

## Fixed Points for Mappings on Product Spaces

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**Abstract:** Existence of fixed points for a particular cyclic type of mappings on a finite product of topological spaces is discussed. Existence of fixed points of a particular cyclic type of set valued mappings on a finite product of metric spaces is derived. Fixed points of shift type mappings are studied.

### 1. Introduction

There are many articles [4, 5, 7] for fixed points of mappings on product spaces. They focus on difficult parts in establishment of existence of fixed points. Simple set theoretic arguments also give some fruitful results which are recorded in the present article. Functions of the type  $F : \prod_{i=1}^n X_i \rightarrow \prod_{i=1}^n X_i$ ,  $F((x_1, x_2, \dots, x_n)) = (f_n(x_n), f_1(x_1), f_2(x_2), \dots, f_{n-1}(x_{n-1}))$  corresponding to given functions  $f_1 : X_1 \rightarrow X_2, f_2 : X_2 \rightarrow X_3, \dots, f_{n-1} : X_{n-1} \rightarrow X_n$  and  $f_n : X_n \rightarrow X_1$  are considered in this article for fixed points. The next section 2 is for such single valued mappings and section 3 is for set valued mappings.

If there is a bijective mapping  $\Lambda$  from an index set  $I$  into  $I$  itself, then for given  $i_1 \in I$ ; (i)  $\Lambda(i_1) = i_1$ ; or (ii) there are distinct  $i_2, i_3, \dots, i_n \in I$  such that  $\Lambda(i_1) = i_2, \Lambda(i_2) = i_3, \dots, \Lambda(i_{n-1}) = i_n, \Lambda(i_n) = i_1$ ; or (iii) there are distinct  $i_2, i_3, \dots, i_0, i_{-1}, i_{-2}, \dots$  in  $I$  such that  $\Lambda(i_j) = i_{j+1}$  for every integer  $j$ . So, for given mappings  $f_i : X_i \rightarrow X_{i+1}, i = 0, \pm 1, \pm 2, \dots$ , fixed points of  $F : \prod_{i=-\infty}^{+\infty} X_i \rightarrow \prod_{i=-\infty}^{+\infty} X_i$  are discussed in section 4. Fixed points of a special type of mappings on sets of the form  $\prod_{i=1}^{\infty} X_i$  are also discussed in section 4.

### 2. Cyclic Single Valued Mappings

**Proposition 2.1.** Let  $F : \prod_{i=1}^n X_i \rightarrow \prod_{i=1}^n X_i$  be a mapping defined by the given mappings  $f_1 : X_1 \rightarrow X_2, f_2 : X_2 \rightarrow X_3, \dots, f_{n-1} : X_{n-1} \rightarrow X_n, f_n : X_n \rightarrow X_1$  and by the relation  $F((x_1, x_2, \dots, x_n)) = (f_n(x_n), f_1(x_1), f_2(x_2), \dots, f_{n-1}(x_{n-1}))$ . Let  $g_i : X_i \rightarrow X_i$  be a mapping defined by  $g_i = f_{i-1} \circ f_{i-2} \circ \dots \circ f_1 \circ f_n \circ f_{n-1} \circ \dots \circ f_i$  for  $i = 2, 3, \dots, n$  and by  $g_1 = f_n \circ f_{n-1} \circ \dots \circ f_2 \circ f_1$  for  $i = 1$ . Let  $G_i = \{x_i \in X_i : g_i(x_i) = x_i\}$  be the fixed point set of  $g_i$  and  $G$  be the fixed point set of  $F$ . If  $G_1 \neq \emptyset$ , then  $G_i \neq \emptyset$  for  $i = 2, 3, \dots, n$ ,  $G \neq \emptyset$  and  $f_1(G_1) = G_2, f_2(G_2) = G_3, \dots, f_{n-1}(G_{n-1}) = G_n, f_n(G_n) = G_1$ . If  $(y_1, y_2, \dots, y_n) \in G \neq \emptyset$ , then  $y_1 \in G_1 \neq \emptyset$ .

