

Conformal Weyl-Euler-Lagrangian equations on 4-Walker manifolds

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Abstract: The main purpose of the present paper is to study almost complex structures conformal Weyl-Euler-Lagrangian equations on 4-dimensional Walker manifolds for (conservative) dynamical systems. In this study, routes of objects moving in space will be modeled mathematically on 4-dimensional Walker manifolds that these are time-dependent partial differential equations. A Walker n -manifold is a semi-Riemannian n -manifold, which admits a field of parallel null r -planes, with $r \leq \frac{n}{2}$. It is well-known that semi-Riemannian geometry has an important tool to describe spacetime events. Therefore, solutions of some structures about 4-Walker manifold can be used to explain spacetime singularities. Then, here we present complex analogues of Lagrangian mechanical systems on 4-Walker manifold. Also, the geometrical-physical results related to complex mechanical systems are also discussed for conformal Weyl-Euler-Lagrangian equations for (conservative) dynamical systems and solution of the motion equations using Maple Algebra software will be made.

Keywords: Walker Manifolds, Weyl theory, holomorphic, symplectic geometry, conformal geometry, Lagrangian, mechanical system, Riemannian manifold, almost complex manifolds.

1 Introduction

It is well-known that an electromagnetic field is a physical field produced by electrically charged objects on the living space. How the movement of objects in electrical, magnetically and gravitational fields force is very important. For instance, on a weather map or the surface wind velocity is describes by assigning a vector to each point on a map. Also, each vector represents the speed and direction of the movement of air or surface at that point on defined space. A classical field theory explains the study of how one or more physical fields interact with matter which is used quantum and classical mechanics of physics branches.

Differential geometry is widely used as a mathematical operative branch for classical field theory. It known using the techniques of differential calculus, integral calculus, linear algebra and multilinear algebra to study problems in geometry. Also, a dynamical system is a concept in mathematics where a fixed rule describes how a point in a geometrical space depends on time. At any given time a dynamical system has a state given by a set of real numbers (a vector) that can be represented by a point in an appropriate state space or a geometrical manifold. In addition, a dynamical systems theory is an area of mathematics used to describe the behavior of complex dynamical systems, usually by employing differential equations or difference equations. The dynamic equations used to model mechanical systems. A model is a precise representation of a system's dynamics used to answer questions via analysis and simulation. The dynamic equations for moving objects are obtained for Lagrangian mechanics by many authors in many areas [1]-[3]. *Kasap* and *Tekkoyun* found Lagrangian and Hamiltonian formalism for mechanical systems using para/pseudo-Kähler manifolds, representing an interesting multidisciplinary field of research [4]. *Kasap* introduced Weyl-Euler-Lagrange equations of motion on flat manifold [5]. *Walker* shown that a Walker n -manifold, they mean a

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semi-Riemannian manifold which admits a field of parallel null r -planes, with $r \leq n/2$ [6]. *Salimov* et al examined that a Walker 4-manifold is a pseudo-Riemannian manifold, (M_4, g) of neutral signature, which admits a field of parallel null 2-plane [7]. *Matsushita* studied that a Walker 4-manifold, endowed with a canonical neutral metric depending on three arbitrary functions, admits a specific almost complex structure and an associated opposite almost complex structure [8]. *Garcia-Rio* et al showed that such a Walker 4-manifold can carry various structures with respect to a certain kind of almost complex structure, e.g., symplectic structures, Kähler structures, Hermitian structures, according as the properties of certain functions which define the canonical form of the metric [9]. *Batat* et al studied the curvature properties of a large class of four-dimensional Walker metrics [10]. *Nadjafikhah* and *Jafari* constructed that Lie symmetry group method is applied to find the lie point symmetries group of a PDE system that is determined general form of four-dimensional Einstein Walker manifold [11]. *Salimov* and *Iscan* showed that a Walker 4-manifold is a semi-Riemannian manifold (M_4, g) of neutral signature, which admits a field of parallel null 2-plane [12]. *Brozos-Vazquez* et al examined commutativity properties of the Ricci operator, of the skew-symmetric curvature operator, and of the Jacobi operator for certain Walker manifolds of signature $(2, 2)$ [13]. *Davidov* showed that any proper almost Hermitian structure on a Walker 4-manifold is isotropic Kähler [14]. *Tekkoyun* showed that a Walker n -manifold is a semi-Riemannian n -manifold, which admits a field of parallel null r -planes, with $r \leq n/2$ [15]. *Law* examined that a four-dimensional Walker geometry is a four-dimensional manifold M with a neutral metric g and a parallel distribution of totally null two-planes [16]. *Ghanam* and *Thompson* submitted that special interest manifolds are Walker manifolds of even dimensions $(n = 2m)$ admitting a field of null planes of maximum dimensionality $(r = m)$ [17].

2 Preliminaries

Definition 1. *Walker manifold* is a triple (M, g, D) where M is an n -dimensional manifold, g an indefinite metric and D an r -dimensional parallel null distribution [15].

Of special interest are those manifolds admitting a field of null planes of maximum dimensionality $r = n/2$. Since the dimension of a null plane is source $r \leq n/2$, the lowest possible case is that of $(+, +, -, -)$ -manifolds admitting a field of parallel null 2-planes.

Definition 2. A *metric tensor* is a non-degenerate, smooth, symmetric, bilinear map which assigns a real number to pairs of tangent vectors at each tangent space of the manifold. Denoting the metric tensor g we can express this as $g : T_p M \times T_p M \rightarrow \mathbb{R}$. The map is symmetric and bilinear so if $X, Y, Z \in T_p M$ are tangent vectors at a point p to the manifold M then we have

- (1) $g(X, Y) = g(Y, X)$,
- (2) $g(aX + Y, Z) = ag(X, Z) + g(Y, Z)$ for any real number $a \in \mathbb{R}$ [6].

Definition 3. A *pseudo-Riemannian manifold* (also called a semi-Riemannian manifold) (M, g) is a differentiable manifold M equipped with a non-degenerate, smooth, symmetric metric tensor g . Such a metric is called a pseudo-Riemannian metric and its values can be positive, negative or zero. The signature of a pseudo-Riemannian metric is (p, q) , where both p and q are non-negative. The model space for a pseudo-Riemannian manifold of signature (p, q) is $\mathbb{R}^{p,q}$ with the metric $g = dx_1^2 + dx_2^2 + \dots + dx_p^2 - dx_{p+1}^2 - \dots - dx_{p+q}^2$ [14].

