STATISTICAL CONVERGENCE OF WEIGHT \( g \) IN A LOCALLY SOLID RIESZ SPACE

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Abstract. In this work, we introduce the notions of statistical convergence and lacunary statistical convergence of weight \( g \) in a locally solid Riesz space and establish some inclusion relations.

1. Introduction

The Riesz space was first introduced by F. Riesz in 1928, at the International Mathematical Congress in Bologna [1]. Soon after, in the mid-thirties, H. Freudental [2] and L. V. Kantrovich [3] independently set up the axiomatic foundation and derived a number of properties dealing with the lattice structure of Riesz space. Riesz space have many applications in measure theory, operator theory and optimization. They also have some applications in economics [4], we may refer to [5, 6, 7, 8].

Recently, Balcerzak et al. [9] show that one can further extend the concept of natural or asymptotic density (as well as natural density of order \( \alpha \)) by considering natural density of weight \( g \) where \( g : N \to [0, \infty] \) is a function with \( \lim_{n \to \infty} g(n) = \infty \) and \( \frac{n}{g(n)} \) does not go to 0 as \( n \to \infty \) (Throughout the paper by N, R and C, we will denote the set of all natural, real and complex numbers, respectively). We denote by \( G \), the set of all such functions \( g \).

In this work, we introduce the notions of statistical convergence and lacunary statistical convergence with respect to weight \( g \) in locally solid Riesz space and establish some inclusion relations.

2. Definitions and Preliminaries

Let \( E \subseteq N \). Then the natural density of \( E \) is denoted by \( \delta(E) \) and defined by

\[
\delta(E) = \lim_{n \to \infty} \frac{|\{k \in E : k \leq n\}|}{n},
\]

where the vertical bars denote the cardinality of the respective set [10].

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\[\text{Key words and phrases.} \quad \text{density, locally solid Riesz space, statistical convergence, topological vector space, weight} \ g.\]
A sequence \( x = (x_k) \) of real numbers is said to be statistically convergent to \( x_0 \) if for arbitrary \( \epsilon > 0 \), the set \( A(\epsilon) = \{ n \in N : |x_n - x_0| \geq \epsilon \} \) has natural density [9].

Let \( g : N \to [0, \infty) \) be a function with \( \lim_{n \to \infty} g(n) = \infty \). The upper density of weight \( g \) was defined by the formula

\[
\delta_g(A) = \limsup_{n \to \infty} \frac{A(1,n)}{g(n)}
\]

for \( A \subseteq N \) where \( A(1,n) \) denotes the cardinality of the set \( A \cap [1,n] \). If the \( \lim_{n \to \infty} A(1,n)/g(n) \) exists then we say that the density of weight \( g \) of the set \( A \) exists and we denote it by \( \delta_g(A) \).

Let \( X \) be a real vector space and "\( \leq \)" be a partial order on this space if it satisfies the following properties:

1. \( \forall x, y \in X \) and \( y \leq x \) imply \( y + z \leq x + z \) for each \( z \in X \),
2. \( \forall x, y \in X \) and \( y \leq x \) imply \( \alpha y \leq \alpha x \) for each \( \alpha \geq 0 \).

In addition, if \( X \) is lattice with respect to the partial order, then \( X \) is said to be a Riesz space (or a vector lattice) [7].

A subset \( S \) of a Riesz space \( X \) is said to be solid if \( y \in S \) and \( |x| \leq |y| \) implies \( x \in X \).

A topological vector space \((X, \tau)\) is a vector space \( X \) which has a linear topology \( \tau \) such that the algebraic operations of additions and scalar multiplication in \( X \) are continuous.

Every linear topology \( \tau \) on a vector space \( X \) has a base \( \mathcal{N}_{sol} \) for the neighbourhoods of zero satisfying the following properties:

1. Each \( Y \in \mathcal{N}_{sol} \) is a balanced set, that is, \( \alpha x \in Y \) holds for all \( x \in Y \) and every \( \alpha \in \mathbb{R} \) with \( |\alpha| \leq 1 \).
2. Each \( Y \in \mathcal{N}_{sol} \) is an absorbing set, that is, for every \( x \in X \) there exists \( \alpha > 0 \) such that \( \alpha x \in Y \).
3. For each \( Y \in \mathcal{N}_{sol} \) there exists some \( W \in Y \) with \( W + W \subseteq Y \).

