New Trends in Mathematical Sciences

http://dx.doi.org/10.20852/ntmsci.2016217022

Tubular surface around a Legendre curve in BCV spaces

Abdullah Yildirim

Department of Mathematics, University of Harran, Sanliurfa-Turkey

Received: 27 July 2015, Revised: 1 August 2015, Accepted: 11 September 2015
Published online: 10 April 2016

Abstract: C. Baikousssis, D.E. Blair[1] made a study of Legendre curves in contact metric manifolds. J. I. Inoguchi, T. Kumamoto, N. Ohsugi, and Y. Suyama[2] studied fundamental properties of Heisenberg 3-spaces. M. Belkhelfa, I.E. Hirica, R. Rosca, L. Verstlraelen[6] obtained a complete characterization of surfaces with paralel second fundamental form in 3-dimensional Bianchi-Cartan-Vranceanu spaces(BCV). In this study, we define the canal surface around Legendre curve with Frenet frame in BCV spaces. Afterwards we investigate tubular surface around Legendre curve curve with Frenet frame. Finally we give some characterizations about special curves lying on tubular surface around Legendre curve.

Keywords: Canal surfaces, tubular surfaces, connections, geodesics, Sasaki spaces, Legendre curve.

1 Introduction

Canal surfaces are useful for representing long thin objects, e.g., pipes, poles, ropes, 3D fonts or intestines of body. Canal surfaces are also frequently used in solid and surface modelling for CAD/CAM(Computer Aided Desing/Computer Aided Manufacturing). Representative examples are natural quadrics, torus, tubular surfaces and Dupin cyclide[8].

Canal surfaces have wide applications in CAGD(Computer-Aided Geometric Design), such as construction of blending surfaces, shape reconstruction,transition surfaces between pipes, robotic path planning or robotic path planning etc. Most of the literature on canal surfaces within the CAGD context has been motivated by the observation that canal surfaces with rational spine curve and rational radius function is rational, and it is therefore natural to ask for methods which allow one to construct a rational parametrization of canal surfaces from its spine curve and radius function. In this paper we shall not be concerned with parametrization but rather with the certain fundamental geometric and algebraic characteristics of canal surfaces[10].

Doğan and Yaylı[8] introduced canal and tubular surfaces. They given some information concerning the curvatures of tubular surface with the Frenet and defined tubular surface with respect to the Bishop frame and then they calculated the curvatures of this new tubular surface and give some characterizations regarding special curves lying on it.

Maekawa *et.all.* [12] researched necessary and sufficient conditions for the regularity of pipe (tubular) surfaces. More recently, Xu *et.al.* [15] studied these conditions for canal surfaces and examined principle geometric properties of these surfaces like computing the area and Gaussian curvature.

Gross [9] gave the concept of generalized tubes (briefy GT) and classifed them in two types as ZGT and CGT. Here, ZGT refers to the spine curve (the axis) that has torsion-free and CGT refers to tube that has circular cross sections. He investigated the properties of GT and showed that parameter curves of a generalized tube are also lines of curvature if and only if the spine curve has torsion free (planar).

* Corresponding author e-mail: abdullahyildirim@harran.edu.tr

Bishop [5] displayed that there exists orthonormal frames which he called relatively paralled adapted frames other than the Frenet frame and compared features of them with the Frenet frame.

2 Tubular surface

A canal surface is defined as the envelope of a family of one parameter spheres. Alternatively, a canal surface is the envelope of a moving sphere with varying radius, defined by the trajectory $\gamma(t)$ of its center and a radius function $r(t)$. This moving sphere $S(t)$ touches the canal surface at a characteristic circle $K(t)$. If the radius function $r(t) = r$ is a constant, then the canal surface is called a tube or pipe surface. Since the canal surface $K(t,\theta)$ is the envelope of a family of one parameter spheres with the center $\gamma(t)$ and radius function $r(t)$, it is parametrized as follows

$$K(t,\theta) = \gamma(t) - r(t)r^{'}(t)\frac{\gamma^{'}(t)}{\left\|\gamma^{'}(t)\right\|} \pm r(t)\frac{\sqrt{\left\|\gamma^{'}(t)\right\|^{2} - r^{'}(t)^{2}}}{\left\|\gamma^{'}(t)\right\|}\left(\cos\theta N(t) + \sin\theta B(t)\right),$$

where $N(t)$ and $B(t)$ are the principal normal and binormal to $\gamma(t)$, respectively. Alternatively, $N(t)$ and $B(t)$ are the basis vectors of the plane containing characteristic circle. If the spine curve $\gamma(s)$ has an arclenght parametrization $\left(\left\|\gamma^{'}(s)\right\| = 1\right)$, then the canal surface is reparametrized as

$$K(s,\theta) = \gamma(s) - r(s)r^{'}(s)T(s) \pm r(s)\sqrt{1 - r^{'}(s)^{2}}\left(\cos\theta N(s) + \sin\theta B(s)\right).$$