Definition 4. Let M be a pseudo-Riemannian manifold of signature (p, q) . We suppose given a splitting of the tangent bundle in the form $TM = V_1 \oplus V_2$ where V_1 and V_2 are smooth subbundles which are called distributions. If M is Riemannian, we can take $V_2 = V_1^\perp$ to be the orthogonal complement of V_1 and in that case V_2 is again parallel. In the pseudo-Riemannian setting, of course, $V_2 \cap V_1$ need not be trivial and there exist examples where although V_1 is parallel, there exists no complementary parallel distribution. Let V_1 be a parallel distribution. The rank of g restricted to V_1 is constant. We can say that V_1 is a **null parallel distribution** if V_1 is parallel and if the metric restricted to V_1 vanishes identically [14].

Proposition 1. A neutral metric g on a 4-manifold M_4 is said to be Walker metric if there exists a 2-dimensional null distribution D on M_4 , which is parallel with respect to g . From Walker theorem there is a system of coordinates with respect to which g takes the local canonical form

$$g(x_1, x_2, x_3, x_4) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}, \quad (1)$$

where a, b, c are **smooth functions** of the coordinates (x_1, x_2, x_3, x_4) . Also, h_{x_1, \dots, x_r} means partial derivatives $\frac{\partial h}{\partial x_1 \dots \partial x_r}$ for any function $h(x_1, x_2, x_3, x_4)$. The parallel null 2-plane D is spanned locally by $\{\partial_{x_1}, \partial_{x_2}\}$, where ∂_{x_i} are abbreviated forms of $\partial_{x_i} = \partial/\partial x_i, \dots, \partial_{x_4} = \partial/\partial x_4$ [6, 15]. Walker [6] studied pseudo-Riemannian manifolds M with a parallel field of null planes D and derived a canonical form. Motivated by this seminal work, one says that a pseudo-Riemannian manifold M which admits a null parallel i.e., (degenerate) distribution D is a Walker manifold.

3 The theory of J -holomorphic curves

Definition 5. J -holomorphic curve is a smooth map from a Riemann surface into an almost complex manifold that satisfies the Cauchy-Riemann equation.

Definition 6. Let ω be a 2-form on a manifold M . For each point $p \in M$, the map $\omega_p : T_p M \times T_p M \rightarrow \mathbb{R}$ is skew-symmetric and bilinear on the tangent space to M at p , and ω_p varies smoothly in p .

Definition 7. The 2-form ω is symplectic if ω is closed (its exterior derivative is $d\omega = 0$) and ω_p is symplectic for all $p \in M$. A symplectic manifold is a pair (M, ω) where M is a manifold and ω is a symplectic form. Symplectic manifolds must be even-dimensional. Let $M = \mathbb{R}^{2n}$ with linear coordinates $x_1, \dots, x_n, y_1, \dots, y_n$. The form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$ is symplectic.

Example 1. An almost complex symplectic manifold is standard Euclidean space $(\mathbb{R}^{2n}, \omega_0)$ with its standard almost complex structure J_0 obtained from the usual identification with \mathbb{C}^n . Thus, one sets $z_j = x_{2j-1} + ix_{2j}$ for $j = 1, \dots, n$ and defines J_0 by $J_0(\partial_{2j-1}) = \partial_{2j}$, $J_0(\partial_{2j}) = -\partial_{2j-1}$ where $\partial_j = \partial/\partial x_j$ is the standard basis of $T_x \mathbb{R}^{2n}$ [18].

Definition 8. Let M be a differentiable manifold of dimension $2n$, and suppose J is a differentiable vector bundle isomorphism $J : TM \rightarrow TM$ such that $J_x : T_x M \rightarrow T_x M$ is a **complex structure** for $T_x M$, i.e. $J^2 = -I$ where I is the identity (unit) operator on V . Then J is called an almost-complex structure for the differentiable manifold M . A manifold with a fixed **almost complex structure** is called an **almost complex manifold** [15].

4 Gauge theory and conformal Weyl geometry

Today, the gauge principle is arguably the most powerful concept in all of modern physics. This gauge principle underlies all of the Yang-Mills theories that it is a key component in string theory and its more recent variant, M theory. The approach for studying conformal field theories is somewhat different from the usual approach for quantum and electromagnetic field theories. Conformal maps can be defined between domains in higher dimensional Euclidean spaces and on a Riemann or semi-Riemann manifold.

H. Weyl made many fundamental and important contributions to physics and he is most famous for his 1929 discovery of quantum-mechanical phase invariance. Phase invariance known more properly as gauge invariance that it is a symmetry and underlies all modern quantum theories. Also, Weyl tensor can be deduced by simply demanding that it be invariant with respect to this transformation. Weyl, using this gauge principle, was able to derive all of electrodynamics from a generalized Einstein-Maxwell Lagrangian.

Definition 9. A *conformal map or transformations* is a function which preserves angles [19].

Definition 10. A *conformal manifold* is a differentiable manifold equipped with an equivalence class of (pseudo) Riemann metric tensors, in which two metrics g_2 and g_1 are equivalent if and only if

$$g_2 = \Psi^2 g_1 \quad (2)$$

where $\Psi > 0$ is a smooth positive function. An equivalence class of such metrics is known as a **conformal metric** or **conformal class** and a manifold with a conformal structure (2) is called a conformal manifold [19].

Definition 11. A change in given by the **global gauge transformation**

$$\Psi(x) \rightarrow e^{i\lambda} \Psi(x) \quad (3)$$

where λ is an arbitrary constant, would have absolutely no effect on the Lagrangian [20].

Definition 12. Action Lagrangians are invariant with respect to the replacement

$$\Psi(x) \rightarrow e^{i\lambda(x)} \Psi(x) \quad (4)$$

can be made without changing anything essential. Where Ψ is a wave function and λ is an arbitrary function of space and time. (4) is called a **local gauge transformation**. Weyl's gauge theory sprang from an even earlier (1918) theory in which Weyl demanded that Einstein's theory of general relativity should be invariant with respect to the similar replacement

$$g_{\mu\nu}(x) \rightarrow e^{\lambda(x)} g_{\mu\nu}(x) \quad (5)$$

which we shall call a **metric gauge transformation** (5) and it has emerged effect of these transformations on Riemannian and non-Riemannian geometry [20].

Definition 13. Two Riemann metrics g_1 and g_2 on M are said to be conformally equivalent iff there exists a smooth function $f : M \rightarrow \mathbb{R}$ with

$$g_2 = e^f g_1. \quad (6)$$

In this case, $g_1 \sim g_2$ [21].

Definition 14. A pair (M, G) , a conformal structure on M is an equivalence class G of Riemann metrics on M , is called a **conformal structure**.

Theorem 1. Let ∇ be a connection on M and $g \in G$ a fixed metric. ∇ is compatible with $(M, G) \iff$ there exists a 1-form ω with $\nabla_X g + \omega(X)g = 0$ [22].

Definition 15. A compatible torsion-free connection is called a **Weyl connection**. The triple (M, G, ∇) is a **Weyl structure**. A **Weyl manifold** is a conformal manifold equipped with a torsion free connection preserving the conformal structure, called a Weyl connection [21].

