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Quasi-Subordinations for Certain Subclasses of Bi-univalent Functions

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Abstract: In this present investigation, a new class of bi-univalent functions associated with quasi-subordination is defined and the bounds on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to this class are derived. Furthermore, several related classes of functions are also indicated.

1. Introduction

Let A be the class of functions f of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \tag{1}$$

which are analytic in the open unit disc $\Lambda = \{z \in C \,|z| < 1\}$ normalized by

$$f(0) = 0,$$

$$f'(0) = 1.$$

Further, by S we shall denote the class of all functions in A which are univalent in Λ. Let $h(z)$ bean analytic function in Λ $|h(z)| \le 1$, such that

$$h(z) = h_0 + h_1 z + h_2 z^2 + \cdots, \tag{2}$$

where all coefficients are real. Also, let the function ϕ be an analytic and univalent function with
positive real part in Λ with $\phi(0) = 1$, $\phi'(0) > 0$ and ϕ maps the unit disc Λ onto a region starlike with
respect to $\phi(0) = 1$ and symmetric with respect to the real axis. Taylor's series expansion for such a
function is of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots, \tag{3}$$

where all coefficients are real and $B_1 > 0$.

For two analytic functions, f and g, such that $f(0) = g(0)$, we say that f is subordinate to g in Λ and write $f(z) \prec g(z),\ z \in \Lambda$, if there exists a Schwarz function $w(z)$ with $w(0) = 0$

and $|w(z)| \le |z|,\ z \in \Lambda$ such that $f(z) = g(w(z)),\ z \in \Lambda$. Furthermore, if the function g is univalent in Λ, then we have the following equivalence;

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$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\Lambda) \subset g(\Lambda).$$

The definition can be found in (Nehari 1952).

In the year 1970, Robertson [18] introduced the concept of quasi-subordination. For two analytic

functions f and g, the function f is said to be quasi-subordinate to g in Λ and written as

$$f(z) \prec_q g(z) \quad (z \in \Lambda)$$

if there exists an analytic function $|h(z)| \le 1,\ z \in \Lambda$ such that $\frac{f(z)}{h(z)}$ analytic in Λ and

$$\frac{f(z)}{h(z)} \prec g(z),\ z \in \Lambda$$

that is, there exists a Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| \le |z|$ such that

$$f(z) = h(z)g(w(z)),\ z \in \Lambda.$$

Observe that if $h(z) = 1$, then $f(z) = g(w(z))$, so that $f(z) \prec g(z)$ in Λ. Also notice that if $w(z) = z$, then $f(z) = h(z)g(z)$, and it is said that f is majorized by g and written $f(z) << g(z)$ in Λ. Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization (see, e.g. [1], [8], [11], [15], [17], [18] for works related to quasi-subordination and subordination).

The Koebe-one quarter theorem [10] ensures that the image of Λ under every univalent function $f \in A$ contains a disc of radius ¼. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z,\ z \in \Lambda$ and $f(f^{-1}(w)) = w,\ (|w| < r_0(f),\ r_0(f) \ge 1/4)$. Indeed, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^3 + \cdots. \tag{4}$$

A function $f \in A$ is said to be bi-univalent in Λ if both f and f^{-1} are univalent in Λ. Let Σ denote the class of bi-univalent functions defined in Λ.

Many researchers have recently introduced and investigated several interesting subclasses of bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ ([3], [4], [7], [9], [20], [21], [22], [24]). However, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions in the literature ([5], [6], [13], [14]). The coefficient estimate problem for each of

$$|a_n|,\ n \in N - \{1,2,3\}\ (N = \{1,2,3,\ldots\})$$

is still an open problem.

Ma and Minda [14] defined a class of starlike functions by using the method of subordination, and studied a class $S^*(\phi)$ which is defined by

$$S^*(\phi) = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \phi(z), \quad z \in \Lambda \right\}.$$

A function f is bi-starlike of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike. This class is denoted by $S_\Sigma^*(\phi)$ (see [2]).

We now define the following:

Definition 1.1. A function $f \in \Sigma$ given by (1) is said to belong to the class

$$S_q(\alpha,\phi) \quad (0 < \alpha \leq 1,\ z, w \in \Lambda)$$

if the following quasi-subordination holds:

$$\left[\frac{1}{2}\left(\frac{zf'(z)}{f(z)} + \left(\frac{zf'(z)}{f(z)} \right)^{\frac{1}{\alpha}} \right) - 1 \right] \prec_q (\phi(z) - 1)$$

and

$$\left[\frac{1}{2}\left(\frac{wg'(w)}{g(w)} + \left(\frac{wg'(w)}{g(w)} \right)^{\frac{1}{\alpha}} \right) - 1 \right] \prec_q (\phi(w) - 1)$$

where the function g is the extension of f^{-1} to Λ.

