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Generalized intuitionistic fuzzy ideals of hemirings

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Abstract: In this paper we generalize the concept of quasi-coincident of an intuitionistic fuzzy point with an intuitionistic fuzzy set and define $(\in, \in \vee q_k)^*$-intuitionistic fuzzy ideals of hemirings and characterize different classes of hemirings by the properties of these ideals.

Keywords: Intuitionistic fuzzy sub-hemiring, intuitionistic fuzzy ideal, fully idemotent hemiring, regular hemiring.

1 introduction

Dedekind introduced the modern definition of the ideal of a ring in 1894 and observed that the family $Id(R)$ of all the ideals of a ring R obeyed most of the rules that the ring$(R, +, \cdot)$ did, but $(Id(R), +, \cdot)$ was not a ring. In 1934, Vandiver [25] studied an algebraic system, which consists of a non-empty set S with two binary operations "$+$" and "$\cdot$" such that S was semigroup under both the operations and $(S, +, \cdot)$ satisfies both the distributive laws but did no satisfy the cancellation law of addition. Vandiver named this system a 'semiring'. Semirings are common generalization of rings and distributive lattices. A hemiring is a semiring in which "$+$" is commutative and it has an absorbing element. Semirings (hemirings) appear in a natural manner in some applications to the theory of automata, formal languages, optimization theory and other branches of applied mathematics (see for example [9, 10, 11, 12, 18, 19]).

Zadeh introduced the concept of fuzzy set in his definitive paper [26] of 1965. Many authors used this concept to generalize basic notions of algebra. In 1971, Rosen feld [22] laid the foundations of fuzzy algebra. He introduced the notions of fuzzy subgroup of a group. Ahsan et al. [3] initiated the study of fuzzy semirings. Murali [20] defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on fuzzy subset and Pu and Liu introduced the concept of quasicoincident of a fuzzy point with a fuzzy set in [21]. Bhakat and Das [5] used these ideas and defined $(\in, \in \vee q)$-fuzzy subgroup of a group which is a generalization of Rosenfeld's fuzzy subgroup. Many researchers used these ideas to define (α, β)-fuzzy substructures of algebraic structures (see [8, 15, 16, 23]).

Generalizing the concept of the quasi-coincident of a fuzzy point with a fuzzy subset, Jun [13] defined $(\in, \in \vee q_k)$-fuzzy subalgebra in BCK/BCI-algebras. In [24] Shabir et al. characterized semigroups by the properties of $(\in, \in \vee q_k)$-fuzzy ideals, quasi-ideal and bi-ideals. Jun et al. in [15] defined $(\in, \in \vee q_k)$-fuzzy ideals of hemirings. Asghar et al. [17], defined $(\in, \in \vee q_k)$-fuzzy bi-ideals in ordered semigroups.

On the other hand Atanassov [4] introduced the notion of intuitionistic fuzzy set which is a generalization of fuzzy set. Intuitionistic fuzzy hemirings are studied by Dudek in [7]. Coker and Demirici [6] introduced the notion of fuzzy point. In [14], Jun introduced the notion of (ϕ, ψ)-intuitionistic fuzzy subgroup of a intuitionistic group where

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$\phi,\ \psi \in \{\in,\ q,\ \in\vee q,\ \in\wedge q\}$ and $\phi \neq \in\wedge q$.

Generalizing the concept of quasi-coincident of an intuitionistic fuzzy point with an intuitionistic fuzzy set we define $(\in,\ \in\vee q_k)^*$-intuitionistic fuzzy ideals of hemirings and characterize different classes of hemirings by the properties of these ideals.

2 Preliminaries

A semiring is a set R together with two binary operations addition "+" and multiplication "·" such that $(R,\ +)$ and $(R,\ \cdot)$ are semigroups, where both algebraic structures are connected by the ring like distributive laws:

$$a(b+c) = ab+ac \quad \text{and} \quad (a+b)c = ac+bc$$

for all a, b and $c \in R$. An element $0 \in R$ is called a zero element of R if $a+0 = 0+a = a$ and $0 \cdot a = a \cdot 0 = 0$ for all $a \in R$. A hemiring is a semiring with zero element, in which "+" is commutative. A hemiring $(R,\ +,\ \cdot)$ is called commutative if multiplication is commutative, that is $ab = ba$ for all $a, b \in R$. An element $1 \in R$ is called an identity element of R if $a \cdot 1 = 1 \cdot a = a$ for all $a \in R$. A non-empty subset I of a hemiring R is called a left (right) ideal of R if I is closed under addition and $RI \subseteq I\ (IR \subseteq I)$. I is called a two-sided ideal or simply an ideal of R if I is both a left ideal and a right ideal of R. A hemiring R is called regular if for each $x \in R$ there exists $a \in R$ such that $x = xax$.

Theorem 1. *[1] A hemiring R is regular if and only if $A \cap B = AB$ for all right ideals A and left ideals B of R. Generalizing the concept of regular hemirings, in [2] right weakly regular hemirings are defined as: A hemiring R is right weakly regular if for each $x \in R$, we have $x \in (xR)^2$. If R is commutative then the concepts of regular and right weakly regular coincides. It is proved in [2].*

Theorem 2. *[2] The following conditions are equivalent for a hemiring R with* 1.

(1) *R is right weakly regular.*
(2) *$A \cap B = AB$ for all right ideals A and two-sided ideals B of R.*
(3) *$A^2 = A$ for every right ideal A of R.*
If R is commutative, then the above conditions are equivalent to
(4) *R is regular.*

Let X be a non-empty fixed set. An intuitionistic fuzzy subset A of X is an object having the form

$$A = \{\langle x, \mu_A(x),\ \lambda_A(x)\ : x \in X\rangle\}$$

where the functions $\mu_A : X \longrightarrow [0,1]$ and $\lambda_A : X \longrightarrow [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\lambda_A(x)$) of each element of $x \in X$ to A, respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$. For the sake of simplicity, we use the symbol $A = (\mu_A, \lambda_A)$ for the intuitionistic fuzzy subset (briefly, IFS) $A = \{\langle x, \mu_A(x),\ \lambda_A(x)\ : x \in X\rangle\}$. If $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ are intuitionistic fuzzy subsets of X, then

(1) $A \subseteq B \Longleftrightarrow \mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x) \quad \forall x \in X$
(2) $A = B \Longleftrightarrow A \subseteq B$ and $B \subseteq A$.
(3) $\overline{A} = (\lambda_A, \mu_A)$. More generally if $\{A_i : i \in I\}$ is a family of intuitionistic fuzzy subset of X, then by the union and intersection of this family we mean an intuitionistic fuzzy subsets
(4) $\bigcup\limits_{i\in I} A_i = \left(\bigvee\limits_{i\in I} \mu_{A_i}, \bigwedge\limits_{i\in I} \lambda_{A_i}\right)$.