Definition 16. Consider a triple (M, g, ∇) where g is a pseudo Riemannian metric on a smooth n dimensional manifold M and where ∇ is a torsion free connection on the tangent bundle TM of M . We suppose $n \geq 2$ henceforth. We say that (M, g, ∇) is a Weyl manifold if the following identity is satisfied: $\nabla g = -2\phi \otimes g$ for some $\phi \in C^\infty(T^*M)$. This notion is conformally invariant. If (M, g, ∇) is a Weyl manifold, then $(M, e^{2f}g, \nabla)$ is again a Weyl manifold where $\phi := \phi - df$. The simultaneous transformation of the pair (g, ϕ) is called a **gauge transformation**, properties of the Weyl geometry that are invariant under gauge transformations are called **gauge invariants** [23].

Theorem 2. To each metric $g \in G$ and 1-form ω , there corresponds a unique Weyl connection ∇ satisfying $\nabla_X g + \omega(X)g = 0$. Here, ∇ is given by the equation [22].

$$g(\nabla_X Y, Z) = \frac{1}{2} \left\{ \begin{aligned} &X(g(Y, Z)) + \omega(X)g(Y, Z) - g([X, Z], Y) + Y(g(X, Z)) \\ &+ \omega(Y)g(Z, X) - g([Y, X], Z) - Z(g(X, Y)) - \omega(Z)g(X, Y) - g([Z, Y], X) \end{aligned} \right\} \quad (7)$$

Definitions 17. Define a function $F : \{ 1\text{-forms on } M \} \times G \rightarrow \{ \text{Weyl connections} \}$ by $F(g, \omega) = \nabla$, where ∇ is the connection guaranteed by **Theorem 2**. We say that ∇ corresponds to (g, ω) [22].

Proposition 2. F is surjective.

Proof. F is surjective by Theorem 1. In fact, Theorem 1. shows that given a compatible, torsion-free connection ∇ , then for every $g \in G$, there exists a 1-form ω with $F(g, \omega) = \nabla$.

Proposition 3. $F(g, \omega) = F(e^f g, \eta)$ iff $\eta = \omega - df$. So

$$F(e^f g) = F(g) - df, \quad (8)$$

where G is a conformal structure. Note that a Riemann metric g and a one-form ω determine a Weyl structure, namely $F : G \rightarrow \wedge^1 M$ where G is the equivalence class of g and $F(e^f g) = \omega - df$ [21, 22].

Proof. Suppose $F(g, \omega) = F(e^f g, \eta) = \nabla$. We have

$$0 = \nabla_X(e^f g) + \eta(X)e^f g = X(e^f)g + e^f \nabla_X g + \eta(X)e^f g = df(X)e^f g + e^f \nabla_X g + \eta(X)e^f g. \quad (9)$$

Therefore $\nabla_X g = -(df(X) + \eta(X))$. On the other hand $\nabla_X g + \omega(X)g = 0$ and $\omega = \eta + df$. Conversely, suppose $\eta = \omega - df$. Set $\nabla = F(g, \omega)$. To show $\nabla = F(e^f g, \eta)$, it suffices, by the uniqueness of Theorem 2., to show $\nabla_X(e^f g) + \eta(X)e^f g = 0$. Let's show the truth of this statement. $\nabla_X(e^f g) = e^f df(X)g + e^f \nabla_X g$ and $\eta(X) = \omega(X) - df(X)$.

$$\begin{aligned} \nabla_X(e^f g) + \eta(X)e^f g &= e^f df(X)g + e^f \nabla_X g + (\omega(X) - df(X))e^f g \\ &= e^f df(X)g + e^f \nabla_X g + \omega(X)e^f g - df(X)e^f g = e^f (\nabla_X g + \omega(X)g) = 0. \end{aligned} \quad (10)$$

Theorem 3. Let (M, g) is conformally flat if for each point x in M , there exists a neighborhood U of x and a smooth function f defined on U such that $(U, e^{2f}g)$ is flat. The function f need not be defined on all of M [24].

Theorem 4. Let $m \geq 6$. If (M, g, J, ∇) is a (para)-Kähler-Weyl structure, then the associated Weyl structure is trivial, i.e. there is a conformally equivalent metric

$$g_1 = e^{2f}g \quad (11)$$

so that (M, g_1, J) is (para)-Kähler and so that $\nabla = \nabla^{g_1}$ [25].

Definition 17. Weyl curvature tensor is a measure of the curvature of spacetime or a pseudo-Riemannian manifold. Like the Riemannian curvature tensor, the Weyl tensor expresses the tidal force that a object feels when moving along a geodesic [26].

Definition 18. Weyl transformation is a local rescaling of the metric tensor: $g_{ab}(x) \rightarrow e^{-2\omega(x)}g_{ab}(x)$ which produces another metric in the same conformal class. A theory or an expression invariant under this transformation is called conformally invariant, or is said to possess Weyl symmetry. The Weyl symmetry is an important **symmetry in conformal field theory** [27].

5 Properties of almost complex structure J

Example 2. Let O be an open subset of \mathbb{R}^4 . Let $a, b, c \in C^\infty(O)$ be smooth function on O . We set $M_{a,b,c} := (O, g_{a,b,c})$ where

$$g_{a,b,c} := 2(dx_1 \circ dx_3 + dx_2 \circ dx_4) + a(x_1, x_2, x_3, x_4) dx_3 \circ dx_3 + b(x_1, x_2, x_3, x_4) dx_3 \circ dx_4 + 2c(x_1, x_2, x_3, x_4) dx_3 \circ dx_4. \tag{12}$$

Example 3. Let $M_{a,b,c} := (O, g_{a,b,c})$ be the pseudo-Riemannian manifold of (12) and we take the induced orthonormal basis;

$$e_1 = \frac{1}{2}(1-a)\partial_{x_1} + \partial_{x_3}, \quad e_2 = -c\partial_{x_2} + \frac{1}{2}(1-b)\partial_{x_4}, \\ e_3 = -\frac{1}{2}(1+a)\partial_{x_1} + \partial_{x_3}, \quad e_4 = -c\partial_{x_2} - \frac{1}{2}(1+b)\partial_{x_4}. \tag{13}$$