*Proof.* Suppose  $G_1 \neq \emptyset$ . Let  $x_1 \in G_1$ . Then  $g_1(x_1) = x_1, f_1 \circ f_n \circ \dots \circ f_2 \circ f_1(x_1) = f_1(x_1), g_2(f_1(x_1)) = f_1(x_1), g_3(f_2 \circ f_1(x_1)) = f_2 \circ f_1(x_1), \dots, g_n(f_{n-1} \circ f_{n-2} \circ \dots \circ f_1(x_1)) = f_{n-1} \circ f_{n-2} \circ \dots \circ f_1(x_1)$  and  $g_1(g_1(x_1)) = g_1(x_1) = x_1$ . Thus  $G_i \neq \emptyset$  for  $i = 2, 3, \dots, n$ . Also, if  $x_i = f_{i-1} \circ f_{i-2} \circ \dots \circ f_1(x_1)$  for  $i = 2, 3, \dots, n$ , then  $f_i(x_i) = f_i \circ f_{i-1} \circ \dots \circ f_1(x_1) = x_{i+1}$  for  $i = 1, 2, \dots, n$ , with  $x_{n+1} = x_1$  and hence  $F((x_1, x_2, \dots, x_n)) = (f_n(x_n), f_1(x_1), \dots, f_{n-1}(x_{n-1})) = (x_1, x_2, \dots, x_n)$ . So,  $G \neq \emptyset$ . By cyclic symmetry, the relations  $f_1(G_1) = G_2, \dots, f_{n-1}(G_{n-1}) = G_n$  and  $f_n(G_n) = G_1$  are also obtained.

Suppose  $(y_1, y_2, \dots, y_n) \in G$ . Then  $f_1(y_1) = y_2, f_2(y_2) = y_3, \dots, f_{n-1}(y_{n-1}) = y_n$  and  $f_n(y_n) = y_1$ . Then  $f_n \circ f_{n-1} \circ \dots \circ f_1(y_1) = y_1$  so that  $g_1(y_1) = y_1$  and hence  $G_1 \neq \emptyset$ . □

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**Corollary 2.2.** *The following are equivalent under the assumptions of the previous proposition 2.1:*

1.  $G \neq \emptyset$ ;
2.  $G_i \neq \emptyset$  for some  $i = 1, 2, \dots, n$ ;
3.  $G_i \neq \emptyset$  for all  $i = 1, 2, \dots, n$ .

*The following are also equivalent:*

1.  $G$  is a singleton set;
2.  $G_i$  is a singleton set for some  $i = 1, 2, \dots, n$ ;
3.  $G_i$  is a singleton set for all  $i = 1, 2, \dots, n$ .

**Some Applications :** (a) Let  $(X, d_i)$  be nonempty metric spaces and  $\alpha_i$  be positive numbers for  $i = 1, 2, \dots, n$ . Suppose  $(X_1, d_1)$  is complete. Assume that  $\alpha_1 \alpha_2 \cdots \alpha_n < 1$ . Let  $f_i : X_i \rightarrow X_{i+1}, i = 1, 2, \dots, n$  (with  $(X_{n+1}, d_{n+1}) = (X_1, d_1)$ ) be mappings such that  $d_{i+1}(f_i(x), f_i(y)) \leq \alpha_i d_i(x, y), \forall x, y \in X_i$ . Define  $F : \prod_{i=1}^n X_i \rightarrow \prod_{i=1}^n X_i$  by  $F((x_1, x_2, \dots, x_n)) = (f_n(x_n), f_1(x_1), \dots, f_{n-1}(x_{n-1}))$ . Since  $\alpha_1 \alpha_2 \cdots \alpha_n < 1$ , by the Banach contraction principle (see p.2 in [6]),  $f_n \circ f_{n-1} \circ \cdots \circ f_1$  has a unique fixed point. Thus  $F$  has a unique fixed point. Some  $(X_i, d_i)$  may not be complete and some  $\alpha_i$  may not be less than 1. So, this is a significant application of Proposition 2.1.

(b) Let  $X_i, i = 1, 2, \dots, n$  be nonempty weakly compact convex subsets of normed spaces with norms  $\|\cdot\|_i, i = 1, 2, \dots, n$ . Let  $d_i$  be the metric induced by  $\|\cdot\|_i, i = 1, 2, \dots, n$ . Let  $\alpha_i, i = 1, 2, \dots, n$  be positive numbers such that  $\alpha_1 \alpha_2 \cdots \alpha_n \leq 1$ . Suppose  $X_1$  has normed structure (see [3]). Let  $f_i$  and  $F$  be as in the previous application (a). A Kirk's theorem (see [3]) asserts that a nonexpansive mapping from a nonempty weakly compact convex subset of a normed space with normal structure into itself has a fixed point. Since  $\alpha_1 \alpha_2 \cdots \alpha_n \leq 1, f_n \circ f_{n-1} \circ \cdots \circ f_1$  is nonexpansive and it has a fixed point in  $X_1$ . Thus  $F$  has a fixed point. Here, some  $X_i$  may not have normal structure. Weak compactness and convexity of  $X_2, X_3, \dots, X_n$  may also be relaxed.