In the event $r(s) = r$ is a constant, the canal surface is called a tube or pipe surface and it turns into the form

$$L(s,\theta) = \gamma(s) + r\left(\cos\theta N(s) + \sin\theta B(s)\right),\ 0 \leq \theta \leq 2\pi.$$

For a regular curve $\gamma : I \longrightarrow M$ is parametrized such that $\left\|\gamma^{'}(s)\right\| = 1$. Then we have

$$T^{'} = \kappa N,\ \ N^{'} = -\kappa T + \tau B,\ \ B^{'} = -\tau N,$$

where κ and τ are curvature and torsion of the $\gamma(s)$, respectively[8].

3 Bcv spaces

$\left\{\mathbb{R}^{3}, g_{\lambda,\mu}\right\}$ is called BCV space which denoted by $\mathfrak{M}^{3}$ or $\mathfrak{M}^{3}_{\lambda,\mu}$ where $g_{\lambda,\mu}$ is Bianchi -Cartan-Vranceanu (BCV) metric in $\mathbb{R}^{3}$ and denoted by

$$g_{\lambda,\mu} = \frac{dx_1^2 + dx_2^2}{\left\{1 + \mu(x_1^2 + x_2^2)\right\}^{2}} + \left(dx_3 + \frac{\lambda}{2}\frac{x_2 dx_1 - x_1 dx_2}{1 + \mu(x_1^2 + x_2^2)}\right)^{2}$$

for $\lambda, \mu \in \mathbb{R}$ such that $1 + \mu\left(x_1^2 + x_2^2\right) \neq 0$. The dimension, of this space is dim $\mathfrak{M}^{3}_{\lambda,\mu} = 3$. If $\mu = 0$, $\lambda = 0$, then the space $\mathfrak{M}^{3}$ is called Euclidean space and denoted by E^{3}. In the special case that $\mu = 0$, $\lambda \neq 0$ the space, $\mathfrak{M}^{3,}$ is called Heisenberg space. Heisenberg space is denoted by N^{3} [11]. In 1894, and later in 1928, L. Bianchi classified Riemannian metrics in the 3−dimensional Euclidean space E^{3} [3,4]. In the same year E. Cartan, [7] and in 1947 G. Vranceanu, [14], published some papers related with these spaces.

According to the metric $g_{\lambda,\mu}$, an orthonormal basis $\phi = \{e_1, e_2, e_3\}$ of $\chi(\mathfrak{M}^3)$ is denoted by

$$\begin{aligned} e_1 &= \left\{1+\mu(x_1^2+x_2^2)\right\}\frac{\partial}{\partial x_1} - \frac{1}{2}\lambda x_2 \frac{\partial}{\partial x_3}, \\ e_2 &= \left\{1+\mu(x_1^2+x_2^2)\right\}\frac{\partial}{\partial x_2} + \frac{1}{2}\lambda x_1 \frac{\partial}{\partial x_3}, \\ e_3 &= \frac{\partial}{\partial x_3}. \end{aligned}$$

The dual basis, θ of ϕ is given by

$$\begin{aligned} \theta^1 &= \frac{dx_1}{1+\mu(x_1^2+x_2^2)}, \\ \theta^2 &= \frac{dx_2}{1+\mu(x_1^2+x_2^2)}, \\ \theta^3 &= dx_3 + \frac{\lambda}{2}\,\frac{x_2dx_1 - x_1dx_2}{1+\mu(x_1^2+x_2^2)}. \end{aligned}$$