A linear topology \( \tau \) on a Riesz space \( X \) is said to be locally solid Riesz space if \( \tau \) has a base at zero consisting of solid sets. A LSRS \((X, \tau)\) is a Riesz space equipped with a locally solid topology \( \tau \).

Throughout the paper, the symbol \( \mathcal{N}_{sol} \) will denote any base at zero consisting of solid sets and satisfying the above conditions (1), (2), (3) in a locally solid Riesz topology \( \tau \). For abbreviation, here and in where follows, we shall write a word "LSRS" instead of a locally solid Riesz space and we mean \( \lim_{k \to \infty} x_k \) by \( \lim x \) for brevity.

Let \((X, \tau)\) be a locally solid Riesz space. A sequence \( x = (x_k) \) in \( X \) is said to be \( S(\tau) \)-convergent to an element \( x_0 \) in \( X \) if for each \( \tau \)-neighbourhood \( V \) of zero

\[
\delta \left( \{ k \in N : x_k - x_0 \notin V \} \right) = 0
\]

i.e.,

\[
\lim_{n \to \infty} \frac{1}{n} \left| \{ k \leq n : x_k - x_0 \notin V \} \right| = 0.
\]

In this case, we write \( S(\tau)\)-lim \( x = x_0 \) or \( (x_k) \xrightarrow{S(\tau)} x_0 \) [8].

By a lacunary sequence, we mean an increasing integer sequence \( \theta = (k_r) \) such that \( k_0 = 0 \) and \( k_r = k_r - k_{r-1} \to \infty \) as \( r \to \infty \). Throughout the paper, the
Definition 3.3. Let \((X, \tau)\) be a Hausdorff LSRS and \((x_n)\) be two sequences in \(X\) such that \(x \in X, \tau\). Choose \(y, z \in Y\) such that \(x \in \{k \in N : x_k - x_0 \notin V\}\), i.e.,

\[
\lim_{n \to \infty} \frac{1}{g(n)} \sum_{k \leq n} |\{k \in N : x_k - x_0 \notin V\}| = 0.
\]

In this case we write \(S_g(\tau)\)-lim \(x = x_0\) or \(S_g(\tau)\)-lim \(x_k \to x_0\) [10].

3. MAIN RESULTS

**Definition 3.1.** Let \((X, \tau)\) be a locally solid Riesz space and \((x_n)\) be a sequence in \(X\). We say that \((x_n)\) is statistically \(\tau\)-convergent of weight \(g\) to \(x_0 \in X\) or \(S_g(\tau)\)-convergent to \(x_0\) provided that for every \(\tau\)-neighbourhood \(U\) of zero,

\[
\lim_{n \to \infty} \frac{1}{g(n)} \sum_{k \leq n} |\{k \leq n : x_k - x_0 \notin U\}| = 0
\]

holds. We denote this by \(S_g(\tau)\)-lim \(x_n = x_0\) or \((x_k) \to x_0\) briefly. The class of all sequences which are statistically \(\tau\)-convergent of weight \(g\) will be denoted by \(S_g(\tau)\).

**Remark 3.2.** For \(g(n) = n^\alpha\) and \(X = R\) the definition given above reduces to statistical convergence of order \(\alpha\) [11].

**Definition 3.3.** Let \((X, \tau)\) be a locally solid Riesz space. We say that \(x = (x_k)\) in \(X\) is said to be \(S_g(\tau)\)-bounded if for every neighbourhood \(V\) of zero, there exists some \(\alpha > 0\) such that,

\[
\lim_{n \to \infty} \frac{1}{g(n)} \sum_{k \leq n} |\{k \leq n : x_k \notin V\}| = 0.
\]

**Theorem 3.4.** Let \((X, \tau)\) be a Hausdorff LSRS and \(x = (x_k), y = (y_k)\) be two sequences in \(X\). Then the followings hold:

(1) If \(S_g(\tau)\) \(-\lim x_k = x_0\) and \(S_g(\tau)\) \(-\lim x_k = y_0\), then \(x_0 = y_0\).

(2) If \(S_g(\tau)\) \(-\lim x_k = x_0\), then \(S_g(\tau)\) \(-\lim \alpha x_k = \alpha x_0\), for every \(\alpha \in R\).