(c) R. Cauty [1] (see also [2]) proved that every continuous function from a nonempty compact convex subset of a Hausdorff topological vector space into itself has a fixed point. Let  $X_2, X_3, \dots, X_n$  be nonempty topological spaces. Let  $X_1$  be a nonempty compact convex subset of a Hausdorff topological vector space. Let  $f_i : X_i \rightarrow X_{i+1}, i = 1, 2, \dots, n$  (with the convention  $X_{n+1} = X_1$ ) be continuous mappings. Since  $f_n \circ f_{n-1} \circ \cdots \circ f_1$  is continuous on  $X_1$ , it has a fixed point in  $X_1$  and hence  $F$  has at least one fixed point.

These applications are illustrations for significance of Proposition 2.1.

### 3. Cyclic Set Valued Mappings

There are difficulties in generalizing Proposition 2.1 to set valued mappings. So, the following fixed point result for contractive type set valued mappings is proposed.

**Proposition 3.1.** *Let  $(X_i, d_i), i = 1, 2, \dots, n$  be nonempty complete metric spaces. Let  $H_i$  be the Hausdorff metric corresponding to the metric  $d_i$  on the collection  $CB(X_i)$  of all closed bounded nonempty subsets of  $X_i$ , for  $i = 1, 2, \dots, n$ . Let  $\alpha \in (0, 1)$ . Let  $f_i : X_i \rightarrow CB(X_{i+1})$  be a mapping such that*

$$H_{i+1}(f_i(x), f_i(y)) \leq \alpha d_i(x, y) \quad (1)$$

$\forall x, y \in X_i, i = 1, 2, \dots, n$ , with the conventions  $(X_{n+1}, d_{n+1}) = (X_1, d_1)$  and  $(CB(X_{n+1}), H_{n+1}) = (CB(X_1), H_1)$ . Define  $F : \prod_{i=1}^n X_i \rightarrow \prod_{i=1}^n CB(X_i)$  by  $F((x_1, x_2, \dots, x_n)) = (f_n(x_n), f_1(x_1), \dots, f_{n-1}(x_{n-1}))$ . Then there is a fixed point  $(x_1^*, x_2^*, \dots, x_n^*)$  of  $F$  in the sense that  $x_1^* \in f_n(x_n^*), x_2^* \in f_1(x_1^*), x_3^* \in f_2(x_2^*), \dots, x_n^* \in f_{n-1}(x_{n-1}^*)$ .

**Proof.** Fix  $x_{n,0} \in X_n$  and fix  $x_{1,1} \in f_n(x_{n,0}), x_{2,1} \in f_1(x_{1,1}), x_{3,1} \in f_2(x_{2,1}), \dots, x_{n,1} \in f_{n-1}(x_{n-1,1})$  successively. Find  $x_{1,j+1} \in f_n(x_{n,j}), x_{2,j+1} \in f_1(x_{1,j+1}), x_{3,j+1} \in f_2(x_{2,j+1}), \dots, x_{n,j+1} \in f_{n-1}(x_{n-1,j+1})$  such that

$$\begin{aligned} d_1(x_{1,j}, x_{1,j+1}) &\leq H_1(f_n(x_{n,j-1}), f_n(x_{n,j})) + \alpha^j, \\ d_2(x_{2,j}, x_{2,j+1}) &\leq H_2(f_1(x_{1,j}), f_1(x_{1,j+1})) + \alpha^j, \\ d_3(x_{3,j}, x_{3,j+1}) &\leq H_3(f_2(x_{2,j}), f_2(x_{2,j+1})) + \alpha^j, \\ &\dots \\ &\dots \\ d_n(x_{n,j}, x_{n,j+1}) &\leq H_n(f_{n-1}(x_{n-1,j}), f_{n-1}(x_{n-1,j+1})) + \alpha^j, \end{aligned}$$

for  $j = 1, 2, \dots$ . Now

$$\begin{aligned}
 d_1(x_{1,m}, x_{1,m+1}) &\leq H_1(f_n(x_{n,m-1}), f_n(x_{n,m})) + \alpha^m \\
 &\leq \alpha d_n(x_{n,m-1}, x_{n,m}) + \alpha^m \\
 &\leq \alpha H_n(f_{n-1}(x_{n-1,m-1}), f_{n-1}(x_{n-1,m})) + 2\alpha^m \\
 &\leq \alpha^2 d_{n-1}(x_{n-1,m-1}, x_{n-1,m}) + 2\alpha^m \\
 &\dots \\
 &\dots
 \end{aligned}$$