For the orthonormal basis $\phi = \{e_1, e_2, e_3\}$ of $\chi\left(\mathfrak{M}^3\right)$ if Levi-Civita connection on $\mathfrak{M}^3$ denoted by ∇, then we have

$$\begin{bmatrix} \nabla_{e_1}e_1 \\ \nabla_{e_1}e_2 \\ \nabla_{e_1}e_3 \end{bmatrix} = \begin{bmatrix} 0 & 2\mu x_2 & 0 \\ -2\mu x_2 & 0 & \frac{\lambda}{2} \\ 0 & -\frac{\lambda}{2} & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

$$\begin{bmatrix} \nabla_{e_2}e_1 \\ \nabla_{e_2}e_2 \\ \nabla_{e_2}e_3 \end{bmatrix} = \begin{bmatrix} 0 & -2\mu x_1 & -\frac{\lambda}{2} \\ 2\mu x_1 & 0 & 0 \\ \frac{\lambda}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

$$\begin{bmatrix} \nabla_{e_3}e_1 \\ \nabla_{e_3}e_2 \\ \nabla_{e_3}e_3 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\lambda}{2} & 0 \\ \frac{\lambda}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

and

$$[e_1, e_2] = -2\mu x_2 e_1 + 2\mu x_1 e_2 + \lambda e_3, \quad [e_3, e_2] = [e_1, e_3] = 0\,.$$

The transformation φ on $\chi(\mathfrak{M}^3)$ given by $\varphi(e_1) = e_2, \varphi(e_2) = -e_1, \varphi(e_3) = 0$ is a linear endomorfizm and the corresponding matrix is given by

$$\varphi = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with respect toorthonormal basis $\phi = \{e_1, e_2, e_3\}$ of $\chi\left(\mathfrak{M}^3\right)$. On the , space $\mathfrak{M}^3$, if $\lambda \neq 0$

$$\eta = \theta^3 = dx_3 + \frac{\lambda}{2}\frac{x_2dx_1 - x_1dx_2}{1+\mu(x_1^2+x_2^2)}$$

and $\xi = e_3$, then we have the following relations for $\forall X, Y \in \chi(\mathfrak{M}^3)$;

$$\left.\begin{array}{l} \varphi(\xi) = 0, \\ \eta(\xi) = 1, \\ d\eta(X,Y) = \dfrac{\lambda}{2} g_{\lambda,\mu}(X, \varphi(Y)), \\ (\nabla_X \varphi)Y = \dfrac{\lambda}{2}\left\{g_{\lambda,\mu}(X,Y)\xi - \eta(Y)X\right\}, \\ \nabla_X \xi = -\dfrac{\lambda}{2}\varphi(X). \end{array}\right\} \tag{1}$$

The structure $(\mathfrak{M}^3, \varphi, \xi, \eta, g_{\lambda,\mu})$ together the equations 1 is a Sasakian manifold [2,16]. From now on for $\lambda \neq 0$ we will call the space as BCV-Sasakian space.

Definition 1. *We denote by $\eta = 0$ the subbundle defined by the subspaces*

$$D_m = \{X \in T_{\mathfrak{M}^3}(m) : \eta(X) = 0\}$$

of $T\mathfrak{M}^3$. The maximum dimension of integral submanifolds is 1. The 1-dimensional integral submanifold of a contact manifold is called a Legendre curve.

Theorem 1. *Let $(\mathfrak{M}^3, \eta, \xi, \varphi, g)$ be BCV-Sasakian manifold. The torsion of its Legendre curve which is not geodesic is equal to $\dfrac{\lambda}{2}$.*

Proof. Let γ be an unite speed Legendre curve which is not geodesic

$$\begin{array}{rll} \gamma : I \longmapsto & D_m \subset \mathfrak{M}^3 & \\ s \longmapsto & \gamma(s) & = (\gamma_1(s), \gamma_2(s), \gamma_3(s)) \end{array}$$

on $\mathfrak{M}^3-$BCV-Sasakian space. Let us calculate Frenet vector fields of γ (in the case $\eta(\dot{\gamma}) = 0$). We know $\dot{\gamma}(s) = T$. We obtain Frenet vector fields of γ as

$$\left\{\dot{\gamma} = T, \varphi\dot{\gamma} = N, \xi = B\right\}.$$

Hence we have

$$\nabla_T T = \kappa\varphi T$$

and

$$\nabla_T T = \kappa N.$$

On the other hand, the directional derivative of φV_1 with respect to V_1 is

$$\nabla_T N = \nabla_T \varphi T = \varphi \nabla_T T + (\nabla_T \varphi) T = \varphi(\kappa\varphi T) + \frac{\lambda}{2}\xi = -\kappa T + \frac{\lambda}{2}B$$