We note that for $h(z) = 1$ and $\alpha = 1$, we get the class $S_\Sigma^*(\phi)$ introduced by Ali et al. [2].

Further, for $h(z) = 1$, we have the class $S_q(\alpha,\phi) = S(\alpha,\phi)$ as defined below.

Definition 1.2. A function $f \in \Sigma$ given by (1) is said to belong to the class

$$S(\alpha,\phi) \quad (0 < \alpha \leq 1,\ z, w \in \Lambda)$$

if the following subordination holds:

$$\frac{1}{2}\left(\frac{zf'(z)}{f(z)}+\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\alpha}}\right)\prec\phi(z)$$

and

$$\frac{1}{2}\left(\frac{wg'(w)}{g(w)}+\left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\alpha}}\right)\prec\phi(w)$$

where the function g is the extension of f^{-1} to Λ.

Putting $\alpha=1$, we have the class $S_q(\alpha,\phi)=S_q(\phi)$ as defined below.

Definition 1.3. [23] A function $f\in\Sigma$ given by (1) is said to belong to the class

$$S_q(\phi)\quad(0<\alpha\le 1,\ z,w\in\Lambda)$$

if the following quasi-subordination holds:

$$\left(\frac{zf'(z)}{f(z)}-1\right)\prec_q\ (\phi(z)-1)$$

and

$$\left(\frac{wg'(w)}{g(w)}-1\right)\prec_q\ (\phi(w)-1)$$

where the function g is the extension of f^{-1} to Λ.

Motivated by the earlier works in Sharma and Raina [19] and Goyal et al. [12], we define and study a new class of functions by the method of quasi-subordination. The coefficient bounds of $|a_2|$ and $|a_3|$ for functions in the class $S_q(\alpha,\phi)$ are obtained. Some interesting results are also pointed out.

2. Coefficient Bounds for $S_q(\alpha,\phi)$

In this section, we derive the resulting estimates for the initial coefficients a_2 and a_3 of functions $f\in S_q(\alpha,\phi)$ given by the Taylor-Maclaurin series expansion (1).

Firstly, we state the following theorem.

Theorem 1. Let $0<\alpha\le 1$. If $f\in A$ of the form (1) belongs to the class $S_q(\alpha,\phi)$, then

$$|a_2|\le\frac{2\alpha|h_0|B_1\sqrt{B_1}}{\sqrt{\left|h_0B_1^2(2\alpha^2+\alpha+1)-(B_2-B_1)(\alpha+1)^2\right|}}$$

and

$$|a_3| \le \frac{\alpha|h_1|B_1}{\alpha+1} + \frac{\alpha|h_0|B_1}{\alpha+1} + \left(\frac{2\alpha|h_0|B_1}{\alpha+1}\right)^2.$$

Proof. Let $f \in S_q(\alpha,\phi)$. In view of the definition 1.1, there are analytic functions $\eta, \mu : \Lambda \to \Lambda$ with $\eta(0) = 0 = \mu(0)$, satisfying

$$\left[\frac{1}{2}\left(\frac{zf'(z)}{f(z)} + \left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\alpha}}\right) - 1\right] \prec h(z)[\phi(\eta(z)) - 1] \tag{5}$$

and

$$\left[\frac{1}{2}\left(\frac{wg'(w)}{g(w)} + \left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\alpha}}\right) - 1\right] \prec h(w)[\phi(\mu(w)) - 1]. \tag{6}$$

Define the functions

$$m(z) = \frac{1+\eta(z)}{1-\eta(z)} = 1 + \eta_1 z + \eta_2 z^2 + \cdots$$

and

$$n(w) = \frac{1+\mu(w)}{1-\mu(w)} = 1 + \mu_1 w + \mu_2 w^2 + \cdots$$

or, equivalently,

$$\eta(z) = \frac{m(z)-1}{m(z)+1} = \frac{1}{2}\left[\eta_1 z + \left(\eta_2 - \frac{\eta_1^2}{2}\right)z^2 + \cdots\right]$$

and

$$\mu(w) = \frac{n(w)-1}{n(w)+1} = \frac{1}{2}\left[\mu_1 w + \left(\mu_2 - \frac{\mu_1^2}{2}\right)w^2 + \cdots\right].$$