We shall let $C_{a,b,c} := (O, g_{a,b,c}, J)$ where J is the proper almost complex structure defined by $J = e_2 \otimes e^1 - e_1 \otimes e^2 + e_4 \otimes e^3 - e_3 \otimes e^4$. Thus, the following expression is obtained: $J : e_1 = e_2, J : e_2 = -e_1, J : e_3 = e_4, J : e_4 = -e_3$. The map J induces a positive $\frac{\pi}{2}$ -rotation on the degenerate parallel field $D := Span\{\partial_{x_1}, \partial_{x_2}\}$:

$$J\partial_{x_1} = \partial_{x_2}, \quad J\partial_{x_3} = -c\partial_{x_1} + \frac{1}{2}(a-b)\partial_{x_2} + \partial_{x_4}, \\ J\partial_{x_2} = -\partial_{x_1}, \quad J\partial_{x_4} = \frac{1}{2}(a-b)\partial_{x_1} + c\partial_{x_2} - \partial_{x_3}. \tag{14}$$

The above structures were taken from [14, 28]. The following result shows that the class of isotropic Kähler structures is larger than might at first sight be expected:

Theorem 5. Let $C_{a,b,c} := O, g_{a,b,c}$ be as given in Example 2. $C_{a,b,c}$ is almost Kähler if and only if $a_1 + b_1 = 0$ and $a_2 + b_2 = 0$, (2) $C_{a,b,c}$ is Hermitian if and only if $a_1 + b_1 = 2c_2$ and $a_2 - b_2 = -2c_1$, (3) $C_{a,b,c}$ is is Kähler if and only if $a_1 = -b_1 = c_2$ and $a_2 = -b_2 = -c_1$ [14].

Definition 19. In three dimensions, the vector from the origin to the point with Cartesian coordinates (x, y, z) can be written as: $r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x\left(\frac{\partial}{\partial x}\right) + y\left(\frac{\partial}{\partial y}\right) + z\left(\frac{\partial}{\partial z}\right)$.

Proposition 4. We recommend transferring the functions of the Weyl conformal factor as follows. J holomorphic property preservation is proposed using Theorems 4 and Definition 19 as follows.

$$(1) J \frac{\partial}{\partial x_1} = e^{2f} \frac{\partial}{\partial x_2}, \quad (3) J \frac{\partial}{\partial x_3} = -ce^{-2f} \frac{\partial}{\partial x_1} + \frac{1}{2}(a-b)e^{2f} \frac{\partial}{\partial x_2} + e^{2f} \frac{\partial}{\partial x_4}, \\ (2) J \frac{\partial}{\partial x_2} = -e^{-2f} \frac{\partial}{\partial x_1}, \quad (4) J \frac{\partial}{\partial x_4} = \frac{1}{2}(a-b)e^{-2f} \frac{\partial}{\partial x_1} + ce^{-2f} \frac{\partial}{\partial x_2} - e^{-2f} \frac{\partial}{\partial x_3}, \tag{15}$$

Proof.

$$(1) J^2 \frac{\partial}{\partial x_1} = e^{2f} J \frac{\partial}{\partial x_2} = -\frac{\partial}{\partial x_1}, \\ (2) J^2 \frac{\partial}{\partial x_2} = -e^{-2f} J \frac{\partial}{\partial x_1} = -\frac{\partial}{\partial x_2}, \\ (3) J^2 \frac{\partial}{\partial x_3} = -ce^{-2f} J \frac{\partial}{\partial x_1} + \frac{1}{2}(a-b)e^{2f} J \frac{\partial}{\partial x_2} + e^{2f} J \frac{\partial}{\partial x_4} = -\frac{\partial}{\partial x_3}, \\ (4) J^2 \frac{\partial}{\partial x_4} = \frac{1}{2}(a-b)e^{-2f} J \frac{\partial}{\partial x_1} + ce^{-2f} J \frac{\partial}{\partial x_2} - e^{-2f} J \frac{\partial}{\partial x_3} = -\frac{\partial}{\partial x_4}. \tag{16}$$

As seen above; holomorphic structures ($J^2 \frac{\partial}{\partial x_i} = -\frac{\partial}{\partial x_i}$ or $J^2 = -I$) are complex.

6 Euler-Lagrange dynamics equations

The Euler-Lagrange equation, Euler’s equation or Lagrange’s equation is a second-order partial differential equation whose solutions are the functions for which a given functional is stationary.

Lemma 1. The closed 2-form (ω) on a vector field (ξ) and 1-form reduction function (i_ξ) on the phase space defined of a mechanical system $(i_\xi \omega)$ is equal to the differential of the energy function 1-form (dE) of the Hamiltonian and the Lagrangian mechanical systems [1].

Definition 20. Let M be an n -dimensional manifold and TM its tangent bundle with canonical projection $\tau_M : TM \rightarrow M$. TM is called the phase space of velocities of the base manifold M . Let $L : TM \rightarrow \mathbb{R}$ be a differentiable function on TM called the **Lagrangian function**. Here, $L = T - V$ such that T is the kinetic energy and V is the potential energy of a mechanical system. In the problem of a mass on the end of a spring, $T = m\dot{x}^2/2$ and $V = kx^2/2$. We consider the closed 2-form and base space (J) on TM given by $\Phi_L = -d\mathbf{d}_J L = -d(J(\mathbf{d}))$. Consider the equation

$$i_\xi \Phi_L = dE_L. \tag{17}$$

Where i_ξ is reduction function and $i_\xi \Phi_L = \Phi_L(\xi)$ is defined in the form. Then ξ is a vector field, we shall see that (17) under a certain condition on ξ is the intrinsical expression of the Euler-Lagrange equations of motion. This equation (17) is named as **Lagrange dynamical equation** [15].

Definition 21. We shall see that for motion in a potential, $E_L = VL - L$ is an energy function and $V = J\xi$ a Liouville vector field. Here dE_L denotes the differential of E . The triple (TM, Φ_L, ξ) is known as **Lagrangian system** on the tangent bundle TM . If it is continued the operations on (17) for any coordinate system then infinite dimension **Lagrange's equation** is obtained the form below. The equations of motion in Lagrangian mechanics are the Lagrange equations of the second kind, also known as the Euler-Lagrange equations [2].

$$\frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0. \tag{18}$$

Definition 22. Newton's second law of motion for the mechanical problem is $F = ma$. Where, the vector sum of the external forces F on an object is equal to the mass m of that object multiplied by the acceleration vector a of the object.

Proposition 5. We have $\partial L/\partial \dot{x} = mx$ and $\partial L/\partial x = -kx$, so eq. (18) gives $m\ddot{x} = -kx$. The Euler-Lagrange equation, eq. (18), gives $m\ddot{x} = -dV/dx$. In a four-dimensional setup written in terms of Cartesian coordinates, the potential takes the form $V(x_1, x_2, x_3, x_4)$, so the Lagrangian is $L = \frac{1}{2}m_i(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2) - V(x_1, x_2, x_3, x_4)$. So, the three Euler-Lagrange equations may be combined into the vector statement $m\ddot{x} = -\nabla V$.