and with1 similarly the derivative of ξ is

$$\nabla_T \xi = -\frac{\lambda}{2}\varphi T = -\frac{\lambda}{2}N$$

Hence

$$\tau = \frac{\lambda}{2}. \tag{2}$$

4 Tubular surface around a Legendre curve

Let $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ be an unite speed Legendre curve which is not geodesic in $\mathfrak{M}^3-$ Sasakian space. Then the canal surface aroud Legendre curve is reparametrized as

$$K(s,\theta) = \gamma(s) - r(s)r^{'}(s)T(s) \pm r(s)\sqrt{1-r^{'}(s)^2}\left(\cos\theta N(s) + \sin\theta B(s)\right).$$

In the event $r(s) = r$ is a constant, the canal surface is called a tube or pipe surface and it turns into the form

$$L(s,\theta) = \gamma(s) + r\left(\cos\theta N(s) + \sin\theta B(s)\right),\ 0 \leq \theta \leq 2\pi.$$

For a regular Legendre curve $\gamma : I \longrightarrow \mathfrak{M}^3$ is parametrized such that $\left\|\gamma^{'}(s)\right\| = 1$. Then we have

$$T^{'} = \kappa N,\quad N^{'} = -\kappa T + \frac{\lambda}{2}B,\quad B^{'} = -\frac{\lambda}{2}N,$$

where κ and $\frac{\lambda}{2}$ are curvature and torsion of the $\gamma(s)$, respectively.

5 The curvatures of tubular surfaces around a Legendre curve

For the tubular surface $L(s,\theta)$ around γ Legendre curve in $\mathfrak{M}^3$, the coefficients of the first and second fundamental form are given by

$$U = \frac{L_s \times L_\theta}{\|L_s \times L_\theta\|} = -\cos\theta N - \sin\theta B \tag{3}$$

$$\begin{aligned}
L_\theta &= r\left(-\sin\theta N + r\cos\theta B\right),\\
L_s &= \left(1 - r\kappa\cos\theta\right)T + L_\theta,\\
L_{\theta\theta} &= -r\cos\theta N - r\sin\theta B,\\
L_{ss} &= \left(-r\kappa^{'}\cos\theta + r\kappa\frac{\lambda}{2}\sin\theta\right)T + \left[\kappa - r\left(\kappa^2 + \left(\frac{\lambda}{2}\right)^2\right)\cos\theta\right]N - r\sin\theta B\\
L_{s\theta} &= r\kappa\sin\theta T - r\frac{\lambda}{2}\cos\theta N - r\frac{\lambda}{2}\sin\theta B,\\
E &= L_s \cdot L_s = \left(1 - r\kappa\cos\theta\right)^2 + r^2\left(\frac{\lambda}{2}\right)^2,\\
F &= L_s \cdot L_\theta = r^2\frac{\lambda}{2},\\
G &= L_\theta \cdot L_\theta = r^2,\\
e &= U \cdot L_{ss} = -r\cos\theta\left(1 - r\kappa\cos\theta\right) + r\left(\frac{\lambda}{2}\right)^2,\\
f &= U \cdot L_{s\theta} = r\frac{\lambda}{2},\\
g &= U \cdot L_{\theta\theta} = r
\end{aligned}$$

and since

$$\|L_s \times L_\theta\|^2 = EG - F^2 = r^2\left(1 - r\kappa\cos\theta\right)^2. \tag{4}$$

Definition 2.[8] *Let Q is any surface. If $EG - F^2 \neq 0$, Q is called a regular surface.*

According to definetion and by Eq 4 $L(s,\theta)$ is a regular tube if and only if $1-r\kappa\cos\theta\neq 0$. Namely

$$\cos\theta\neq\frac{1}{r\kappa} \tag{5}$$

The Gaussian and mean curvature for $L(s,\theta)$ are computed as

$$\left.\begin{aligned} K&=\frac{eg-f^2}{EG-F^2}=\frac{-\kappa\cos\theta}{r(1-r\kappa\cos\theta)},\\ H&=\frac{eG-2fF+gE}{2(EG-F^2)}=\frac{1}{2}\left[\frac{1}{r}+rK\right].\end{aligned}\right\} \tag{6}$$