(3) If \(S_g(\tau)\) \(-\lim x_k = x_0\) and \(S_g(\tau)\) \(-\lim y_k = y_0\), then \(S_g(\tau)\) \(-\lim x_k + y_k = x_0 + y_0\).

**Proof.**

(1) Suppose that \(S_g(\tau)\) \(-\lim x_k = x_0\) and \(S_g(\tau)\) \(-\lim x_k = y_0\). Let \(V\) be any \(\tau\)-neighbourhood of zero. Then there exists a \(Y \in \mathcal{N}_{sol}\) such that \(Y \subseteq V\). Choose \(W \in \mathcal{N}_{sol}\) such that \(W + W \subseteq Y\). Since \(S_g(\tau)\) \(-\lim x_k = x_0\) and \(S_g(\tau)\) \(-\lim x_k = y_0\), then we have \(d_g(G_1) = d_g(G_2) = 1\) where

\[
G_1 = \{k \leq n : x_k - x_0 \in W\}, \quad G_2 = \{k \leq n : x_k - y_0 \in W\}.
\]

Now let \(G = G_1 \cap G_2\). Then we have

\[
x_0 - y_0 = x_0 - x_k + x_k - y_0 \in W + W \subseteq Y \subseteq V
\]

for every \(k \in G\). Hence for every \(\tau\)-neighbourhood \(V\) of zero we have \(x_0 - y_0 \in V\). Since \((X, \tau)\) is Hausdorff, the intersection of all \(\tau\)-neighbourhood \(V\) of zero is the singleton set \(\{\theta\}\). Thus we get \(x_0 - y_0 = \theta\), i.e., \(x_0 = y_0\).
(2) Let $S_g(\tau) - \lim x_k = x_0$ and let $V$ be an arbitrary $\tau-$neighbourhood of zero. Then there exists $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq V$. Since $S_g(\tau) - \lim x_k = x_0$, we have
\[
\lim_{n \to \infty} \frac{1}{g(n)} |\{k \leq n : x_k - x_0 \in Y\}| = 1.
\]
Let $|\alpha| \leq 1$. Since $Y$ is balanced, $x_n - x_0 \in Y$ implies that $\alpha(x_n - x_0) \in Y$ for every $\alpha \in \mathbb{R}$ with $|\alpha| \leq 1$. Hence we have
\[
\{k \leq n : x_k - x_0 \in Y\} \subseteq \{k \leq n : \alpha(x_n - x_0) \in Y\} 
\subseteq \{k \leq n : \alpha(x_k - x_0) \in V\}.
\]
Thus we get
\[
\lim_{n \to \infty} \frac{1}{g(n)} |\{k \leq n : \alpha(x_k - x_0) \in Y\}| = 1,
\]
for each $\tau$-neighbourhood $V$ of zero. Now let $|\alpha| > 1$ and $[\alpha]$ be the smallest integer greater than or equal to $\alpha$. There exists a $W \in \mathcal{N}_{sol}$ such that $[\alpha]W \subseteq Y$. Since $S_g(\tau) - \lim x_n = x_0$, we have $d(K) = 1$, where
\[
K = \{k \leq n : x_k - x_0 \in W\}.
\]
Then we have
\[
|\alpha x_n - \alpha y_0| = |\alpha| |x_n - x_0| \leq [\alpha] |x_n - x_0| \in W \subseteq Y \subseteq V
\]
for each $n \in K$. Since the set $V$ is solid, we have $\alpha x_n - \alpha x_0 \in Y$ and so $\alpha x_n - \alpha x_0 \in V$ for each $n \in K$. Thus we get
\[
\lim_{n \to \infty} \frac{1}{g(n)} |\{k \leq n : x_k - x_0 \in V\}| = 1
\]
for each $\tau$-neighbourhood $V$ of zero. Hence $S_g(\tau) - \lim \alpha x_k = \alpha x_0$ for every $\alpha \in \mathbb{R}$.