7 Conformal Weyl-Euler-Lagrange equations

In this section we, using (17), will introduced the movement equation on Walker manifold. Let M_4 be a Walker manifold and $\{x_1, x_2, x_3, x_4\}$ be its coordinate. Let ξ be a the semispray vector field determined by

$$\xi = \sum_{i=1}^4 X^i \frac{\partial}{\partial x_i}. \tag{19}$$

Also, $X^i = \dot{x}_i$ the dot indicates the derivative with respect to time t . Euler-Lagrange equations are obtained by using. By means of the proper almost complex structure J given by (15), the vector field is defined by

$$V = J(\xi) = X^1 e^{2f} \frac{\partial}{\partial x_2} - X^2 e^{-2f} \frac{\partial}{\partial x_1} + X^3 \left(-ce^{-2f} \frac{\partial}{\partial x_1} + \frac{1}{2}(a-b)e^{2f} \frac{\partial}{\partial x_2} + e^{2f} \frac{\partial}{\partial x_4} \right) + X^4 \left(\frac{1}{2}(a-b)e^{-2f} \frac{\partial}{\partial x_1} + ce^{-2f} \frac{\partial}{\partial x_2} - e^{-2f} \frac{\partial}{\partial x_3} \right), \tag{20}$$

which is named Liouville vector field on the Walker manifold M_4 . The maps given by $T, P : M_4 \rightarrow \mathbb{R}$ such that $T = m_i(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2)/2$, $V = m_i gh$ are said to be the kinetic energy and the potential energy of the system, respectively. Here m_i, g and h stand for mass of a mechanical system having m particles, the gravity acceleration and distance to the origin of a mechanical system on Walker manifold M_4 , respectively. Then $L : M_4 \rightarrow \mathbb{R}$ is a map that satisfies the conditions; (i) $L = T - V$ is a Lagrangian function, (ii) the function determined by $E_L = VL - L$, is energy function. The function i_J induced by J and denoted by

$$i_J \omega(X_1, X_2, \dots, X_r) = \sum_{i=1}^r J(X_i, \dots, JX_i, \dots, X_r), \tag{21}$$

is called vertical derivation, where $\omega \in \wedge^r M_4$, $X_i \in \chi(M_4)$. The vertical differentiation d_J is given by $\mathbf{d}_J = [i_J, d] = i_J d - d i_J$, where d is the usual exterior derivation. For the almost complex structure J given by (15), the form on Walker manifold M_4 is the closed 2-form determined by $\Phi_L = -d\mathbf{d}_J L$ such that $d_J : \mathcal{F}(M_4) \rightarrow \wedge^1 M_4$,

$$\begin{aligned} \mathbf{d} &= \sum_{i=1}^4 \frac{\partial}{\partial x_i} dx_i, \text{ and} \\ \mathbf{d}_J &= J(\mathbf{d}) = e^{2f} \frac{\partial}{\partial x_2} dx_1 - e^{-2f} \frac{\partial}{\partial x_1} dx_2 + \left(-ce^{-2f} \frac{\partial}{\partial x_1} + \frac{1}{2}(a-b)e^{2f} \frac{\partial}{\partial x_2} + e^{2f} \frac{\partial}{\partial x_4} \right) dx_3 \\ &\quad + \left(\frac{1}{2}(a-b)e^{-2f} \frac{\partial}{\partial x_1} + ce^{-2f} \frac{\partial}{\partial x_2} - e^{-2f} \frac{\partial}{\partial x_3} \right) dx_4. \end{aligned} \quad (22)$$

Now, we calculate the first part ($i_\xi \Phi_L$) of (17). Through a direct computation using (22), the closed 2-form $\Phi_L = -d(J(\mathbf{d}))$ is seen to be as follows:

$$\Phi_L = \sum_{i=1}^4 \left[\begin{aligned} &\left(e^{2f} \frac{\partial^2 L}{\partial x_i \partial x_2} + 2e^{2f} \frac{\partial f}{\partial x_i} \frac{\partial L}{\partial x_2} \right) dx_1 \wedge dx_i + \left(-e^{-2f} \frac{\partial^2 L}{\partial x_i \partial x_1} + 2e^{-2f} \frac{\partial f}{\partial x_i} \frac{\partial L}{\partial x_1} \right) dx_2 \wedge dx_i \\ &+ \left(-ce^{-2f} \frac{\partial^2 L}{\partial x_i \partial x_1} + 2ce^{-2f} \frac{\partial f}{\partial x_i} \frac{\partial L}{\partial x_1} + \frac{1}{2}(a-b)e^{2f} \frac{\partial^2 L}{\partial x_i \partial x_2} \right. \\ &\quad \left. + 2\frac{1}{2}(a-b)e^{2f} \frac{\partial f}{\partial x_i} \frac{\partial L}{\partial x_2} + e^{2f} \frac{\partial^2 L}{\partial x_i \partial x_4} + 2e^{2f} \frac{\partial f}{\partial x_i} \frac{\partial L}{\partial x_4} \right) dx_3 \wedge dx_i \\ &+ \left(\frac{1}{2}(a-b)e^{-2f} \frac{\partial^2 L}{\partial x_i \partial x_1} - 2\frac{1}{2}(a-b)e^{-2f} \frac{\partial f}{\partial x_i} \frac{\partial L}{\partial x_1} + ce^{-2f} \frac{\partial^2 L}{\partial x_i \partial x_2} \right. \\ &\quad \left. - 2ce^{-2f} \frac{\partial f}{\partial x_i} \frac{\partial L}{\partial x_2} - e^{-2f} \frac{\partial^2 L}{\partial x_i \partial x_3} + 2e^{-2f} \frac{\partial f}{\partial x_i} \frac{\partial L}{\partial x_3} \right) dx_4 \wedge dx_i \end{aligned} \right]. \quad (23)$$

We obtain, ξ vector based processing on (23), the following statement $\Phi_L(\xi)$;