Theorem 2. [10] *A curve δ lying on a surface is an asymptotic curve if and only if the acceleration vector δ'' is tangent to the surface that is $U\cdot\delta''=0$.*

Theorem 3. *Let $L(s,\theta)$ be a regular tube around a Legendre curve. Then the following holds:*
(1) The s-parameter curves of $L(s,\theta)$ are asymptotic curves if and only if $\gamma(s)$ Legendre curve is circular helix.
(2) The $\theta-$parameter curves of $L(s,\theta)$ cannot be asymptotic curves.

Proof. (1) For the $s-$parameter curves we have

$$e=U\cdot L_{ss}=-\kappa\cos\theta(1-r\kappa\cos\theta)+r\left(\frac{\lambda}{2}\right)^2=0.$$

From this, we get

$$\kappa=\frac{1}{2r\cos\theta}\left[1\pm\sqrt{1-(\lambda r)^2}\right]$$

where $0<r\leq\frac{1}{\sqrt{\lambda}}$ for $s-$parameter curves. Since κ depends $s-$parameter, κ is constant and before we obtained $\tau=\frac{\lambda}{2}$. According to this $\frac{\tau}{\kappa}=constant$. Namely γ Legendre curve is circular helix.

(2) For the $\theta-$parameter curves we have

$$g=U\cdot L_{\theta\theta}=r\neq 0,$$

$\theta-$parameter curves cannot be asymptotic.

Theorem 4. [13] *A curve δ lying on a surface is a geodesic curve if and only if the acceleration vector δ'' is normal to the surface. This means that δ'' and the surface normal U are linearly dependent, namely $U\times\delta''=0$.*

Theorem 5. *Let $L(s,\theta)$ be a regular tube around a Legendre curve. Then the following holds:*

(1) The $\theta-$parameter curves of $L(s,\theta)$ are geodesic curves.

(2) The $s-$parameter curves of $L(s,\theta)$ are geodesic curves if and only if the curvatures of $\gamma(s)$

$$\kappa=constant,$$

(3) The s-parameter curves of $L(s,\theta)$ are geodesic curves if and only if $\alpha(s)$ is a circular helix.

Proof. For the $s-$ and $\theta-$ parameter curves we conclude

$$U\times L_{\theta\theta}=(-\cos\theta N-\sin\theta B)\times(-r\cos\theta N-r\sin\theta B),$$
$$U\times L_{ss}=[\kappa\sin\theta\,(1-r\kappa\cos\theta)]\,T+\left[r\kappa'\sin\theta\cos\theta-r\kappa\frac{\lambda}{2}\sin^2\theta\right]N+\left[-r\kappa'\cos^2\theta+r\kappa\frac{\lambda}{2}\sin\theta\cos\theta\right]B,$$

(1) As immediately seen above, $\theta-$parameter curves of $L(s,\theta)$ are geodesics.
(2) Since $\{T,N,B\}$ is an orthonormal basis, $U\times L_{ss}=0$ if and only if

$$\begin{aligned}\kappa\sin\theta\,(1-r\kappa\cos\theta)&=0,\\ r\sin\theta\left(\kappa'\cos\theta-\kappa\frac{\lambda}{2}\sin\theta\right)&=0,\\ r\cos\theta\left(\kappa'\cos\theta-\kappa\frac{\lambda}{2}\sin\theta\right)&=0.\end{aligned}\tag{7}$$

By the first two equations we have

$$\kappa'\cos\theta=\kappa\frac{\lambda}{2}\sin\theta$$

and

$$\kappa=ce^{\frac{\lambda\tan\theta}{2}s}\tag{8}$$

where c is constant. If the equation 8 is solved with the equation 7 it concludes that $\theta=0$ thus

$$\kappa=ce=\text{constant}.$$

(3) Because κ and $\tau=\frac{\lambda}{2}$ are constant, the case is obvious.