(5) $\bigcap_{i \in I} A_i = \left(\bigwedge_{i \in I} \mu_{A_i}, \bigvee_{i \in I} \lambda_{A_i} \right)$.
Let a be a point in a non-empty set X. If $\alpha \in (0,1]$ and $\beta \in [0,1)$ are two real numbers such that $0 \leq \alpha + \beta \leq 1$ then IFS.

$$a(\alpha,\beta) = \langle x, a_\alpha, 1 - a_{1-\beta} \rangle$$

is called an intuitionistic fuzzy point(IFP) in X, where α and β is the degree of membership and nonmembership of $a(\alpha, \beta)$ respectively and $a \in X$ is the support of $a(\alpha, \beta)$.

Let $a(\alpha, \beta)$ be an IFP in X, and $A = (\mu_A, \lambda_A)$ be an IFS in X. Then $a(\alpha, \beta)$is said to belong to A, written $a(\alpha, \beta) \in A$, if $\mu_A(a) \geq \alpha$ and $\lambda_A(a) \leq \beta$ and quasi-coincident with A, written $a(\alpha, \beta)qA$, if $\mu_A(a) + \alpha > 1$, and $\lambda_A + \beta < 1$. $a(\alpha, \beta) \in \vee qA$, means that $a(\alpha, \beta) \in A$ or $a(\alpha, \beta)qA$ and $a(\alpha, \beta) \in \wedge qA$, means that $a(\alpha, \beta) \in A$ and $a(\alpha, \beta)qA$ and $a(\alpha, \beta)\overline{\in \vee q}A$, means that $a(\alpha, \beta) \in \vee qA$ doesn't hold.

Let $x(t, s)$ be an IFP in X, and $A = (\mu_A, \lambda_A)$ be an IFS in R, Then for all $x, y \in R$ and $t \in (0,1]$, $s \in [0,1)$, we define the following:

(i) $x(t,s)q_k A$ if $\mu_A(x) + t + k > 1$ and $\lambda_A(x) + s + k < 1$.
(ii) $x(t,s) \in \vee q_k A$ if $x(t,s) \in A$ or $x(t,s)q_k A$.
(iii) $x(t,s) \in \wedge q_k A$ if $x(t,s) \in A$ and $x(t,s)q_k A$.
(iv) $x(t,s)\overline{\in \vee q_k}A$ means that $x(t,s) \in \vee q_k A$ doesn't hold, where $k \in [0,1)$.

3 (α,β)-intuitionistic fuzzy ideals

Throughout the remaining paper $k \in [0,1)$, α any one of $\in$, q_k, $\in \vee q_k$ and β any one of $\in$, q_k, $\in \vee q_k$, $\in \wedge q_k$ unless otherwise specified.

Definition 1. *An IFS $A = (\mu_A, \lambda_A)$ of a hemiring R is called an (α, β)-intuitionistic fuzzy sub-hemiring of R, if $\forall$ $x, y \in R$ and $t_1, t_2 \in (0,1]$, $s_1, s_2 \in [0,1)$,*

(1) $x(t_1,s_1), y(t_2,s_2)\alpha A \Rightarrow (x+y)(\min(t_1,t_2), \max(s_1,s_2))\beta A$,
(2) $x(t_1,s_1), y(t_2,s_2)\alpha A \Rightarrow (xy)(\min(t_1,t_2), \max(s_1,s_2))\beta A$.

Definition 2. *An IFS $A = (\mu_A, \lambda_A)$ of a hemiring R is called an (α, β)-intuitionistic fuzzy left (right) ideal of R, if $\forall$ $x, y \in R$ and $t_1, t_2 \in (0,1]$, $s_1, s_2 \in [0,1)$,*

(1) $x(t_1,s_1), y(t_2,s_2)\alpha A \Rightarrow (x+y)(\min(t_1,t_2), \max(s_1,s_2))\beta A$
(2) $y(t_1,s_1)\alpha A$, $x \in R \Rightarrow (xy)(t_1,s_1)\beta A$ $((yx)(t_1,s_1)\beta A)$.

An IFS $A = (\mu_A, \lambda_A)$ of a hemiring R is called an (α, β)-intuitionistic fuzzy ideal of R, if it is both (α, β)-intuitionistic fuzzy left ideal and (α, β)-intuitionistic fuzzy right ideal of R.

Theorem 3. *Let $A = (\mu_A, \lambda_A)$ be an (α, β)-intuitionistic fuzzy ideal of R . Then the set*

$$R_{(0,1)} = \{x \in R : \mu_A(x) > 0 \text{ and } \lambda_A(x) < 1\} \neq \phi$$

is an ideal of R.

Proof. Let $x,y \in R_{(0,1)}$. Then $\mu_A(x) > 0$ and $\lambda_A(x) < 1$, $\mu_A(y) > 0$ and $\lambda_A(y) < 1$. Assume that $\mu_A(x+y) = 0$ or $\lambda_A(x+y) = 1$. If $\alpha \in \{\in, \in \vee q_k\}$, then, $x(\mu_A(x),\ \lambda_A(x))\,\alpha A$ and $y(\mu_A(y),\ \lambda_A(y))\,\alpha A$ but $(x+y)(\min\{\mu_A(x),\ \mu_A(y)\}, \max\{\lambda_A(x),\ \lambda_A(y)\})\overline{\beta}A$, for every $\beta \in \{\in, q_k, \in \vee q_k, \in \wedge q_k\}$, a contradiction. Also $x(1,0)q_kA$ and $y(1,0)q_kA$ but $(x+y)\,(1,0)\overline{\beta}A$ for every $\beta \in \{\in, q_k, \in \vee q_k, \in \wedge q_k\}$, a contradiction. Thus $\mu_A(x+y) > 0$ and $\lambda_A(x+y) < 1$. Therefore, $x+y \in R_{(0,1)}$.

Let $x \in R_{(0,1)}$ and $y \in R$. Then $\mu_A(x) > 0$ and $\lambda_A(x) < 1$. suppose that $\mu_A(xy) = 0$ or $\lambda_A(xy) = 1$. If $\alpha \in \{\in, \in \vee q_k\}$, then $x(\mu_A(x),\ \lambda_A(x))\,\alpha A$ but $(xy)(\mu_A(x), \lambda_A(x))\overline{\beta}A$ for every $\beta \in \{\in, q_k, \in \vee q_k, \in \wedge q_k\}$, a contradiction. Also $x(1,0)q_kA$ but $(xy)\,(1,0)\overline{\beta}A$ for every $\beta \in \{\in, q_k, \in \vee q_k, \in \wedge q_k\}$, a contradiction. Thus $\mu_A(xy) > 0$ and $\lambda_A(xy) < 1$. Therefore, $xy \in R_{(0,1)}$. Similarly $yx \in R_{(0,1)}$. This completes the proof.