$$\begin{aligned} &-X^1 \left(e^{2f} \frac{\partial^2 L}{\partial x_1 \partial x_2} + 2e^{2f} \frac{\partial f}{\partial x_1} \frac{\partial L}{\partial x_2} \right) dx_1 - X^1 \left(-e^{-2f} \frac{\partial^2 L}{\partial x_1 \partial x_1} + 2e^{-2f} \frac{\partial f}{\partial x_1} \frac{\partial L}{\partial x_1} \right) dx_2 \\ &-X^1 \left(-ce^{-2f} \frac{\partial^2 L}{\partial x_1 \partial x_1} + 2ce^{-2f} \frac{\partial f}{\partial x_1} \frac{\partial L}{\partial x_1} + \frac{1}{2}(a-b)e^{2f} \frac{\partial^2 L}{\partial x_1 \partial x_2} \right. \\ &\quad \left. + 2\frac{1}{2}(a-b)e^{2f} \frac{\partial f}{\partial x_1} \frac{\partial L}{\partial x_2} + e^{2f} \frac{\partial^2 L}{\partial x_1 \partial x_4} + 2e^{2f} \frac{\partial f}{\partial x_1} \frac{\partial L}{\partial x_4} \right) dx_3 \\ &-X^1 \left(\frac{1}{2}(a-b)e^{-2f} \frac{\partial^2 L}{\partial x_1 \partial x_1} - 2\frac{1}{2}(a-b)e^{-2f} \frac{\partial f}{\partial x_1} \frac{\partial L}{\partial x_1} + ce^{-2f} \frac{\partial^2 L}{\partial x_1 \partial x_2} \right. \\ &\quad \left. - 2ce^{-2f} \frac{\partial f}{\partial x_1} \frac{\partial L}{\partial x_2} - e^{-2f} \frac{\partial^2 L}{\partial x_1 \partial x_3} + 2e^{-2f} \frac{\partial f}{\partial x_1} \frac{\partial L}{\partial x_3} \right) dx_4 \\ &-X^2 \left(-ce^{-2f} \frac{\partial^2 L}{\partial x_2 \partial x_1} + 2ce^{-2f} \frac{\partial f}{\partial x_2} \frac{\partial L}{\partial x_1} + \frac{1}{2}(a-b)e^{2f} \frac{\partial^2 L}{\partial x_2 \partial x_2} \right. \\ &\quad \left. + 2\frac{1}{2}(a-b)e^{2f} \frac{\partial f}{\partial x_2} \frac{\partial L}{\partial x_2} + e^{2f} \frac{\partial^2 L}{\partial x_2 \partial x_4} + 2e^{2f} \frac{\partial f}{\partial x_2} \frac{\partial L}{\partial x_4} \right) dx_3 \\ &-X^2 \left(\frac{1}{2}(a-b)e^{-2f} \frac{\partial^2 L}{\partial x_2 \partial x_1} - 2\frac{1}{2}(a-b)e^{-2f} \frac{\partial f}{\partial x_2} \frac{\partial L}{\partial x_1} \right. \\ &\quad \left. + ce^{-2f} \frac{\partial^2 L}{\partial x_2 \partial x_2} - 2ce^{-2f} \frac{\partial f}{\partial x_2} \frac{\partial L}{\partial x_2} - e^{-2f} \frac{\partial^2 L}{\partial x_2 \partial x_3} + 2e^{-2f} \frac{\partial f}{\partial x_2} \frac{\partial L}{\partial x_3} \right) dx_4 \\ &-X^3 \left(e^{2f} \frac{\partial^2 L}{\partial x_3 \partial x_2} + 2e^{2f} \frac{\partial f}{\partial x_3} \frac{\partial L}{\partial x_2} \right) dx_1 - X^3 \left(-e^{-2f} \frac{\partial^2 L}{\partial x_3 \partial x_1} + 2e^{-2f} \frac{\partial f}{\partial x_3} \frac{\partial L}{\partial x_1} \right) dx_2 \\ &-X^3 \left(-ce^{-2f} \frac{\partial^2 L}{\partial x_3 \partial x_1} + 2ce^{-2f} \frac{\partial f}{\partial x_3} \frac{\partial L}{\partial x_1} + \frac{1}{2}(a-b)e^{2f} \frac{\partial^2 L}{\partial x_3 \partial x_2} \right. \\ &\quad \left. + 2\frac{1}{2}(a-b)e^{2f} \frac{\partial f}{\partial x_3} \frac{\partial L}{\partial x_2} + e^{2f} \frac{\partial^2 L}{\partial x_3 \partial x_4} + 2e^{2f} \frac{\partial f}{\partial x_3} \frac{\partial L}{\partial x_4} \right) dx_3 \\ &-X^3 \left(\frac{1}{2}(a-b)e^{-2f} \frac{\partial^2 L}{\partial x_3 \partial x_1} - 2\frac{1}{2}(a-b)e^{-2f} \frac{\partial f}{\partial x_3} \frac{\partial L}{\partial x_1} + ce^{-2f} \frac{\partial^2 L}{\partial x_3 \partial x_2} \right. \\ &\quad \left. - 2ce^{-2f} \frac{\partial f}{\partial x_3} \frac{\partial L}{\partial x_2} - e^{-2f} \frac{\partial^2 L}{\partial x_3 \partial x_3} + 2e^{-2f} \frac{\partial f}{\partial x_3} \frac{\partial L}{\partial x_3} \right) dx_4 \\ &-X^4 \left(e^{2f} \frac{\partial^2 L}{\partial x_4 \partial x_2} + 2e^{2f} \frac{\partial f}{\partial x_4} \frac{\partial L}{\partial x_2} \right) dx_1 - X^4 \left(-e^{-2f} \frac{\partial^2 L}{\partial x_4 \partial x_1} + 2e^{-2f} \frac{\partial f}{\partial x_4} \frac{\partial L}{\partial x_1} \right) dx_2 \\ &-X^4 \left(-ce^{-2f} \frac{\partial^2 L}{\partial x_4 \partial x_1} + 2ce^{-2f} \frac{\partial f}{\partial x_4} \frac{\partial L}{\partial x_1} + \frac{1}{2}(a-b)e^{2f} \frac{\partial^2 L}{\partial x_4 \partial x_2} \right. \\ &\quad \left. + 2\frac{1}{2}(a-b)e^{2f} \frac{\partial f}{\partial x_4} \frac{\partial L}{\partial x_2} + e^{2f} \frac{\partial^2 L}{\partial x_4 \partial x_4} + 2e^{2f} \frac{\partial f}{\partial x_4} \frac{\partial L}{\partial x_4} \right) dx_3 \\ &-X^4 \left(\frac{1}{2}(a-b)e^{-2f} \frac{\partial^2 L}{\partial x_4 \partial x_1} - 2\frac{1}{2}(a-b)e^{-2f} \frac{\partial f}{\partial x_4} \frac{\partial L}{\partial x_1} + ce^{-2f} \frac{\partial^2 L}{\partial x_4 \partial x_2} \right. \\ &\quad \left. - 2ce^{-2f} \frac{\partial f}{\partial x_4} \frac{\partial L}{\partial x_2} - e^{-2f} \frac{\partial^2 L}{\partial x_4 \partial x_3} + 2e^{-2f} \frac{\partial f}{\partial x_4} \frac{\partial L}{\partial x_3} \right) dx_4. \end{aligned} \quad (25)$$

