $Y\in\mathcal{N}_{sol}$ such that $Y\subseteq V$. Choose $W\in\mathcal{N}_{sol}$ such that $W+W\subseteq Y$. From the condition (2), we have $D_1,D_2\in F(I)$ where

$$E_1=\left\{r\in\mathbb{N}:\frac{1}{H_r}\left|\left\{k\in I_r^{'}:p_k\left(x_k-x_0\right)\notin W\right\}\right|<\delta\right\}\in F\left(I\right),$$

and

$$E_2=\left\{r\in\mathbb{N}:\frac{1}{H_r}\left|\left\{k\in I_r^{'}:p_k\left(z_k-z_0\right)\notin W\right\}\right|<\delta\right\}\in F\left(I\right).$$

We also get $E=E_1\cap E_2\in F(I)$ and from (i), we have

$$p_k(x_k-x_0)\leq p_k(y_k-y_0)\leq p_k(z_k-z_0)$$

for all $k\in\mathbb{N}$. This implies that for all $k\in K_1\cap K_2$,

$$\Longrightarrow |y_k-y_0|\leq|x_k-x_0|+|z_k-z_0|\in W+W\subseteq Y\subseteq V.$$

Since Y is solid we have $p_k(y_k-y_0)\in Y\subseteq V$. Thus

$$\left\{r\in\mathbb{N}:\frac{1}{H_r}\left|\left\{k\in I_r^{'}:p_k\left(y_k-x_0\right)\notin W\right\}\right|<\delta\right\}\in F\left(I\right).$$

Hence $S^I_{(\bar{N},\theta)}(\tau)-\lim y=x_0$. This completes the proof of the theorem. □

Theorem 3.8. Let (X,τ) be a LSRS and $\theta=(k_r)$ be a lacunary sequence. If $\lim\inf_r Q_r>1$ then $S^I_{(\bar{N})}(\tau)\subseteq S^I_{(\bar{N},\theta)}(\tau)$.

Proof. Suppose that $\liminf_r Q_r > 1$, then there exists $\lambda > 0$ such that $Q_r \geq 1+\lambda$ for sufficiently large values of r which implies that $\frac{H_r}{P_{k_r}} = 1 - \frac{P_{k_r}}{P_{k_{r-1}}} = 1 - \frac{1}{Q_r} \geq \frac{\lambda}{1+\lambda}$. Let $S^I_{(\bar{N})}(\tau) - \lim x = x_0$ and V be an arbitrary τ-neighbourhood of zero. Since $S^I_{(\bar{N})}(\tau) - \lim x = x_0$, we have

$$\left\{ r \in \mathbb{N} : \frac{1}{P_{k_r}} \left| \{ k \in P_{k_r} : p_k (x_k - x_0) \notin V \} \right| < \delta \right\}.$$

Then for all $r > r_0$ we have

$$\begin{aligned} & \tfrac{1}{P_{k_r}} \left| \{ k \in P_{k_r} : p_k (x_k - x_0) \notin V \} \right| \\ & \geq \tfrac{1}{P_{k_r}} \left| \{ k_{r-1} < k \leq k_r : p_k (x_k - x_0) \notin V \} \right| \\ & = \tfrac{H_r}{P_{k_r}} \left(\tfrac{1}{H_r} \left| \left\{ k \in I'_r : p_k (x_k - x_0) \notin V \right\} \right| \right) \\ & \geq \tfrac{\lambda}{\lambda+1} \left(\tfrac{1}{H_r} \left| \left\{ k \in I'_r : p_k (x_k - x_0) \notin V \right\} \right| \right) \end{aligned}$$

i.e., for any $\delta > 0$

$$\left\{ r \in \mathbb{N} : \tfrac{\lambda}{\lambda+1} \tfrac{1}{H_r} \left| \left\{ k \in I'_r : p_k (x_k - x_0) \notin V \right\} \right| < \delta \right\}$$

$$\subseteq \left\{ r \in \mathbb{N} : \tfrac{1}{P_{k_r}} \left| \{ k \leq P_{k_r} : p_k (x_k - x_0) \notin V \} \right| < \delta \right\}.$$

Since $S^I_{(\bar{N})}(\tau) - \lim x = x_0$, then the left side of inequality belongs to $F(I)$. Consequently $S^I_{(\bar{N},\theta)}(\tau) - \lim x = x_0$. Hence $S^I_{(\bar{N})}(\tau) \subseteq S^I_{(\bar{N},\theta)}(\tau)$. □

Theorem 3.9. Let (X, τ) be a LSRS and a sequence $x = (x_k)$ in X. For any lacunary sequence $\theta = (k_r)$, the following statements are true:

(1) If $p_k \leq 1$ for all $k \in N$ then $S^I_{(\theta)}(\tau) \subseteq S^I_{(\bar{N},\theta)}(\tau)$ and $S^I_{(\theta)}(\tau) - \lim x = S^I_{(\bar{N},\theta)}(\tau) - \lim x = x_0$.