Let

$$\begin{aligned}\gamma:I&\longmapsto\ \mathfrak{M}^3\\ s&\longmapsto\gamma(s)=(\gamma_1(s),\gamma_2(s),\gamma_3(s))\end{aligned}$$

$\gamma(s)=(\gamma_1(s),\gamma_2(s),\gamma_3(s))$ be an unite speed Legendre curve in $\mathfrak{M}^3$ and its velocity vector is

$$\dot\gamma(s)=(\dot\gamma_1(s),\dot\gamma_2(s),\dot\gamma_3(s))=\left(\dot\gamma_1(s)\frac{\partial}{\partial x}+\dot\gamma_2(s)\frac{\partial}{\partial y}+\dot\gamma_3(s)\frac{\partial}{\partial z}\right)_{\gamma(s)}$$

According to the orthonormal basis (e_1,e_2,e_3) its form is

$$\dot\gamma(s)=\frac{\dot\gamma_1(s)}{1+\mu(x_1^2+x_2^2)}e_1+\frac{\dot\gamma_2(s)}{1+\mu(x_1^2+x_2^2)}e_2+(\dot\gamma_3(s)+\frac{\lambda}{2}\frac{x_2\dot\gamma_1(s)-x_1\dot\gamma_2(s)}{1+\mu(x_1^2+x_2^2)})e_3$$

If γ is an Legendre curve in $\mathfrak{M}^3$, $\eta\left(\dot\gamma(s)\right)=0$ and

$$\dot\gamma_3(s)=-\frac{\lambda}{2}\frac{\gamma_2(s)\dot\gamma_1(s)-\gamma_1(s)\dot\gamma_2(s)}{1+\mu((\gamma_1(s))^2+(\gamma_2(s))^2)}\tag{9}$$

Solution of γ_3 is integral of the right side of equation 9.

Example 1. Let

$$\begin{aligned} \gamma : I \longmapsto \quad & \mathbb{R}^3 \\ s \longmapsto \gamma(s) = & \left(2\sin s, 2\cos s, \frac{\lambda}{2(1+4\mu)} s\right) \end{aligned}$$

be an Legendre curve in $\mathfrak{M}^3$ for $\lambda = 4$ and $\mu = \frac{1}{4}$ as shown Figure 1.

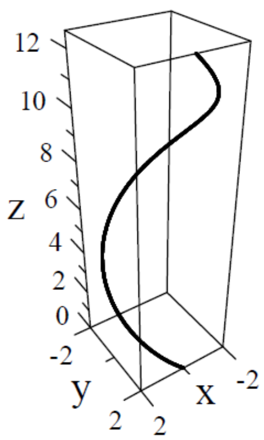

Fig. 1: γ Legendre curve.

Its velocity vector is

$$\dot{\gamma}(s) = -\sin s e_1 + \cos s e_2$$

See that $\eta\left(\dot{\gamma}(s)\right) = 0$. Now then, if the Frenet vector of γ curve is $\{T, N, B\}$

$$T = \dot{\gamma}(s) = -\sin s e_1 + \cos s e_2, \quad N = \varphi\dot{\gamma}(s) = \cos s e_1 + \sin s e_2, \quad B = \xi = e_{3.}$$

Now than for $r = \frac{1}{2}$, equation of tubular surface around γ Legendre curve in $\mathfrak{M}^3$ is

$$L(s,\theta) = \gamma(s) + \frac{1}{2}\left(\cos\theta N(s) + \sin\theta B(s)\right), \; 0 \leq \theta \leq 2\pi. \tag{10}$$

Let $L(s,\theta) = \left(x(s,\theta)\frac{\partial}{\partial x} + y(s,\theta)\frac{\partial}{\partial y} + z(s,\theta)\frac{\partial}{\partial z}\right)$ now than

$$x(s,\theta) = 2\sin s + \cos\theta\cos s, \quad y(s,\theta) = 2\cos s + \cos\theta\sin s, \quad z(s,\theta) = s - 2\cos\theta\cos 2s + \frac{1}{2}\sin\theta.$$

Tubular surface around γ Legendre curve is shown in following Figure 2 where $0 \leq \theta, s \leq 2\pi$

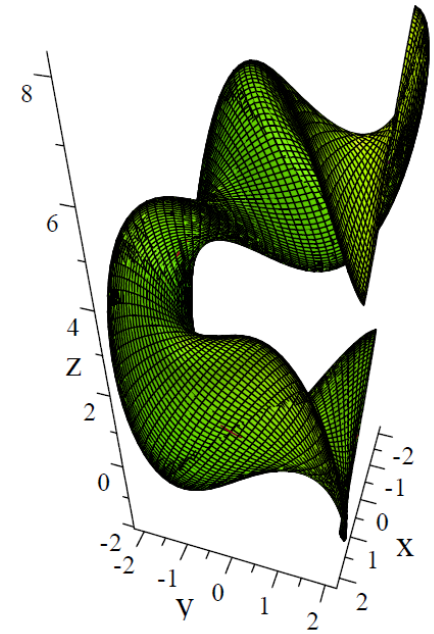

Fig. 2: Tubular surface around γ Legendre curve.