Then the energy function E_L is found as follows:

$$E_L = V(L) - L = X^1 e^{2f} \frac{\partial L}{\partial x_2} - X^2 e^{-2f} \frac{\partial L}{\partial x_1} + X^3 \left(-c e^{-2f} \frac{\partial L}{\partial x_1} + \frac{1}{2} (a - b) e^{2f} \frac{\partial L}{\partial x_2} + e^{2f} \frac{\partial L}{\partial x_4} \right) + X^4 \left(\frac{1}{2} (a - b) e^{-2f} \frac{\partial L}{\partial x_1} + c e^{-2f} \frac{\partial L}{\partial x_2} - e^{-2f} \frac{\partial L}{\partial x_3} \right) - L. \tag{26}$$

Now, we calculate the second part of (17). Thus, the differential energy function is as follows:

$$dE_L = \sum_{i=1}^4 \left[\begin{aligned} & X^1 \left[e^{2f} \frac{\partial^2 L}{\partial x_i \partial x_2} dx_i + 2e^{2f} \frac{\partial f}{\partial x_i} \frac{\partial L}{\partial x_2} dx_i \right] - X^2 \left[e^{-2f} \frac{\partial^2 L}{\partial x_i \partial x_1} dx_i - 2e^{-2f} \frac{\partial f}{\partial x_i} \frac{\partial L}{\partial x_1} dx_i \right] \\ & + X^3 \left[-c e^{-2f} \frac{\partial^2 L}{\partial x_i \partial x_1} dx_i + 2c e^{-2f} \frac{\partial f}{\partial x_i} \frac{\partial L}{\partial x_1} dx_i + \frac{1}{2} (a - b) e^{2f} \frac{\partial^2 L}{\partial x_i \partial x_2} dx_i \right. \\ & \quad \left. + \frac{1}{2} (a - b) 2e^{2f} \frac{\partial f}{\partial x_i} \frac{\partial L}{\partial x_2} dx_i + e^{2f} \frac{\partial^2 L}{\partial x_i \partial x_4} dx_i + 2e^{2f} \frac{\partial f}{\partial x_i} \frac{\partial L}{\partial x_4} dx_i \right] \\ & + X^4 \left[\frac{1}{2} (a - b) e^{-2f} \frac{\partial^2 L}{\partial x_i \partial x_1} dx_i - \frac{1}{2} (a - b) 2e^{-2f} \frac{\partial f}{\partial x_i} \frac{\partial L}{\partial x_1} dx_i + c e^{-2f} \frac{\partial^2 L}{\partial x_i \partial x_2} dx_i \right. \\ & \quad \left. - c 2e^{-2f} \frac{\partial f}{\partial x_i} \frac{\partial L}{\partial x_2} dx_i - e^{-2f} \frac{\partial^2 L}{\partial x_i \partial x_3} dx_i + 2e^{-2f} \frac{\partial f}{\partial x_i} \frac{\partial L}{\partial x_3} dx_i \right] - \frac{\partial L}{\partial x_i} dx_i \end{aligned} \right]. \tag{27}$$

Definition 23. Suppose that $\xi : \mathbb{R} \rightarrow M_4$ is a vector field: that is, a vector-valued function with Cartesian coordinates (ξ_1, \dots, ξ_n) ; and $\mathbf{x}(t)$ a parametric curve with Cartesian coordinates $(x_1(t), \dots, x_n(t))$. Then $\mathbf{x}(t)$ is an **integral curve** of ξ if it is a solution of the following autonomous system of ordinary differential equations: $\frac{dx_1}{dt} = \xi_1(x_1, \dots, x_n), \dots, \frac{dx_n}{dt} = \xi_n(x_1, \dots, x_n)$. Such a system may be written as a single vector equation

$$\xi(\mathbf{x}(t)) = \mathbf{x}'(t) = \frac{d}{dt}(\mathbf{x}(t)). \tag{28}$$

According to Theorem 5, using (24,25) and(27), then we find the following first equations:

$$\begin{aligned} & -X^1 \left(e^{2f} \frac{\partial^2 L}{\partial x_1 \partial x_2} + 2e^{2f} \frac{\partial f}{\partial x_1} \frac{\partial L}{\partial x_2} \right) dx_1 - X^2 \left(e^{2f} \frac{\partial^2 L}{\partial x_2 \partial x_2} + 2e^{2f} \frac{\partial f}{\partial x_2} \frac{\partial L}{\partial x_2} \right) dx_1 \\ & - X^3 \left(e^{2f} \frac{\partial^2 L}{\partial x_3 \partial x_2} + 2e^{2f} \frac{\partial f}{\partial x_3} \frac{\partial L}{\partial x_2} \right) dx_1 - X^4 \left(e^{2f} \frac{\partial^2 L}{\partial x_4 \partial x_2} + 2e^{2f} \frac{\partial f}{\partial x_4} \frac{\partial L}{\partial x_2} \right) dx_1 = -\frac{\partial L}{\partial x_1} dx_1, \\ & - \left(X^1 \frac{\partial}{\partial x_1} + X^2 \frac{\partial}{\partial x_2} + X^3 \frac{\partial}{\partial x_3} + X^4 \frac{\partial}{\partial x_4} \right) \left(e^{2f} \frac{\partial L}{\partial x_2} \right) + \frac{\partial L}{\partial x_1} = 0, \\ & -\xi \left(e^{2f} \frac{\partial L}{\partial x_2} \right) + \frac{\partial L}{\partial x_1} = 0, \\ & -\frac{\partial}{\partial t} \left(e^{2f} \frac{\partial L}{\partial x_2} \right) + \frac{\partial L}{\partial x_1} = 0. \end{aligned} \tag{29}$$

Performed a similar procedure as in (29) for (17) are obtained equations below:

$$\begin{aligned} dif1 : & -\frac{\partial}{\partial t} \left(e^{2f} \frac{\partial L}{\partial x_2} \right) + \frac{\partial L}{\partial x_1} = 0, \\ dif2 : & \frac{\partial}{\partial t} \left(e^{-2f} \frac{\partial L}{\partial x_1} \right) + \frac{\partial L}{\partial x_2} = 0, \\ dif3 : & c \frac{\partial}{\partial t} \left(e^{-2f} \frac{\partial L}{\partial x_1} \right) - \frac{1}{2} (a - b) \frac{\partial}{\partial t} \left(e^{2f} \frac{\partial L}{\partial x_2} \right) - \frac{\partial}{\partial t} \left(e^{2f} \frac{\partial L}{\partial x_4} \right) + \frac{\partial L}{\partial x_3} = 0, \\ dif4 : & -\frac{1}{2} (a - b) \frac{\partial}{\partial t} \left(e^{-2f} \frac{\partial L}{\partial x_1} \right) - c \frac{\partial}{\partial t} \left(e^{-2f} \frac{\partial L}{\partial x_2} \right) + \frac{\partial}{\partial t} \left(e^{-2f} \frac{\partial L}{\partial x_3} \right) + \frac{\partial L}{\partial x_4} = 0. \end{aligned} \tag{30}$$

The equations calculated in (30) are named conformal Weyl-Euler-Lagrange equations constructed on Walker manifold M_4 and thus the triple (M_4, Φ_L, ξ) is named a **conformal Weyl-Euler-Lagrange mechanical system** on Walker-Weyl manifold (M_4, g, D, ∇, J) .