(3) Let $V$ be an arbitrary $\tau-$neighbourhood of zero. Then there exists $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq V$. We choose $W \in \mathcal{N}_{sol}$ such that $W + W \subseteq Y$. Since $S_g(\tau) - \lim x_n = \alpha x_0$ and $S'_g(\tau) - \lim y = y_0$, we have $d(B_1) = d(B_2) = 1$ where
\[
B_1 = \{k \leq n : x_k - x_0 \in W\}
\]
and
\[
B_2 = \{k \leq n : x_k - x_0 \in W\}.
\]
Now let $B = B_1 \cap B_2$. Hence we have $d(B) = 1$ and
\[
(x_n + y_n) - (x_0 - y_0) = (x_n - x_0) + (y_n - y_0) \in W + W \subseteq Y \subseteq V
\]
for each $n \in B$. Thus we get
\[
\lim_{n \to \infty} \frac{1}{g(n)} |\{k \leq n : (x_n - x_0) + (y_n - y_0) \in V\}| = 1.
\]
Since $V$ is arbitrary, we have $S_g(\tau) - \lim x_k + y_k = x_0 + y_0$.

\begin{theo}
Let $(X, \tau)$ be a locally solid Riesz space. If a sequence $x = (x_k)$ is $S_g(\tau)$-convergent, then $x = (x_k)$ is $S_g(\tau)$-bounded.
\end{theo}
Proof. Let \( S_g(\tau) - \lim x_k = x_0 \). Let \( V \) be an arbitrary \( \tau \)-neighbourhood of zero. Then there exists a \( Y \in \mathcal{N}_{sol} \) such that \( Y \subseteq V \). Let us choose \( W \in \mathcal{N}_{sol} \) such that \( W + W \subseteq Y \). Since \( x_k \xrightarrow{S_g(\tau)} x_0 \), we have \( d(K) = 0 \), where
\[
K = \{ \{ k \leq n : x_k - x_0 \in V \} \}.
\]
Since \( W \) is absorbing, there exists a \( a > 0 \) such that \( ax_0 \in W \). Let \( b \) be such that \( b \leq 1 \) and \( b \leq a \). Since \( W \) is solid and \( |bx_0| \leq |ax_0| \), we have \( bx_0 \in W \). Since \( W \) is balanced, \( x_k - x_0 \in W \) implies that \( b(x_k - x_0) \in W \). Then we have
\[
bx_n = b(x_n - x_0) + bx_0 \in W + W \subseteq Y \subseteq V
\]
for each \( n \in \mathbb{N} \setminus K \) and thus we get
\[
\lim_{n \to \infty} \frac{1}{g(n)} |\{ k \leq n : bx_k \notin V \}| = 0.
\]
Consequently, \((x_n)\) is \( S_g(\tau)\)-bounded. \( \square \)

**Theorem 3.6.** Let \((X, \tau)\) be a locally solid Riesz space. If \((x_k), (y_k), (z_k)\) are sequences such that;

1. \( x_k \leq y_k \leq z_k \), for all \( k \in \mathbb{N} \);
2. \( S_g(\tau) - \lim x_k = x_0 = S_g(\tau) - \lim z_k \), then \( S_g(\tau) - \lim x_k = x_0 \).

Proof. Let \( V \) be an arbitrary \( \tau \)-neighbourhood of zero. Then there exists \( Y \in \mathcal{N}_{sol} \) such that \( Y \subseteq V \). Choose \( W \in \mathcal{N}_{sol} \) such that \( W + W \subseteq Y \). From the condition (2), we have \( S_g(\tau) (E_1) = 1 = S_g(\tau) (E_2) \) where
\[
E_1 = \{ k \leq n : x_k - x_0 \in W \}
\]
and
\[
E_2 = \{ k \leq n : z_k - x_0 \in W \}.
\]
Also, we get \( S_g(\tau) (E_1 \cap E_2) = 1 \) and from (1) we have
\[
x_k - x_0 \leq y_k - x_0 \leq z_k - x_0 \]
for all \( k \in \mathbb{N} \). This implies that for all \( k \in E_1 \cap E_2 \), we get
\[
|y_k - x_0| \leq |z_k - x_0| + |x_k - x_0| \in W + W \subseteq Y \subseteq V.
\]
Since \( Y \) is solid, we have \( y_k - y_0 \in V \). Thus
\[
\lim_{n \to \infty} \frac{1}{g(n)} |\{ k \leq n : y_k - x_0 \in V \}| = 1
\]
for each \( \tau \)-neighbourhood \( V \) of zero. Hence \( S_g(\tau) - \lim x_k = x_0 \). \( \square \)

4. LACUNARY STATISTICAL CONVERGENCE OF WEIGHT \( g \) IN A LOCALLY SOLID RIESZ SPACE

In this section, we define lacunary statistical convergence of weight \( g \) in a locally solid Riesz space and we examine some inclusion relations.