Then, $m(z)$ and $n(w)$ analytic in Λ with $m(0) = 1 = n(0)$. Since the functions $m(z)$ and $n(w)$ have a positive real part in Λ, $|\eta_i| \le 2$ and $|\mu_i| \le 2$. Now,

$$h(z)[\phi(\eta(z)) - 1] = \frac{1}{2}h_0B_1\eta_1 z + \left[\frac{1}{2}h_1B_1\eta_1 + \frac{1}{2}h_0B_1\left(\eta_2 - \frac{1}{2}\eta_1^2\right) + \frac{1}{4}h_0B_2\eta_1^2\right]z^2 + \ldots \tag{7}$$

and

$$h(w)[\phi(\mu(w)) - 1] = \frac{1}{2}h_0B_1\mu_1 w + \left[\frac{1}{2}h_1B_1\mu_1 + \frac{1}{2}h_0B_1\left(\mu_2 - \frac{1}{2}\mu_1^2\right) + \frac{1}{4}h_0B_2\mu_1^2\right]w^2 + \ldots. \tag{8}$$

In the light of (5), (6) and (7), (8), we obtain

$$\frac{\alpha+1}{2\alpha}a_2 = \frac{1}{2}h_0B_1\eta_1 \tag{9}$$

$$\frac{\alpha+1}{2\alpha}(2a_3 - a_2^2) + \frac{1-\alpha}{4\alpha^2}a_2^2 = \frac{1}{2}h_1B_1\eta_1 + \frac{1}{2}h_0B_1\left(\eta_2 - \frac{1}{2}\eta_1^2\right) + \frac{1}{4}h_0B_2\eta_1^2 \tag{10}$$

and

$$-\frac{\alpha+1}{2\alpha}a_2 = \frac{1}{2}h_0 B_1 \mu_1 \tag{11}$$

$$\frac{\alpha+1}{2\alpha}(3a_2^2 - 2a_3) + \frac{1-\alpha}{4\alpha^2}a_2^2 = \frac{1}{2}h_1 B_1 \mu_1 + \frac{1}{2}h_0 B_1\left(\mu_2 - \frac{1}{2}\mu_1^2\right) + \frac{1}{4}h_0 B_2 \mu_1^2. \tag{12}$$

From (9) and (11), it follows that

$$\eta_1 = -\mu_1 \tag{13}$$

and

$$\frac{2(\alpha+1)^2}{\alpha^2}a_2^2 = h_0^2 B_1^2(\eta_1^2 + \mu_1^2). \tag{14}$$

Now, by adding (10) and (12), we obtain

$$\left(\frac{\alpha+1}{\alpha} + \frac{1-\alpha}{2\alpha^2}\right)a_2^2 = \frac{1}{2}h_0 B_1(\eta_2 + \mu_2) + \frac{1}{4}h_0(B_2 - B_1)(\eta_1^2 + \mu_1^2). \tag{15}$$

Using (14) in (15), we get

$$a_2^2 = \frac{\alpha^2 h_0^2 B_1^3(\eta_2 + \mu_2)}{h_0 B_1^2(2\alpha^2 + \alpha + 1) - (B_2 - B_1)(\alpha+1)^2}. \tag{16}$$

Applying $|\eta_i| \le 2$ and $|\mu_i| \le 2$ for the coefficients η_2 and μ_2, we immediately have

$$|a_2|^2 \le \frac{4\alpha^2 |h_0|^2 B_1^3}{\left|h_0 B_1^2(2\alpha^2 + \alpha + 1) - (B_2 - B_1)(\alpha+1)^2\right|}.$$

This gives the bound on $|a_2|$ as asserted in Theorem 1.