Theorem 4. *Let $A = (\mu_A, \lambda_A)$ be an $(\alpha,\ \beta)$-intuitionistic fuzzy sub-hemiring of R. Then the set*

$$R_{(0,1)} = \{x \in R : \mu_A(x) > 0 \text{ and } \lambda_A(x) < 1\} \neq \phi$$

is a sub-hemiring of R.

Proof. Let $x,y \in R_{(0,1)}$. Then $\mu_A(x) > 0$ and $\lambda_A(x) < 1$, $\mu_A(y) > 0$ and $\lambda_A(y) < 1$. Assume that $\mu_A(x+y) = 0$ or $\lambda_A(x+y) = 1$. If $\alpha \in \{\in, \in \vee q_k\}$, then, $x(\mu_A(x),\ \lambda_A(x))\,\alpha A$ and $y(\mu_A(y),\ \lambda_A(y))\,\alpha A$ but, $(x+y)(\min\{\mu_A(x),\ \mu_A(y)\}, \max\{\lambda_A(x),\ \lambda_A(y)\})\overline{\beta}A$, for every $\beta \in \{\in, q_k, \in \vee q_k, \in \wedge q_k\}$, a contradiction. Also $x(1,0)q_kA$ and $y(1,0)q_kA$ but $(x+y)\,(1,0)\overline{\beta}A$ for every $\beta \in \{\in, q_k, \in \vee q_k, \in \wedge q_k\}$, a contradiction. Thus $\mu_A(x+y) > 0$ and $\lambda_A(x+y) < 1$. Therefore, $x+y \in R_{(0,1)}$.

Let $x,y \in R_{(0,1)}$. Then $\mu_A(x) > 0$ and $\lambda_A(x) < 1$, $\mu_A(y) > 0$ and $\lambda_A(y) < 1$. Suppose that $\mu_A(xy) = 0$ or $\lambda_A(xy) = 1$. If $\alpha \in \{\in, \in \vee q_k\}$, then $x(\mu_A(x), \lambda_A(x))\,\alpha A$ and $y(\mu_A(y),\ \lambda_A(y))\,\alpha A$ but, $(xy)\,(\min\{\mu_A(x),\ \mu_A(y)\},\ \max\{\lambda_A(x),\ \lambda_A(y)\})\overline{\beta}A$ for every $\beta \in \{\in, q_k, \in \vee q_k, \in \wedge q_k\}$, a contradiction. Also $x(1,0)q_kA$ and $y(1,0)q_kA$ but $(xy)\,(1,0)\overline{\beta}A$ for every $\beta \in \{\in, q_k, \in \vee q_k, \in \wedge q_k\}$, a contradiction. Thus $\mu_A(xy) > 0$ and $\lambda_A(xy) < 1$. Therefore, $xy \in R_{(0,1)}$. This completes the proof.

4 $(\in, \in \vee q_k)^*$-intuitionistic fuzzy ideals

Definition 3. *An IFS $A = (\mu_A, \lambda_A)$ of a hemiring R is called an $(\in,\ \in \vee q_k)$-intuitionistic fuzzy sub-hemiring of R, if $\forall$ $x,y \in R$ and $t_1,t_2 \in (0,1]$, $s_1,s_2 \in [0,1)$,*

(1a) $x(t_1,s_1), y(t_2,s_2) \in A \Rightarrow (x+y)(\min(t_1,t_2), \max(s_1,s_2)) \in \vee q_kA$.
(2a) $x(t_1,s_1), y(t_2,s_2) \in A \Rightarrow (xy)(\min(t_1,t_2), \max(s_1,s_2)) \in \vee q_kA$.

Definition 4. *An IFS $A = (\mu_A, \lambda_A)$ of a hemiring R is called an $(\in,\ \in \vee q_k)$-intuitionistic fuzzy left $(right)$ ideal of R, if $\forall$ $x,y \in R$ and $t_1,t_2 \in (0,1]$, $s_1,s_2 \in [0,1)$,*

(1a) $x(t_1,s_1), y(t_2,s_2) \in A \Rightarrow (x+y)(\min(t_1,t_2), \max(s_1,s_2)) \in \vee q_kA$.
(3a) $y(t_1,s_1) \in A,\ x \in R \Rightarrow (xy)(t_1,s_1) \in \vee q_kA\ ((yx)(t_1,s_1) \in \vee q_kA)$.

An IFS $A = (\mu_A, \lambda_A)$ of a hemiring R is called an $(\in,\ \in \vee q_k)$-intuitionistic fuzzy ideal of R, if it is both $(\in,\ \in \vee q_k)$-intuitionistic fuzzy left ideal and $(\in,\ \in \vee q_k)$-intuitionistic fuzzy right ideal of R.

Theorem 5. *Let A be an intuitionistic fuzzy subset of a hemiring R. Then $(1a) \Longrightarrow (1b)$, $(2a) \Longrightarrow (2b)$, $(3a) \Longrightarrow (3b)$, where $\forall$ $x,y \in R$ and $k \in [0,1)$,* (1b) $\mu_A(x+y) \geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}$ *and* $\lambda_A(x+y) \leq \max\left\{\lambda_A(x), \lambda_A(y), \frac{1-k}{2}\right\}$.

(2b) $\mu_A(xy) \geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}$ *and* $\lambda_A(xy) \leq \max\left\{\lambda_A(x), \lambda_A(y), \frac{1-k}{2}\right\}$.

(3b) $\mu_A(xy) \geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\}$ *and* $\lambda_A(xy) \leq \max\left\{\lambda_A(y), \frac{1-k}{2}\right\}$.

Proof. $(1a) \Rightarrow (1b)$ Let A be an intuitionistic fuzzy subset of a hemiring R, and $(1a)$ holds. Suppose that $(1b)$ doesn't hold then there exist $x, y \in R$ such that $\mu_A(x+y) < \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}$ or $\lambda_A(x+y) > \max\left\{\lambda_A(x), \lambda_A(y), \frac{1-k}{2}\right\}$. So there exits three possible cases.

(i) $\mu_A(x+y) < \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}$ and $\lambda_A(x+y) \leq \max\left\{\lambda_A(x), \lambda_A(y), \frac{1-k}{2}\right\}$,
(ii) $\mu_A(x+y) \geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}$ and $\lambda_A(x+y) > \max\left\{\lambda_A(x), \lambda_A(y), \frac{1-k}{2}\right\}$,
(iii) $\mu_A(x+y) < \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}$ and $\lambda_A(x+y) > \max\left\{\lambda_A(x), \lambda_A(y), \frac{1-k}{2}\right\}$.