(2) If $1 \leq p_k$ for all $k \in N$ and $\frac{H_r}{h_r}$ is upper bounded then $S^I_{(\bar{N},\theta)}(\tau) \subseteq S^I_{(\theta)}(\tau)$ and $S^I_{(\bar{N},\theta)}(\tau) - \lim x = S^I_{(\theta)}(\tau) - \lim x = x_0$.

Proof. (1) If $p_k \leq 1$ for all $k \in \mathbb{N}$ then $H_r \leq h_r$ for all $r \in \mathbb{N}$. So there exists M constant such that $0 < M \leq \frac{H_r}{h_r} < 1$ for all $r \in \mathbb{N}$. Let a sequence $x = (x_k)$ in a LSRS (X, τ) and assume that $S^I_{(\theta)}(\tau) - \lim x = x_0$. For an arbitrary τ-neighborhood V of zero, then we have

$$\begin{aligned} & \tfrac{1}{H_r} \left| \left\{ k \in I'_r : p_k (x_k - x_0) \notin V \right\} \right| \\ & = \tfrac{1}{H_r} \left| \left\{ P_{k_{r-1}} < k \leq P_{k_r} : p_k (x_k - x_0) \notin V \right\} \right| \\ & \leq \tfrac{1}{M} \tfrac{1}{h_r} \left| \{ k_{r-1} < k \leq k_r : (x_k - x_0) \notin V \} \right| \\ & = \tfrac{1}{M} \tfrac{1}{h_r} \left| \{ k \in I_r : (x_k - x_0) \notin V \} \right|. \end{aligned}$$

i.e.,

$$\frac{1}{H_r} \left| \left\{ k \in I'_r : p_k (x_k - x_0) \notin V \right\} \right| \leq \frac{1}{M} \frac{1}{h_r} \left| \{ k \in I_r : (x_k - x_0) \notin V \} \right|.$$

For any $\delta > 0$

$$\left\{r \in \mathbb{N} : \tfrac{1}{H_r} \left|\left\{k \in I_r^{'} : p_k\left(x_k - x_0\right) \notin V\right\}\right| \geq \delta\right\}$$
$$\subseteq \left\{r \in \mathbb{N} : \tfrac{1}{M} \tfrac{1}{h_r} \left|\left\{k \in I_r : \left(x_k - x_0\right) \notin V\right\}\right| \geq \delta\right\}.$$

Since $S^I_{(\theta)}(\tau) - \lim x = x_0$, on the right side of inequality belongs to I. In this case, on the left side of inequality belongs I. This shows that $S^I_{(\bar{N},\theta)}(\tau) - \lim x = x_0$.

(2) If $1 \leq p_k$ for all $k \in \mathbb{N}$, then $h_r \leq H_r$ for all $r \in \mathbb{N}$. Since $\frac{H_r}{h_r}$ is upper bounded, there exists N constant such that $1 \leq \frac{H_r}{h_r} \leq N$ for all $r \in \mathbb{N}$. Assume that $S^I_{(\bar{N},\theta)}(\tau) - \lim x = x_0$. For an arbitrary τ-neighborhood V of zero we have

$$\tfrac{1}{h_r} \left|\left\{k \in I_r : \left(x_k - x_0\right) \notin V\right\}\right|$$
$$= \tfrac{1}{h_r} \left|\left\{k_{r-1} < k \leq k_r : \left(x_k - x_0\right) \notin V\right\}\right|$$
$$\leq \tfrac{1}{N} \tfrac{1}{H_r} \left|\left\{P_{k_{r-1}} < k \leq P_{k_r} : p_k\left(x_k - x_0\right) \notin V\right\}\right|$$
$$= \tfrac{1}{N} \tfrac{1}{H_r} \left|\left\{k \in I_r^{'} : p_k\left(x_k - x_0\right) \notin V\right\}\right|,$$

that is,

$$\frac{1}{h_r} \left|\left\{k \in I_r : \left(x_k - x_0\right) \notin V\right\}\right| \leq \frac{1}{N} \frac{1}{H_r} \left|\left\{k \in I_r^{'} : p_k\left(x_k - x_0\right) \notin V\right\}\right|.$$

Consequently, for any $\delta > 0$

$$\left\{r \in \mathbb{N} : \tfrac{1}{h_r} \left|\left\{k \in I_r : \left(x_k - x_0\right) \notin V\right\}\right| \geq \delta\right\}$$
$$\subseteq \left\{r \in \mathbb{N} : \tfrac{1}{N} \tfrac{1}{H_r} \left|\left\{k \in I_r^{'} : p_k\left(x_k - x_0\right) \notin V\right\}\right| \geq \delta\right\}.$$

Since $S^I_{(\bar{N},\theta)}(\tau) - \lim x = x_0$, then the left side of inequality belongs to I. This shows that $S^I_{(\theta)}(\tau) - \lim x = x_0$.

□

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(Şükran Konca) Bitlis Eren University, Mathematics Institute of Science, 13000, Bitlis, Turkey
E-mail address, Şükran Konca: skonca@beu.edu.tr

(Ergın Genç) Bitlis Eren University, Mathematics Institute of Science, 13000, Bitlis, Turkey
E-mail address, Ergın Genç: ergingenc4444@gmail.com