This procedure of calculation and the inequalities of the type

$$\begin{aligned}
 d_1(x_{1,m}, x_{1,m+k}) &\leq d_1(x_{1,m}, x_{1,m+1}) + d_1(x_{1,m+1}, x_{1,m+2}) \\
 &\quad + \dots + d_1(x_{1,m+k-1}, x_{1,m+k})
 \end{aligned}$$

imply that  $(x_{1,j})_{j=1}^\infty, (x_{2,j})_{j=1}^\infty, \dots, (x_{n,j})_{j=1}^\infty$  and  $(f_n(x_{n,j}))_{j=1}^\infty, (f_{n-1}(x_{n-1,j}))_{j=1}^\infty, \dots, (f_1(x_{1,j}))_{j=1}^\infty$  are Cauchy sequences in their respective spaces, because  $n$  is fixed and  $0 < \alpha < 1$ . Also, if  $(x_{i,j})_{j=1}^\infty$  converges to  $x_i^*$ , then  $(f_i(x_{i,j}))_{j=1}^\infty$  converges to  $f_i(x_i^*)$ , by the inequality (1) and in this case  $x_{i+1}^*$  belongs to  $f_i(x_i^*)$ , because  $x_{i+1,j} \in f_i(x_{i,j})$  (with the cyclic convention  $n+1 \mapsto 1$ ). This completes the proof.  $\square$

Note that, if we consider  $(f_n(x_n), f_1(x_1), \dots, f_{n-1}(x_{n-1}))$  as the subset  $f_n(x_n) \times f_1(x_1) \times \dots \times f_{n-1}(x_{n-1})$  of  $\prod_{i=1}^n X_i$ , then  $F$  in Proposition 3.1 becomes an usual set valued mapping.

#### 4. Shift Type Mappings

Consider the mappings  $f_i : X_i \rightarrow X_{i+1}, i = 0, \pm 1, \pm 2, \dots$  between nonempty sets. Define  $F : \prod_{i=-\infty}^{+\infty} X_i \rightarrow \prod_{i=-\infty}^{+\infty} X_i$  by  $F((x_i)_{i=-\infty}^{+\infty}) = ((f_i(x_i))_{i=-\infty}^{+\infty})$ . Then  $(x_i)_{i=-\infty}^{+\infty}$  is a fixed point of  $F$  if and only if  $x_{i+1} = f_i(x_i)$  for every  $i = 0, \pm 1, \pm 2, \dots$ . So,  $F$  has a fixed point if and only if there is an integer  $m$  such that (or, for every integer  $m$  we have)  $\bigcap_{n \leq m} f_{m-1} \circ f_{m-2} \circ \dots \circ f_{n+1} \circ f_n(X_n) \neq \emptyset$ .

**Some Applications :** (a) If  $f_n$  is surjective for all  $n \leq n_0$ , for some integer  $n_0$ , then  $F$  has a fixed point.

(b) If each  $X_n$  is compact and each  $f_n$  is continuous for all  $n \leq n_0$ , for some integer  $n_0$ , then  $F$  has a fixed point.

(c) Suppose there is an integer  $n_0$  such that

1.  $(X_{n_0+1}, d_{n_0+1})$  is a complete metric space,
2.  $(X_n, d_n)$  is a metric space for  $n \leq n_0$  and
3. there are positive numbers  $\alpha_n$  and  $M_n$  such that  $d_{n+1}(f_n(x), f_n(y)) \leq \alpha_n d_n(x, y)$  and  $d(x, y) \leq M_n$ , for all  $x, y \in X_n$  and for every  $n \leq n_0$  and such that  $M_{n_0} \alpha_{n_0} M_{n_0-1} \alpha_{n_0-1} \dots M_{n_0-k} \alpha_{n_0-k} \rightarrow 0$  as  $k \rightarrow +\infty$ .

Then  $F$  has a fixed point, because diameter of  $f_{n_0} \circ f_{n_0-1} \circ \dots \circ f_{n_0-k}(X_{n_0-k})$  tends to zero as  $k \rightarrow +\infty$ .

Consider a sequence of nonempty sets  $X_1, X_2, \dots$  and a sequence of mappings  $f_i : X_i \rightarrow X_{i+1}, i = 1, 2, \dots$ . Let  $f_0 : \prod_{i=1}^\infty X_i \rightarrow X_1$  be a function. Define  $F((x_1, x_2, \dots)) = (f_0((x_1, x_2, \dots)), f_1(x_1), f_2(x_2), f_3(x_3), \dots)$ . There are some examples of this type functions without fixed points (see [3] and see p.16, p.36 in [6]). Now, this type functions with fixed points are to be discussed. Define  $G : X_1 \rightarrow X_1$  by  $G(x_1) = f_0((x_1, f_1(x_1), f_2 \circ f_1(x_1), f_3 \circ f_2 \circ f_1(x_1), \dots))$ .