For the first case, there exist $t \in (0,1]$ such that $\mu_A(x+y) < t < \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}$. Now choose $s = 1-t$, then clearly $x(t,s) \in A$ and $y(t,s) \in A$ but $(x+y)(t,s)\overline{\in \vee q_k}A$. Which is a contradiction. Second case is similar to this case.

Now consider case (iii), *i.e* $\mu_A(x+y) < \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}$ and $\lambda_A(x+y) > \max\left\{\lambda_A(x), \lambda_A(y), \frac{1-k}{2}\right\}$. Then there exist $t \in (0,1]$ and $s \in [0,1)$, such that $\mu_A(x+y) < t \leq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}$ and $\lambda_A(x+y) > s \geq \max\left\{\lambda_A(x), \lambda_A(y), \frac{1-k}{2}\right\}$

$\Longrightarrow x(t,s) \in A$ and $y(t,s) \in A$ but $(x+y)(t,s)\overline{\in \vee q_k}A$. Which is again a contradiction. So our supposition is wrong. Hence $(1b)$ holds.

Similarly we can prove $(2a) \Longrightarrow (2b)$, $(3a) \Longrightarrow (3b)$.

Definition 5. *Let $A = (\mu_A, \lambda_A)$ be an IFS of a hemiring R. Then A is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy sub-hemiring of R if it satisfies the conditions $(1b)$ and $(2b)$.*

Definition 6. *Let $A = (\mu_A, \lambda_A)$ be an IFS of a hemiring R. Then A is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left ideal of R if it satisfies the conditions $(1b)$ and $(3b)$.*

Remark. Every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left ideal (right ideal, sub-hemiring) $A = (\mu_A, \lambda_A)$ of R need not be an $(\in, \in \vee q_k)$-intuitionistic fuzzy left ideal (right ideal, sub-hemiring) of R.

Example 1. Let $\mathbb{N}$ be the set of all non negative integers and $A = \langle \mu_A, \ \lambda_A \rangle$ be an IFS of $\mathbb{N}$ defined as follows:

$$\mu_A(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.5 & \text{if } 1 \leq x \leq 4, \\ 0.4 & \text{if } 4 < x \end{cases} \qquad \lambda_A(x) = \begin{cases} 0 & \text{if } x = 0 \\ 0.5 & \text{if } 1 \leq x \leq 4 \\ 0.4 & \text{if } 4 < x \end{cases}$$

For all $x, y \in R$,

(1) $\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y), 0.4\}$ and $\lambda_A(x+y) \leq \max\{\lambda_A(x), \lambda_A(y), 0.4\}$,
(2) $\mu_A(xy) \geq \min\{\mu_A(y), 0.4\}$ and $\lambda_A(xy) \leq \max\{\lambda_A(y), 0.4\}$,
(3) $\mu_A(xy) \geq \min\{\mu_A(x), 0.4\}$ and $\lambda_A(xy) \leq \max\{\lambda_A(x), 0.4\}$.

Thus $A = (\mu_A, \lambda_A)$ is an $(\in, \in \vee q_{0.2})^*$-intuitionistic fuzzy ideal of $\mathbb{N}$. But $2(0.45, \ 0.55)$, $3(0.45, \ 0.55) \in A \Longrightarrow (2.3)(0.45, \ 0.55)\overline{\in \vee q_{0.2}}A$. Thus $A = (\mu_A, \lambda_A)$ is not an $(\in, \in \vee q_{0.2})$-intuitionistic fuzzy ideal of $\mathbb{N}$.

Definition 7. *For any intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in R and $t \in (0,1]$, $s \in [0,1)$ and $k \in [0,1)$ we define $U_{(t,s)} = \{x \in R : x(t,s) \in A\}$, $A_{(t,s)_k} = \{x \in R : x(t,s) q_k A\}$ and $[A]_{(t,s)_k} = \{x \in R : x(t,s) \in \vee q_k A\}$.*

Obviously, $[A]_{(t,s)_k} = A_{(t,s)_k} \cup U_{(t,s)}$, *where* $U_{(t,s)}$, $A_{(t,s)_k}$ *and* $[A]_{(t,s)_k}$ *are called* $\in$*-level set,* q_k*-level set and* $\in \vee q_k$*-level set of* $A = (\mu_A, \lambda_A)$, *respectively.*

Lemma 1. *Every intuitionistic fuzzy subset* $A = (\mu_A, \lambda_A)$ *of a hemiring* R *satisfies the following condition:*

$t \in (0, \frac{1-k}{2}]$, $s \in [\frac{1-k}{2}, 1) \implies [A]_{(t,s)_k} = U_{(t,s)}$.

Proof. Let $t \in (0, \frac{1-k}{2}]$, and $s \in [\frac{1-k}{2}, 1)$. It is clear that $U_{(t,s)} \subseteq [A]_{(t,s)_k}$. Let $x \in [A]_{(t,s)_k}$. If $x \notin U_{(t,s)}$, then $\mu_A(x) < t$, or $\lambda_A(x) > s$ and so $\mu_A(x) + t < 2t \leq 1-k$, or $\lambda_A(x) + s > 2s \geq 1-k$. This shows that $x(t,s)\overline{q_k}A.i.e., x \notin A_{(t,s)_k}$ and thus $x \notin U_{(t,s)} \cup A_{(t,s)_k} = [A]_{(t,s)_k}$. This is a contradiction. Thus $x \in U_{(t,s)}$.Therefore $[A]_{(t,s)_k} \subseteq U_{(t,s)}$.

Theorem 6. *If* A *is an* $(\in, \in \vee q_k)^*$*-intuitionistic fuzzy ideal of* R, *then the set* $A_{(t,s)_k}$ *is an ideal of* R *when it is non-empty for all* $t \in (\frac{1-k}{2}, 1]$, $s \in [0, \frac{1-k}{2})$.