Additionaly, in order to calculate the bound on $|a_3|$, by subtracting (12) from (10), we obtain

$$\frac{2(\alpha+1)}{\alpha}(a_3 - a_2^2) = \frac{1}{2}h_1 B_1(\eta_1 - \mu_1) + \frac{1}{2}h_0 B_1(\eta_2 - \mu_2) + \frac{1}{4}h_0(B_2 - B_1)(\eta_1^2 - \mu_1^2). \tag{17}$$

Using (13) and (14) in (17), we get

$$a_3 = \frac{\alpha h_1 B_1(\eta_1 - \mu_1)}{4(\alpha+1)} + \frac{\alpha h_0 B_1(\eta_2 - \mu_2)}{4(\alpha+1)} + \frac{\alpha^2 h_0^2 B_1^2(\eta_1^2 + \mu_1^2)}{2(\alpha+1)^2}.$$

Applying $|\eta_i| \le 2$ and $|\mu_i| \le 2$ once again for the coefficients η_2 and μ_2, we readily get the bound on $|a_3|$ as asserted in Theorem 1.

3. Concluding Corollaries

For the function ϕ is given by

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\xi} = 1 + 2\xi z + 2\xi^2 z^2 + \cdots \quad (0 < \xi \le 1)$$

which gives

$$B_1 = 2\xi \text{ and } B_2 = 2\xi^2,$$

Theorem 1 reduces to:

Corollary 1. Let $f \in S_q\left[\alpha, \left(\frac{1+z}{1-z}\right)^{\xi}\right]$. Then

$$|a_2| \leq \frac{4\alpha|h_0|\xi}{\sqrt{\left|\xi\left[(4h_0-1)\alpha^2+2(h_0-1)\alpha+2h_0-1\right]+(\alpha+1)^2\right|}}$$

and

$$|a_3| \leq \frac{2\alpha|h_1|\xi}{\alpha+1}+\frac{2\alpha|h_0|\xi}{\alpha+1}+\left(\frac{4\alpha|h_0|\xi}{\alpha+1}\right)^2.$$

For $\alpha=1$ and $-1\leq B\leq A<1$ if we take

$$\phi(z)=\frac{1+Az}{1+Bz}=1+(A-B)z-B(A-B)z^2+\cdots$$

Then we have

$$B_1=A-B \text{ and } B_2=-B(A-B).$$

Hence, we get the following corollary.

Corollary 2. Let $f\in S_q\left(\alpha,\frac{1+Az}{1+Bz}\right)$. Then

$$|a_2| \leq \frac{2\alpha|h_0|(A-B)}{\sqrt{\left|h_0(A-B)(2\alpha^2+\alpha+1)+(1+B)(\alpha+1)^2\right|}}$$

and

$$|a_3| \leq \frac{\alpha|h_1|(A-B)}{\alpha+1}+\frac{\alpha|h_0|(A-B)}{\alpha+1}+\left(\frac{2\alpha|h_0|(A-B)}{\alpha+1}\right)^2.$$

Further, by taking $\alpha=1$ and

$$\phi(z)=\frac{1+(1-2\beta)z}{1-z}=1+2(1-\beta)z+2(1-\beta)z^2+\cdots\ (0\leq\beta<1)$$

we get

$$B_1=B_2=2(1-\beta).$$

Hence, we get the following corollary.

Corollary 3. Let $f\in S_q\left(\alpha,\frac{1+(1-2\beta)z}{1-z}\right)$. Then

$$|a_2| \leq \frac{2\alpha\sqrt{2|h_0|(1-\beta)}}{\sqrt{2\alpha^2+\alpha+1}}$$

and

$$|a_3| \leq \frac{2\alpha|h_1|(1-\beta)}{\alpha+1}+\frac{2\alpha|h_0|(1-\beta)}{\alpha+1}+\left(\frac{4\alpha|h_0|(1-\beta)}{\alpha+1}\right)^2.$$

Finally, by taking $\alpha=1$ and

$$\phi(z)=\frac{1+z}{1-z}=1+2z+2z^2+\cdots$$

we get

$$B_1=B_2=2.$$

Hence, we get the following corollary.

Corollary 4. Let $f \in S_q\left(\alpha, \frac{1+z}{1-z}\right)$. Then

$$|a_2| \le \frac{2\alpha\sqrt{2|h_0|}}{\sqrt{2\alpha^2+\alpha+1}}$$

and

$$|a_3| \le \frac{2\alpha|h_1|}{\alpha+1} + \frac{2\alpha|h_0|}{\alpha+1} + \left(\frac{4\alpha|h_0|}{\alpha+1}\right)^2.$$

References

[1] A.Akgül, On second-order differential subordinations for a class of analytic functions defined by convolution. Journal of Nonlinear Science and Applications. 2017 (**10**): 954-963.

[2] R.M. Ali, S.K. Lee,V. Ravichandran, S. Supramaniam, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions. Applied Mathematics Letters. 2012 (**25**): 344-351.