**Proposition 4.1.** Suppose  $X_1, X_2, \dots, f_0, f_1, f_2, \dots, F$  and  $G$  be as above. Then  $G$  has a fixed point  $x_1^*$  if and only if  $F$  has a fixed point, which is, in this case, of the form  $(x_1^*, f_1(x_1^*), f_2 \circ f_1(x_1^*), f_3 \circ f_2 \circ f_1(x_1^*), \dots)$ .

**Proof.** If  $G$  has a fixed point  $x_1^*$ , then  $x_1^* = f_0((x_1^*, f_1(x_1^*), f_2 \circ f_1(x_1^*), f_3 \circ f_2 \circ f_1(x_1^*), \dots))$ . Now  $F((x_1^*, f_1(x_1^*), f_2 \circ f_1(x_1^*), f_3 \circ f_2 \circ f_1(x_1^*), \dots))$

$$\begin{aligned}
 &= (f_0((x_1^*, f_1(x_1^*), f_2 \circ f_1(x_1^*), \dots)), f_1(x_1^*), f_2 \circ f_1(x_1^*), f_3 \circ f_2 \circ f_1(x_1^*), \dots) \\
 &= (x_1^*, f_1(x_1^*), f_2 \circ f_1(x_1^*), f_3 \circ f_2 \circ f_1(x_1^*), \dots).
 \end{aligned}$$

Conversely, assume that  $(x_1^*, x_2^*, \dots)$  is a fixed point of  $F$ . Then  $x_1^* = f_0((x_1^*, x_2^*, \dots)), x_2^* = f_1(x_1^*), x_3^* = f_2(x_2^*), \dots$ . So,  $x_1^* = f_0((x_1^*, f_1(x_1^*), f_2 \circ f_1(x_1^*), \dots))$  or  $x_1^* = G(x_1^*)$ .  $\square$

**Some Applications :** (a) Suppose  $(X_1, d_1)$  is a complete metric space (or a compact metric space). Suppose further that  $f_1, f_2, \dots$  are functions such that  $G : X_1 \rightarrow X_1$  satisfies the relation  $d_1(G(x), G(y)) \leq \alpha d_1(x, y)$ ,  $\forall x, y \in X_1$  and for some  $\alpha \in (0, 1)$  (or  $d_1(G(x), G(y)) < d_1(x, y)$  for  $x, y \in X_1$  satisfying  $x \neq y$ ). Banach contraction principle (or another known result (see p.38 in [6])) implies a unique fixed point  $x_1^*$  of  $G$ . Thus  $F$  has a unique fixed point  $(x_1^*, f_1(x_1^*), f_2 \circ f_1(x_1^*), \dots)$ .

(b) Suppose  $X_1$  is a weakly compact convex subset of a normed space with normal structure and norm  $\| \cdot \|$ . Suppose  $f_1, f_2, \dots$  are functions such that  $G : X_1 \rightarrow X_1$  satisfies the relation  $\|G(x) - G(y)\| \leq \|x - y\|$ ,  $\forall x, y \in X_1$ . Then  $G$  has a fixed point and hence  $F$  has a fixed point.

## References

- [1] R. Cauty, Solution du problème de point fixe de Schauder, *Fund. Math.* 2001(**170**): 231-246 (French)
- [2] T. Dobrowolski, Revisiting Cauty's proof of the Schauder Conjecture, *Abstract and applied analysis.* 2003 (**7**): 407-433.
- [3] W. A. Kirk, A fixed point theorem for mappings which do not increase distances, *Amer. Math. Monthly.* 1965 (**72**): 1004-1006.
- [4] W. A. Kirk, An iteration process for nonexpansive mappings with applications to fixed point theory in product spaces, *Proc. Amer. Math. Soc.*, 1989(**107**): 411-415.
- [5] W. A. Kirk and C. M. Yanez, Nonexpansive and locally nonexpansive mappings in product spaces, *Nonlinear analysis, Theory, Methods and Applications*, 1988(**12**): 719-725.
- [6] D. R. Smart, *Fixed point theorems*, Cambridge University press, Cambridge, 1974.
- [7] P. Vijayaraju, Fixed point theorems for asymptotically nonexpansive mappings in product spaces, *Twiwanese J. Mathematics.* 1998(**2**): 97-105.