Proof. Assume that A is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy ideal of R, and let $t \in (\frac{1-k}{2}, 1]$, $s \in [0, \frac{1-k}{2})$ be such that $A_{(t,s)_k} \neq \phi$. Let $x, y \in A_{(t,s)_k}$. Then $\mu_A(x) + t + k > 1$, $\lambda_A(x) + s + k < 1$ and $\mu_A(y) + t + k > 1$, $\lambda_A(y) + s + k < 1$. As $\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\}$, $\lambda_A(x+y) \leq \max\{\lambda_A(x), \lambda_A(y), \frac{1-k}{2}\}$. We have $\mu_A(x+y) \geq \min\{1-t-k, \frac{1-k}{2}\}$, $\lambda_A(x+y) \leq \max\{1-s-k, \frac{1-k}{2}\}$. Since $t \in (\frac{1-k}{2}, 1]$, and $s \in [0, \frac{1-k}{2})$, so $1-t-k < \frac{1-k}{2}$ and $1-s-k > \frac{1-k}{2}$, thus $\mu_A(x+y) > 1-t-k$ and $\lambda_A(x+y) < 1-s-k$. Hence $x+y \in A_{(t,s)_k}$. Let $x \in A_{(t,s)_k}$ and $y \in R$. Then $\mu_A(x) + t + k > 1$, $\lambda_A(x) + s + k < 1$. Then $\mu_A(x) > 1-t-k$, $\lambda_A(x) < 1-s-k$. Since A is an $(\in, \in \vee q)^*$-intuitionistic fuzzy ideal of R, we have $\mu_A(xy) \geq \min\{\mu_A(x), \frac{1-k}{2}\}$, $\lambda_A(x+y) \leq \max\{\lambda_A(x), \frac{1-k}{2}\}$. Implies that $\mu_A(xy) \geq \min\{1-t-k, \frac{1-k}{2}\}$, $\lambda_A(xy) \leq \max\{1-s-k, \frac{1-k}{2}\}$. Since $t \in (\frac{1-k}{2}, 1]$, and $s \in [0, \frac{1-k}{2})$, so $1-t-k < \frac{1-k}{2}$ and $1-s-k > \frac{1-k}{2}$, thus $\mu_A(xy) > 1-t-k$ and $\lambda_A(xy) < 1-s-k$. This implies $xy \in A_{(t,s)}$. Similarly $xy \in A_{(t,s)_k}$. Hence $A_{(t,s)_k}$ is an ideal of R.

Theorem 7. *For any intuitionistic fuzzy subset* A *of* R, *the following are equivalent:*

(i) *A is an* $(\in, \in \vee q_k)^*$*-intuitionistic fuzzy ideal of* R.
(ii) *For all* $t \in (0, \frac{1-k}{2}]$, *and* $s \in [\frac{1-k}{2}, 1), U_{(t,s)} \neq \phi \implies U_{(t,s)}$ *is an ideal of* R.

Proof. Let A be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy ideal of R and $x, y \in U_{(t,s)}$ for some $t \in (0, \frac{1-k}{2}]$, $s \in [\frac{1-k}{2}, 1)$. Then $\mu_A(x+y) \geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} \geq \min\left\{t, \frac{1-k}{2}\right\} = t$ *and* $\lambda_A(x+y) \leq \max\left\{\lambda_A(x), \lambda_A(y), \frac{1-k}{2}\right\} \leq \max\left\{s, \frac{1-k}{2}\right\} = s$, which implies $x+y \in U_{(t,s)}$. Now, if $x \in U_{(t,s)}$ and $y \in R$ then $\mu_A(xy) \geq \min\left\{\mu_A(x), \frac{1-k}{2}\right\} \geq \min\left\{t, \frac{1-k}{2}\right\} = t$ *and* $\lambda_A(xy) \leq \max\left\{\lambda_A(x), \frac{1-k}{2}\right\} \leq \max\left\{s, \frac{1-k}{2}\right\} = s$, which implies $xy \in U_{(t,s)}$. Similarly $yx \in U_{(t,s)}$. This shows that $U_{(t,s)}$ is an ideal of R.

Conversely, assume that for every $t \in (0, \frac{1-k}{2}]$, and $s \in [\frac{1-k}{2}, 1)$, each non-empty $U_{(t,s)}$ is an ideal of R. Suppose A is not an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy ideal of R, then there exist $x, y \in R$ such that one of the following three cases is true.

(i) $\mu_A(x+y) < \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}$ *and* $\lambda_A(x+y) \leq \max\left\{\lambda_A(x), \lambda_A(y), \frac{1-k}{2}\right\}$.
(ii) $\mu_A(x+y) \geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}$ *and* $\lambda_A(x+y) > \max\left\{\lambda_A(x), \lambda_A(y), \frac{1-k}{2}\right\}$.
(iii) $\mu_A(x+y) < \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}$ *and* $\lambda_A(x+y) > \max\left\{\lambda_A(x), \lambda_A(y), \frac{1-k}{2}\right\}$.

For the first case, $t \in (0, \frac{1-k}{2}]$ such that $\mu_A(x+y) < t \leq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}$. Now choose $s = 1-t$, then clearly x ,$y \in U_{(t,s)}$ but $x+y \notin U_{(t,s)}$. Which is a contradiction. Case (ii) is similar to the case (i).

Now consider case (iii), then there exist $t \in (0, \frac{1-k}{2}]$, and $s \in [\frac{1-k}{2}, 1)$, such that $\mu_A(x+y) < t \leq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}$ *and* $\lambda_A(x+y) > s \geq \max\left\{\lambda_A(x), \lambda_A(y), \frac{1-k}{2}\right\} \implies x, y \in U_{(t,s)}$ but $x+y \notin U_{(t,s)}$. Which is a contradiction. So our supposition is wrong, hence $\mu_A(x+y) \geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}$ *and* $\lambda_A(x+y) \leq \max\left\{\lambda_A(x), \lambda_A(y), \frac{1-k}{2}\right\}$ for all

$x, y \in R$.

In a similar way we can show that $\mu_A(xy) \geq \min\{\mu_A(x), \frac{1-k}{2}\}$ *and* $\lambda_A(xy) \leq \max\{\lambda_A(x), \frac{1-k}{2}\}$, $\mu_A(xy) \geq \min\{\mu_A(y), \frac{1-k}{2}\}$ *and* $\lambda_A(xy) \leq \max\{\lambda_A(y), \frac{1-k}{2}\}$ for all $x, y \in R$.

Theorem 8. *Let $\{A_i : i \in I\}$ be a family of $(\in, \in \vee q_k)^*$-intuitionistic fuzzy sub-hemiring of R. Then $A = \cap_{i\in I} A_i$ is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy sub-hemiring of R.*

Proof. Straightforward.

Theorem 9. *Let $\{A_i : i \in I\}$ be a family of $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left $(right)$ ideals of R. Then $A = \cap_{i\in I} A_i$ is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left $(right)$ ideal of R.*

Proof. Straightforward.

5 Regular and idempotent hemirings

Definition 8. *Let A and B be two intuitionistic fuzzy subsets of a hemiring R, then $A \cdot_k B$ is defined as, $A \cdot_k B = \langle \mu_A \cdot_k \mu_B,\ \lambda_A \cdot_k \lambda_B \rangle$ where*

$$(\mu_A \cdot_k \mu_B)(x) = \begin{cases} \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} [\mu_A(y_i) \wedge \mu_B(z_i)] \right] \wedge \frac{1-k}{2} \\ 0 \quad \text{if } x \text{ cannot be expressed as } x = \sum_{i=1}^{p} y_i z_i \end{cases}$$

$$(\lambda_A \cdot_k \lambda_B)(x) = \begin{cases} \bigwedge_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigvee_{1\leq i\leq p} [\lambda_A(y_i) \vee \lambda_B(z_i)] \right] \vee \frac{1-k}{2} \\ 1 \quad \text{if } x \text{ cannot be expressed as } x = \sum_{i=1}^{p} y_i z_i \end{cases}$$

where $x \in R$.