8 Weyl-Euler-Lagrangian equations for conservative dynamical systems

Proposition 6. We choose $F = \mathbf{i}_\xi$, $g = \Phi_L$ and $f \rightarrow 2f$ at (17) and we consider the equation $F(e^f g) = F(g) - df$, we can write Weyl-Lagrangian dynamic equation as follows:

$$\mathbf{i}_\xi(e^{2f} \Phi_L) = \mathbf{i}_\xi(\Phi_L) - d(2f). \quad (31)$$

The second part (17), according to the law of conservation of energy [2], will not change for conservative dynamical systems and $\mathbf{i}_\xi(\Phi_L) = \Phi_L(\xi)$, $\Phi_L(\xi) - 2df = dE_L$, $\Phi_L(\xi) = dE_L + 2df = d(E_L + 2f)$. The above equation (30) $L \rightarrow L + 2f$. So, we can write

$$\begin{aligned} dif5: & -\frac{\partial}{\partial t} \left(e^{2f} \frac{\partial(L+2f)}{\partial x_2} \right) + \frac{\partial(L+2f)}{\partial x_1} = 0, \\ dif6: & \frac{\partial}{\partial t} \left(e^{-2f} \frac{\partial(L+2f)}{\partial x_1} \right) + \frac{\partial(L+2f)}{\partial x_2} = 0, \\ dif7: & c \frac{\partial}{\partial t} \left(e^{-2f} \frac{\partial(L+2f)}{\partial x_1} \right) - \frac{1}{2}(a-b) \frac{\partial}{\partial t} \left(e^{2f} \frac{\partial(L+2f)}{\partial x_2} \right) - \frac{\partial}{\partial t} \left(e^{2f} \frac{\partial(L+2f)}{\partial x_4} \right) + \frac{\partial(L+2f)}{\partial x_3} = 0, \\ dif8: & -\frac{1}{2}(a-b) \frac{\partial}{\partial t} \left(e^{-2f} \frac{\partial(L+2f)}{\partial x_1} \right) - c \frac{\partial}{\partial t} \left(e^{-2f} \frac{\partial(L+2f)}{\partial x_2} \right) + \frac{\partial}{\partial t} \left(e^{-2f} \frac{\partial(L+2f)}{\partial x_3} \right) + \frac{\partial(L+2f)}{\partial x_4} = 0, \end{aligned} \quad (32)$$

and these differential equations (32) is named a **conformal Weyl-Euler-Lagrange mechanical system** (M_4, Φ_L, ξ, F) on Walker manifold on Walker-Weyl manifold $(M_4, g, D, \nabla, J, F)$

9 Computer solution of equations and graph

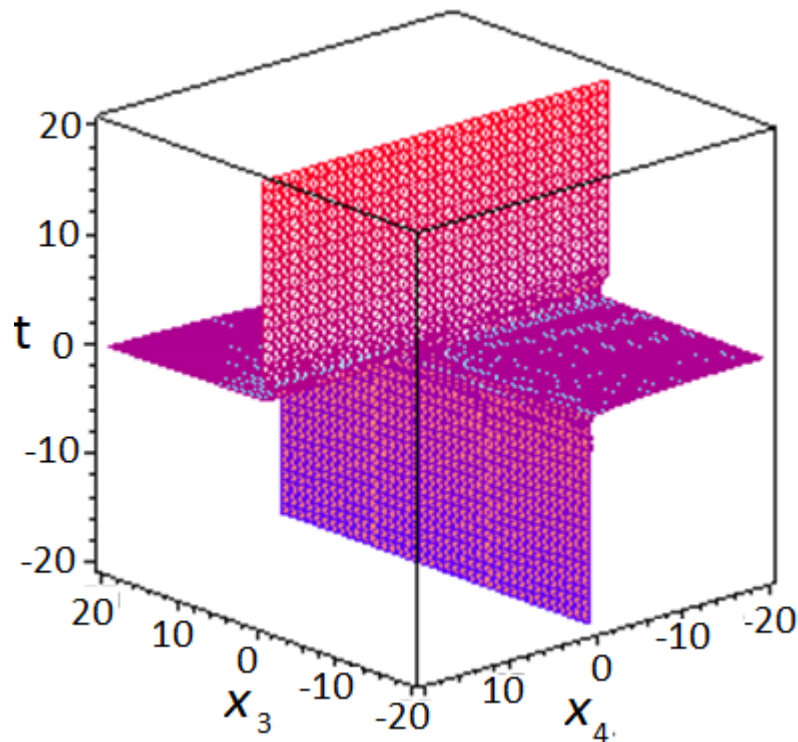
The location of each object in space represented by three dimensions in physical space. Three-dimensional space is a geometric three-parameter model of the physical universe in which all known matter exists. These three dimensions can be labeled by a combination of three chosen from the terms length, width, height, depth, mass, density and breadth. Any three directions can be chosen, provided that they do not all lie in the same plane. So, each vector represents the speed and direction of the movement of air at that point. These found (30) are partial differential equation. We can solve these equations systems of motion for the routes of moving objects (30) using Maple Algebra software. This solution is implicit and will be made according to the situation in Theorem 5 and the graphics will be drawn.

Example 4. a_i, b_i and c_i

$$L(x_1, x_2, x_3, x_4, t) := x_4 * ((t-1) * c_2 + c_1) * \exp(-t) + \exp(t) * (c_1 + c_2 * t) * x_3 + F_3(t). \quad (33)$$

The number of dimensions of the equation (33) will be reduced to three and behind the graphics will be drawn. First, closed function at (33) will be selected as a special. After, the figure of the equation (33) has been drawn for the route of the movement of objects in the electromagnetic field. We choose at (33) for special case of $f(x_1, x_2, x_3, x_4, t) = t$, $F_3(t) = t$ and the conditions in the Theorem 8. of a_i, b_i, c_i ;

$$L(x_1, x_2, x_3, x_4, t) = x_4 * t * \exp(-t) + \exp(t) * (1+t) * x_3 + t. \quad (34)$$



10 Discussion

The most important advantage of this study is to obtain geodesic on 4-Walker manifolds. Thus, geodesics is to allow the calculation of linear or nonlinear distance for the orbits of moving objects. In addition, in the equations implicit solutions (33) were found using Maple Algebra software. Also, conformal Weyl-Euler-Lagrange mechanical equations (30)-(32) derived on a generalized on 4-Walker manifolds may be suggested to deal with problems in electrical, magnetically and gravitational fields force for the path of movement in above figure of defined space moving objects [29,30].

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