Example 2. Let an other Legendre curve be $\alpha(s) = \left(s^2, \frac{1}{s}, -\frac{\lambda}{2}\frac{3s^2}{s^2+\mu(s^6+1)}\right)$ in $\mathfrak{M}^3$ for $\lambda = 4$ and $\mu = 1$ as shown Figure 3.

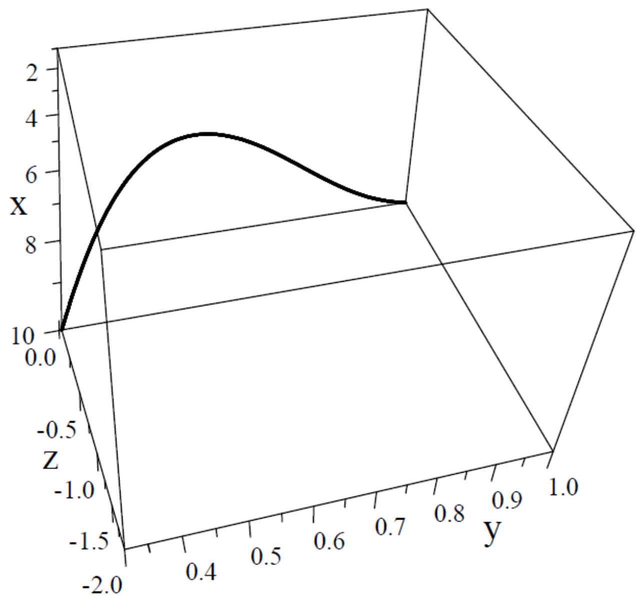

Fig. 3: α Legendre curve.

Its velocity vector is

$$\dot{\alpha}(s) = \frac{2s^3}{s^6+s^2+1}e_1 - \frac{1}{s^6+s^2+1}e_2$$

See that $\eta\left(\dot{\alpha}(s)\right) = 0$. Now then, if the Frenet vector of α curve is $\{T,N,B\}$

$$T = \frac{\dot{\alpha}(s)}{\left\|\dot{\alpha}(s)\right\|} = \frac{2s^3}{\sqrt{4s^6+1}}e_1 - \frac{1}{\sqrt{4s^6+1}}e_2, \quad B = \xi = e_3, \quad N = B \wedge T = \frac{1}{\sqrt{4s^6+1}}e_1 + \frac{2s^3}{\sqrt{4s^6+1}}e_2.$$

Now than for $r = 1$, equation of tubular surface around α Legendre curve is

$$L(s,\theta) = \alpha(s) + (\cos\theta N(s) + \sin\theta B(s)),\ 0 \leq \theta \leq 2\pi. \tag{11}$$

Let $L(s,\theta) = \left(x(s,\theta)\frac{\partial}{\partial x} + y(s,\theta)\frac{\partial}{\partial y} + z(s,\theta)\frac{\partial}{\partial z}\right)$ now than

$$x(s,\theta) = s^2 + \frac{s^6+s^2+1}{s^2\sqrt{4s^6+1}}\cos\theta, \quad y(s,\theta) = \frac{1}{s} + 2\frac{s^7+s^3+s}{\sqrt{4s^6+1}}\cos\theta, \quad z(s,\theta) = \frac{4s^6-2}{s\sqrt{4s^6+1}}\cos\theta + \sin\theta.$$

Tubular surface around α Legendre curve is shown in following Figure 4 where $(0 \leq \theta \leq 2\pi)$ and $(1 \leq s \leq 3,16)$.