Definition 9. *let A and B an intuitionistic fuzzy subsets of R. We define the intuitionistic fuzzy subsets A_k, $A \cap_k B$, $A \cup_k B$ and $A \cdot_k B$ of R as follows:*

$$A_k = \left(\mu_A \wedge \frac{1-k}{2}, \lambda_B \vee \frac{1-k}{2} \right),$$
$$A \cap_k B = (A \cap B)_k = (\mu_A \wedge_k \mu_B, \lambda_A \vee_k \lambda_B),$$
$$A \cup_k B = (A \cup B)_k = (\mu_A \vee_k \mu_B, \lambda_A \wedge_k \lambda_B).$$

Theorem 10. *Let A be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy sub-hemiring of R. Then A_k is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy sub-hemiring of R.*

Proof. Suppose A is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy sub-hemiring of R and $x, y \in R$. Then

$$\begin{aligned} (\mu_A \wedge \frac{1-k}{2})(x+y) &= \mu_A(x+y) \wedge \frac{1-k}{2} \\ &\geq \left(\min\left\{ \mu_A(x), \mu_A(y), \frac{1-k}{2} \right\} \right) \wedge \frac{1-k}{2} \\ &= \min\left\{ \mu_A(x) \wedge \frac{1-k}{2}, \mu_A(y) \wedge \frac{1-k}{2}, \frac{1-k}{2} \right\} \\ &= \min\left\{ (\mu_A \wedge \frac{1-k}{2})(x), (\mu_A \wedge \frac{1-k}{2})(y), \frac{1-k}{2} \right\}, \end{aligned}$$

and

$$\begin{aligned}(\lambda_A \vee \frac{1-k}{2})(x+y) &= \lambda_A(x+y) \vee \frac{1-k}{2} \\ &\leq (\max\{\lambda_A(x), \lambda_A(y), \frac{1-k}{2}\}) \vee \frac{1-k}{2}. \\ &= \max\{\lambda_A(x) \vee \frac{1-k}{2}, \lambda_A(y) \vee \frac{1-k}{2}, \frac{1-k}{2}\}. \\ &= \max\{(\lambda_A \vee \frac{1-k}{2})(x), (\lambda_A(y) \vee \frac{1-k}{2})(y), \frac{1-k}{2}\}.\end{aligned}$$

Similarly we can show that

$$(\mu_A \wedge \frac{1-k}{2})(xy) \geq \min\left\{(\mu_A \wedge \frac{1-k}{2})(x), (\mu_A \wedge \frac{1-k}{2})(y), \frac{1-k}{2}\right\},$$

and

$$(\lambda_A \vee \frac{1-k}{2})(xy) \leq \max\{(\lambda_A \vee \frac{1-k}{2})(x), (\lambda_A(y) \vee \frac{1-k}{2})(y), \frac{1-k}{2}\}.$$

This shows that $A_k = A \cap \frac{1-k}{2}$ is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy sub-hemiring of R.

Theorem 11. *Let A be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy ideal of R. Then A_k is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy ideal of R.*

Proof. This proof is similar to the proof of the theorem 10,

Remark. let A and B be intuitionistic fuzzy subsets of R.Then the following hold.

(i) $A \cap_k B = (A_k \cap B_k)$.
(ii) $A \cup_k B = (A_k \cup B_k)$.
(iii) $A \cdot_k B = (A_k \cdot B_k)$.

Proof. let $x \in R$,
(1) $(\mu_A \wedge_k \mu_B)(x) = (\mu_A \wedge \mu_B)(x) \wedge \frac{1-k}{2} = \mu_A(x) \wedge \mu_B(x) \wedge \frac{1-k}{2} = (\mu_A(x) \wedge \frac{1-k}{2}) \wedge (\mu_B(x) \wedge \frac{1-k}{2})$
$= \mu_{A_k}(x) \wedge \mu_{B_K}(x) = (\mu_{A_k} \wedge \mu_{B_K})(x)$
and
$(\lambda_A \vee_k \lambda_B)(x) = (\lambda_A \vee \lambda_B)(x) \vee \frac{1-k}{2}) = \lambda_A(x) \vee \lambda_B(x) \vee \frac{1-k}{2} = (\lambda_A(x) \vee \frac{1-k}{2}) \vee (\lambda_B(x) \vee \frac{1-k}{2})$
$= \lambda_{A_k}(x) \vee \lambda_{B_K}(x) = (\lambda_{A_k} \vee \lambda_{B_K})(x)$.
Hence (1) holds. Similarly we can prove (2).

(3) If x is not expressible as $x = \sum_{i=1}^{p} y_i z_i$ where $y_i, z_i \in R$, then $(\mu_A \cdot \mu_B)(x) = 0$.
Thus $(\mu_A \cdot_k \mu_B)(x) = (\mu_A \cdot \mu_B)(x) \wedge \frac{1-k}{2} = 0$. As x is not expressible as $x = \sum_{i=1}^{p} y_i z_i$ so $(\mu_{A_k} \cdot \mu_{B_k})(x) = 0 \implies \mu_A \cdot_k \mu_B = \mu_{A_k} \cdot \mu_{B_k}$ and $(\lambda_A \cdot \lambda_B)(x) = 1$, thus $(\lambda_A \cdot_k \lambda_B)(x) = (\lambda_A \cdot \lambda_B)(x) \vee \frac{1-k}{2} = 1$ as x is not expressible as $x = \sum_{i=1}^{p} y_i z_i$ so $(\lambda_{A_k} \cdot \lambda_{B_k})(x) = 1 \implies \lambda_A \cdot_k \lambda_B = \lambda_{A_k} \cdot \lambda_{B_k}$. Hence (3) holds.