[3] Ş. Altınkaya, S. Yalçın, Initial coefficient bounds for a comprehensive subclass of Sakaguchi type functions. Acta Universitatis Matthiae Belii, series Mathematics. 2015 (**23**): 27-33.

[4] Ş. Altınkaya, S. Yalçın, On a new subclass of bi-univalent functions satisfying subordinate conditions. Acta Universitatis Sapientiae, Mathematica. 2015 (**7**): 5-14.

[5] Ş. Altınkaya, S. Yalçın, Faber polynomial coefficient bounds for a subclass of bi-univalent functions. C. R. Acad. Sci. Paris, Ser. I. 2015 (**353**): 1075-1080.

[6] Ş. Altınkaya, S. Yalçın, Coefficient bounds for a subclass of bi-univalent functions.TWMS Journal of Pure and Applied Mathematics. 2015 (**6**): 180-185.

[7] Ş. Altınkaya, S. Yalçın, Coefficient estimates for a certain subclass of bi-univalent functions. Le Matematiche. 2016 (**71**): 53--61.

[8] O. Altıntaş,S. Owa, Majorizations and quasi-subordinations for certain analytic functions. Proc. Japan Acad. Ser. A Math. Sci. 1992(68): 181-185.

[9] D. A Brannan, T. S.Taha, On some classes of bi-univalent functions. Studia Universitatis Babeş-Bolyai Mathematica. 1986 (**31**): 70-77.

[10] P.L. Duren, Univalent Functions. Grundlehren der Mathematischen Wissenschaften, Bd. 259, 1983, Springer-Verlag, Berlin, Heidelberg, New York and Tokyo.

[11] R. El-Ashwah,S. Kanas, Fekete-Szegö inequalities for quasi-subordination functions classes of complex order. Kyungpook Mathematical Journal. 2015 (**55**): 679-688.

[12] S.P. Goyal, O. Singh, R. Mukherjee, Certain results on a subclass of analytic and bi-univalent functions associated with coefficient estimates and quasi-subordination. Palestine Journal of Mathematics. 2016 (**5**): 79-85.

[13] S.G. Hamidi,J.M. Jahangiri, Faber polynomial coefficient estimates for analytic bi-close-to-convex functions. C. R. Acad. Sci. Paris, Ser. I. 2014 (**352**): 17-20.

[14] W. Ma, D. Minda, A unified treatment of some special classes of univalent functions. in: Proceedings of the Conference on Complex Analysis, Tianjin, 1992, 157-169, Conf. Proc. Lecture Notes Anal. I, 1994, Int. Press, Cambridge, MA.

[15] T.H. Macgregor, Majorization by univalent functions. Duke Math. J. 1967 (**34**): 95-102.

[16] Z. Nehari, Conformal Mappings.1952, McGraw-Hill, New York.

[17] F. Ren, S. Owa, S. Fukui, Some inequalities on quasi-subordinate functions. Bull. Aust. Math. Soc. 1991 (**43**): 317-324.

[18] M.S. Robertson, Quasi-subordination and coefficients conjectures. Bull. Amer. Math. Soc. 1970 (**76**): 1-9.

[19] P. Sharma, R.K. Raina, On Sakaguchi type class of analytic functions associated with quasi-subordination. Commentarii Mathematici Universitatis Sancti Pauli. 2015 (**64**): 59-70.

[20] S. Sivasubramanian, R. Sivakumar,S. Kanas, S.-A. Kim, Verification of Brannan and Clunie's conjecture for certain subclasses of bi-univalent functions.Ann. Polonici Mathematici. 2015 (**113**): 295-304.

[21] H.M. Srivastava, A.K. Mishra, P.Gochhayat.Certain subclasses of analytic and bi-univalent functions. Appl. Math. Lett. 2010 (**23**): 1188-1192.

[22] H.M. Srivastava, S. Sümer Eker, R.M. Ali, Coefficient bounds for a certain class of analytic and bi-univalent functions. Filomat. 2015 (**29**): 1839-1845.

[23] P.P. Vyas, S. Kant, Certain subclasses of bi-univalent functions associated with quasi-subordination. J. Rajasthan Acad. Phys. Sci. 2015 (**15**): 315-325.

[24] Q.-H. Xu, Y.-C. Gu, H.M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions. Appl. Math. Lett. 2012 (**25**): 990-994.