**Definition 4.1.** Let \( \theta \) be a lacunary sequence. A sequence \( x = (x_k) \) is said to be lacunary statistically convergent of weight \( g \) to \( x_0 \) or \( S^\theta_g(\tau) \)-convergent to \( x_0 \) if for every \( \tau \)-neighbourhood \( U \) of zero,
\[
\lim_{n \to \infty} \frac{1}{g(h_\tau)} |\{ k \in I_\tau : x_k - x_0 \notin U \}| = 0
\]
holds. We denote this by $S^\theta_g(\tau)$-lim $x_n = x_0$ (or $x_k \to x_0$ briefly). The class of all sequences which are lacunary statistically $\tau$-convergent of weight $g$ will be denoted by $S^\theta_g(\tau)$.

**Definition 4.2.** Let $(X, \tau)$ be a locally solid Riesz space and $\theta$ be a lacunary sequence. We say that $x = (x_k)$ in $X$ is said to be $S^\theta_g(\tau)$-bounded if for every neighbourhood $V$ of zero, there exists some $\alpha > 0$ such that,

$$\lim_{r \to \infty} \frac{1}{g(\tau)} |\{k \leq I_r : x_k \notin V\}| = 0$$

We leave the proofs of the following two theorems to the reader. They can be done in a similar manner as the proofs of Theorem 3.4 and Theorem 3.5.

**Theorem 4.3.** Let $(X, \tau)$ be a Hausdorff locally solid Riesz space and $\theta$ be a lacunary sequence and $x = (x_k)$, $y = (y_k)$ be two sequences in $X$. Then the followings hold:

1. If $S^\theta_g(\tau) - \lim x_k = x_0$ and $S^\theta_g(\tau) - \lim x_k = y_0$, then $x_0 = y_0$.
2. If $S^\theta_g(\tau) - \lim x_k = x_0$, then $S^\theta_g(\tau) - \lim \alpha x_k = \alpha x_0$, for every $\alpha \in \mathbb{R}$.
3. If $S^\theta_g(\tau) - \lim x_k = x_0$ and $S^\theta_g(\tau) - \lim y_k = y_0$, then $S^\theta_g(\tau) - \lim x_k + y_k = x_0 + y_0$.

**Theorem 4.4.** Let $(X, \tau)$ be a locally solid Riesz space and $\theta$ be a lacunary sequence. If a sequence $x = (x_k)$ is $S^\theta_g(\tau)$-convergent, then $x = (x_k)$ is $S^\theta_g(\tau)$-bounded.

**Theorem 4.5.** Let $(X, \tau)$ be a locally solid Riesz space and $\theta$ be a lacunary sequence. If $\liminf_{r} \frac{g(h_r)}{g(\tau)} > 1$, then $S^\theta_g(\tau) \subset S^\theta_g(\tau)$

**Proof.** Since $\liminf_{r} \frac{g(h_r)}{g(\tau)} > 1$, so we can find a $\delta > 0$ such that $\frac{g(h_r)}{g(\tau)} \geq 1 + \delta$ for sufficiently large values of $r$. Assume that $x_k \overset{S^\theta_g(\tau)}{\to} x_0$, hence for every $U$ neighborhood of zero and for sufficiently large values of $r$ we have

$$\frac{1}{g(\tau)} |\{k \leq k_r : x_k - x_0 \notin U\}| \geq \frac{1}{g(\tau)} |\{k \in I_r : x_k - x_0 \notin U\}|$$

$$= \frac{g(h_r)}{g(\tau)} \frac{1}{g(\tau)} |\{k \in I_r : x_k - x_0 \notin U\}|$$

$$\geq (1 + \delta) \frac{1}{g(\tau)} |\{k \in I_r : x_k - x_0 \notin U\}| .$$

Hence we have $x_k \overset{S^\theta_g(\tau)}{\to} x_0$ while taking limit as $n \to \infty$. \qed

5. **Conclusion**

In this work, we introduce the notions of statistical convergence and lacunary statistical convergence with respect to weight $g$ in a locally solid Riesz space and establish some inclusion relations. One can see that if $\limsup_r \frac{g(h_r)}{g(\tau)} < \infty$ the inverse of the Theorem 4.5 holds.
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