Theorem 12. *If A and B are $(\in, \in \vee q_k)^*$-intuitionistic fuzzy ideals of R then $A \cdot_k B$ is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy ideal of R.*

Proof. Let $x, y \in R$ be such that $x = \sum_{i=1}^{p} a_i b_i$, and $y = \sum_{j=1}^{q} a_i' b_i'$. Then

$$(\mu_A \cdot_k \mu_B)(x) = \bigvee_{x=\sum_{i=1}^{p} a_i b_i} \left[\bigwedge_{1 \leq i \leq p} [\mu_A(a_i) \wedge \mu_B(b_i)]\right] \wedge \frac{1-k}{2}$$

and

$$(\mu_A \cdot_k \mu_B)(x') = \bigvee_{x'=\sum_{j=1}^{q} a'_i b'_i} \left[\bigwedge_{1\le i\le p} [\mu_A(a'_i) \wedge \mu_B(b'_i)] \right] \wedge \frac{1-k}{2}.$$

Thus

$$\begin{aligned}
(\mu_A \cdot_k \mu_B)(x) \wedge (\mu_A \cdot_k \mu_B)(x') \wedge \frac{1-k}{2} &= \begin{cases} \left[\bigvee_{x=\sum_{i=1}^{p} a_i b_i} \left[\bigwedge_{1\le i\le p} [\mu_A(a_i) \wedge \mu_B(b_i)]\right] \wedge \frac{1-k}{2}\right] \wedge \\ \left[\bigvee_{x'=\sum_{j=1}^{q} a'_i b'_i} \left[\bigwedge_{1\le i\le p} [\mu_A(a'_i) \wedge \mu_B(b'_i)]\right] \wedge \frac{1-k}{2}\right] \wedge \frac{1-k}{2} \end{cases} \\
&= \left[\bigvee_{x=\sum_{i=1}^{p} a_i b_i} \bigvee_{x'=\sum_{j=1}^{q} a'_i b'_i} \left[\begin{array}{c} [\bigwedge_{1\le i\le p} [\mu_A(a_i) \wedge \mu_B(b_i)]] \wedge \\ [\bigwedge_{1\le j\le q} [\mu_A(a'_i) \wedge \mu_B(b'_i)]] \end{array} \right] \wedge \frac{1-k}{2} \right] \\
&\le \left[\bigvee_{x+x'=\sum_{k=1}^{s} a'' b''} \left[\bigwedge_{1\le k\le s} [\mu_A(a'') \wedge \mu_B(b'')] \right] \wedge \frac{1-k}{2} \right] \\
&= (\mu_A \cdot_k \mu_B)(x+x')
\end{aligned}$$

and

$$(\lambda_A \cdot_k \lambda_B)(x) = \left[\bigwedge_{x=\sum_{i=1}^{p} a_i b_i} \left[\bigvee_{1\le i\le p} [\lambda_A(a_i) \vee \lambda_B(b_i)] \right] \vee \frac{1-k}{2} \right],$$

$$(\lambda_A \cdot_k \lambda_B)(x') = \left[\bigwedge_{x'=\sum_{j=1}^{q} a'_i b'_i} \left[\bigvee_{1\le i\le p} [\lambda_A(a'_i) \vee \lambda_B(b'_i)] \right] \vee \frac{1-k}{2} \right].$$

Thus

$$\begin{aligned}
(\lambda_A \cdot_k \lambda_B)(x) \vee (\lambda_A \cdot_k \lambda_B)(x') \vee \frac{1-k}{2} &= \begin{cases} \left[\bigwedge_{x=\sum_{i=1}^{p} a_i b_i} \left[\bigvee_{1\le i\le p} [\lambda_A(a_i) \vee \lambda_B(b_i)]\right] \vee \frac{1-k}{2}\right] \vee \\ \left[\bigwedge_{x'=\sum_{j=1}^{q} a'_i b'_i} \left[\bigvee_{1\le i\le p} [\lambda_A(a'_i) \vee \lambda_B(b'_i)]\right] \vee \frac{1-k}{2}\right] \vee \frac{1-k}{2} \end{cases} \\
&= \left[\bigwedge_{x=\sum_{i=1}^{p} a_i b_i} \bigwedge_{x'=\sum_{j=1}^{q} a'_i b'_i} \left[\begin{array}{c} [\bigvee_{1\le i\le p} [\lambda_A(a_i) \vee \lambda_B(b_i)]] \wedge \\ [\bigvee_{1\le j\le q} [\lambda_A(a'_i) \vee \lambda_B(b'_i)]] \end{array} \right] \vee \frac{1-k}{2} \right]
\end{aligned}$$

$\ge \left[\bigwedge_{x+x'=\sum_{k=1}^{s} a'' b''} \left[\bigvee_{1\le k\le s} [\lambda_A(a'') \vee \lambda_B(b'')]\right] \vee \frac{1-k}{2}\right]$

$= (\lambda_A \cdot_k \lambda_B)(x+x')$

$\Longrightarrow \{(\lambda_A \cdot_k \lambda_B)(x) \vee (\lambda_A \cdot_k \lambda_B)(x') \vee \frac{1-k}{2}\} \ge (\lambda_A \cdot_k \lambda_B)(x+x')$. Also, $(\mu_A \cdot_k \mu_B)(x) \wedge \frac{1-k}{2}$

$= \left[\left[\bigvee_{x=\sum_{i=1}^{p} a_i b_i} \left[\bigwedge_{1\le i\le p} [\mu_A(a_i) \wedge \mu_B(b_i)]\right]\right] \wedge \frac{1-k}{2}\right] \frac{1-k}{2}$

$= \left[\bigvee_{x=\sum_{i=1}^{p} a_i b_i} \left[\bigwedge_{1\le i\le p} [\mu_A(a_i) \wedge \mu_B(b_i) \frac{1-k}{2}]\right]\right] \wedge \frac{1-k}{2}$

$\le \left[\bigvee_{x=\sum_{i=1}^{p} a_i b_i} \left[\bigwedge_{1\le i\le p} [\mu_A(a_i) \wedge \mu_B(b_i r)]\right]\right] \wedge \frac{1-k}{2}$

$\leq \left[\bigvee_{xr=\sum_{j=1}^{q} a_i' b_i'} \left[\bigwedge_{1\leq j\leq q}[\mu_A(a_i')\wedge\mu_B(b_i')]\right]\right] \wedge \frac{1-k}{2}$

$= (\mu_A \cdot_k \mu_B)(xr).$

Thus $\left\{(\mu_A \cdot_k \mu_B)(x) \wedge \frac{1-k}{2}\right\} \leq (\mu_A \cdot_k \mu_B)(xr).$

Similarly we can prove $(\lambda_A \cdot_k \lambda_B)(xr) \leq \left\{(\lambda_A \cdot_k \lambda_B)(x) \vee \frac{1-k}{2}\right\} \implies A \cdot_k B$ is an $(\in, \in\vee q_k)^*$-intuitionistic fuzzy right ideal of R. On the same line it can be proved that $\left\{(\mu_A \cdot_k \mu_B)(x) \wedge \frac{1-k}{2}\right\} \leq (\mu_A \cdot_k \mu_B)(rx)$ and $(\lambda_A \cdot_k \lambda_B)(rx) \leq \left\{(\lambda_A \cdot_k \lambda_B)(xr) \vee \frac{1-k}{2}\right\}$. Thus $A \cdot_k B$ is an $(\in, \in\vee q_k)^*$-intuitionistic fuzzy ideal of R.