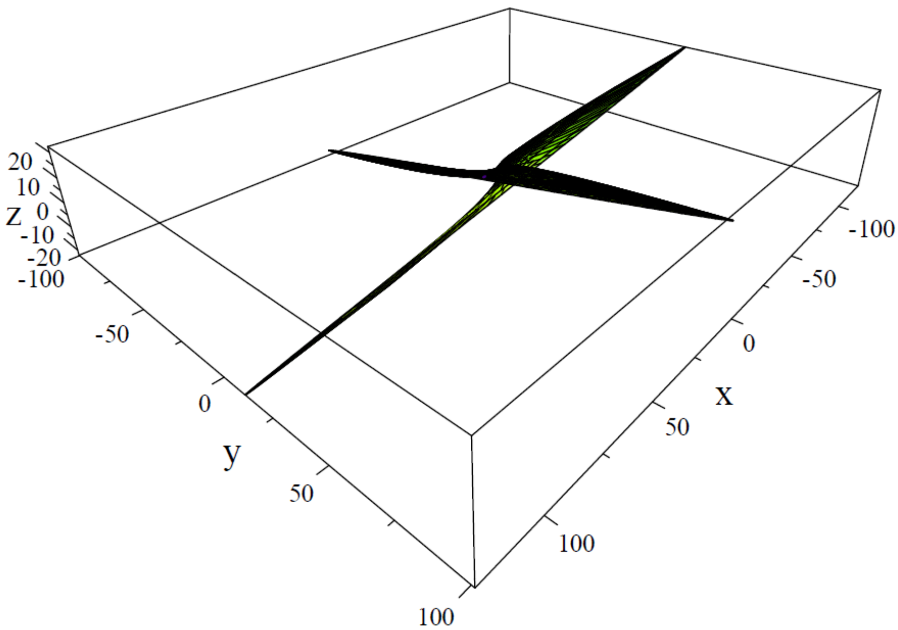

Fig. 4: Tubular surface around α Legendre curve.

References

[1] Baikousssis, C., Blair, D.E. On Legendre Curves in Contact 3-manifolds. Geometriae Dedicata 49; 135-142, (1994).

[2] Belkhelfa, M., Hırıca, E., Rosca, R. and Verstraelen, L., On Legendre Curves In Rıemannıan And Lorentzıan Sasakı Spaces, Soochow Journal Of Mathematıcs, Volume 28, No. 1, pp. 81-91, January (2002).

[3] Bianchi, L. , Lezioni Di Geometrie Differenziale, E. Spoerri Librario-Editore. (1894).

[4] Bianchi, L. , Lezioni Sulla Teoria Dei Gruppi Continui E Finiti Di Transformazioni, Ed. Zanichelli, (1928).
[5] Bishop, R. L., There is more than one way to frame a curve, Amer. Math. Monthly, Volume 82, Issue 3 (1975), 246-251.
[6] Blair, D.E., Riemannian Geometry of Contact and Symplectic Manifolds. Boston (2001).
[7] Cartan, E. Lecons Sur La Geometria Des Espace De Riemann. Gauthier Villars, Paris, (1894).
[8] Doğan, F. and Yaylı, Y., On the curvatures of tubular surface with Bishop frame, Commun. Fac. Sci. Univ. Ank. Series A1 Volume 60, 1 (2011), 59-69.
[9] Gross, A., Analyzing generalized tubes, Siam Rev. 30, 1-68, (1988).
[10] Hacısalihoğlu, H. H., Differensiyel Geometri II, Hacısalihoğlu Yayınları(2000).
[11] Inoguchi, J.I., Kumamoto, T., Ohsugi, N., Suyama, Y. Differential Geometry of Curves and Surfaces in 3-dimensional Homogeneous Spaces 1. Fukuoka University Science, Reports, 29(2); 155-182, (1999).
[12] Maekawa, T., Patrikalakis, M. N., Sakkalis, T. and Yu, G., Yu, Analysis and applications of pipe surfaces, Computer Aided Geometric Design 15 (1998), 437-458.
[13] Oprea, J., Differential Geometry and Its Applications, Prentice-Hall Inc., (1997).
[14] Vranceanu, G., Leçons De Geometrie Differentielle I.Ed.Acad.Rep.Pop. Roum, Bucarest, (1947).
[15] Xu, Z., Feng, R. and Sun, G.J., Analytic and algebraic properties of canal surfaces, Journal of Computational and Applied Mathematics, 195 (2006), 220-228.
[16] Yıldırım, A., Differential Geometry of Curves on Homogeneous Spaces, Phd Thesis, University of Ankara, (2005).