Theorem 13. *If A and B are $(\in, \in\vee q)^*$-intuitionistic fuzzy left(right) ideals of R, then so is $A \cap_k B$.*

Proof. We only consider the case of $(\in, \in\vee q_k)^*$-intuitionistic fuzzy left ideals.

Let $x, y \in R$. Then

$$\begin{aligned}(\mu_A \wedge_k \mu_B)(x+y) &= \min\{\mu_A(x+y), \mu_B(x+y), \frac{1-k}{2}\} \\ &\geq \min\left\{\min\{\mu_A(x),\mu_A(y),\frac{1-k}{2}\}, \min\{\mu_B(y),\mu_B(x),\frac{1-k}{2}\}, \frac{1-k}{2}\right\} \\ &= \min\left\{\min\{\mu_A(x),\mu_B(x),\frac{1-k}{2}\}, \min\{\mu_A(y),\mu_B(y),\frac{1-k}{2}\}, \frac{1-k}{2}\right\} \\ &= \min\left\{(\mu_A \wedge_k \mu_B)(x), (\mu_A \wedge_k \mu_B)(y), \frac{1-k}{2}\right\}\end{aligned}$$

and

$$\begin{aligned}(\lambda_A \vee_k \lambda_B)(x+y) &= \max\left\{\lambda_A(x+y), \lambda_B(x+y), \frac{1-k}{2}\right\} \\ &\leq \max\left\{\max\{\lambda_A(x),\lambda_A(y),\frac{1-k}{2}\}, \max\{\lambda_B(x),\lambda_B(y),\frac{1-k}{2}\}, \frac{1-k}{2}\right\} \\ &= \max\left\{\max\{\lambda_A(x),\lambda_B(x),\frac{1-k}{2}\}, \max\{\lambda_A(y),\lambda_B(y),\frac{1-k}{2}\}, \frac{1-k}{2}\right\} \\ &= \max\left\{(\lambda_A \vee_k \lambda_B)(x), (\lambda_A \vee_k \lambda_B)(y), \frac{1-k}{2}\right\}.\end{aligned}$$

Now

$$\begin{aligned}(\mu_A \wedge_k \mu_B)(x.y) &= \min\left\{\mu_A(x.y), \mu_B(x.y), \frac{1-k}{2}\right\} \\ &\geq \min\left\{\min\{\mu_A(y),\frac{1-k}{2}\}, \min\{\mu_B(y),\frac{1-k}{2}\}, \frac{1-k}{2}\right\} \\ &= \min\left\{\min\{\mu_A(y),\mu_B(y),\frac{1-k}{2}\}, \frac{1-k}{2}\right\} = \min\left\{(\mu_A \wedge_k \mu_B)(y), \frac{1-k}{2}\right\}\end{aligned}$$

and

$$\begin{aligned}(\lambda_A \vee_k \lambda_B)(x.y) &= \max\{\lambda_A(x.y), \lambda_B(x.y), \frac{1-k}{2}\} \\ &\leq \max\left\{\max\{\lambda_A(y), \frac{1-k}{2}\}, \max\{\lambda_B(y), \frac{1-k}{2}\}, \frac{1-k}{2}\right\} \\ &= \max\left\{\max\{\lambda_A(y), \lambda_B(y), \frac{1-k}{2}\}, \frac{1-k}{2}\right\} = \max\left\{(\lambda_A \vee_k \lambda_B)(y), \frac{1-k}{2}\right\}.\end{aligned}$$

Thus $A \cap_k B$ is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left ideal of R.

Theorem 14. *If A is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right ideal, and B is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left ideal of R, then $A \cdot_k B \subseteq A \cap_k B$.*

Proof. Let A and B be $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right and left ideals of R respectively. For any $x \in R$,

$$\begin{aligned}(\mu_A \cdot_k \mu_B)(x) &= \bigvee_{x=\sum_{i=1}^{p} a_i b_i} \left[\bigwedge_{1\leq i\leq p} [\mu_A(a_i) \wedge \mu_B(b_i)]\right] \wedge \frac{1-k}{2} \\ &= \bigvee_{x=\sum_{i=1}^{p} a_i b_i} \left[\bigwedge_{1\leq i\leq p} [\mu_A(a_i) \wedge \frac{1-k}{2}] \wedge [\mu_B(b_i) \wedge \frac{1-k}{2}]\right] \wedge \frac{1-k}{2} \\ &\leq \bigvee_{x=\sum_{i=1}^{p} a_i b_i} \left[\bigwedge_{1\leq i\leq p} [\mu_A(a_i b_i) \wedge \mu_B(a_i b_i)]\right] \wedge \frac{1-k}{2} \\ &= \bigvee_{x=\sum_{i=1}^{p} a_i b_i} \left[\left(\bigwedge_{1\leq i\leq p} \mu_A(a_i b_i)\right) \wedge \left(\bigwedge_{1\leq i\leq p} \mu_B(a_i b_i)\right)\right] \wedge \frac{1-k}{2} \\ &\leq \left[\bigvee_{x=\sum_{i=1}^{p} a_i b_i} [\mu_A(x) \wedge \mu_B(x)]\right] \wedge \frac{1-k}{2} = (\mu_A \wedge_k \mu_B)(x),\end{aligned}$$

and

$$\begin{aligned}(\lambda_A \cdot_k \lambda_B)(x) &= \bigwedge_{x=\sum_{i=1}^{p} a_i b_i} \left[\bigvee_{1\leq i\leq p} [\lambda_A(a_i) \vee \lambda_B(b_i)]\right] \vee \frac{1-k}{2} \\ &= \bigwedge_{x=\sum_{i=1}^{p} a_i b_i} \left[\bigvee_{1\leq i\leq p} [\lambda_A(a_i) \vee \frac{1-k}{2}] \vee [\lambda_B(b_i) \vee \frac{1-k}{2}]\right] \vee \frac{1-k}{2} \\ &\geq \bigwedge_{x=\sum_{i=1}^{p} a_i b_i} \left[\bigvee_{1\leq i\leq p} [\lambda_A(a_i b_i) \vee \lambda_B(a_i b_i)]\right] \vee \frac{1-k}{2} \\ &= \bigwedge_{x=\sum_{i=1}^{p} a_i b_i} \left[\left(\bigvee_{1\leq i\leq p} \lambda_A(a_i b_i)\right) \vee \left(\bigvee_{1\leq i\leq p} \lambda_B(a_i b_i)\right)\right] \vee \frac{1-k}{2} \\ &\geq \left[\bigwedge_{x=\sum_{i=1}^{p} a_i b_i} [\lambda_A(x) \vee \lambda_B(x)]\right] \vee \frac{1-k}{2} = (\lambda_A \vee_k \lambda_B)(x).\end{aligned}$$

Thus $A \cdot_k B \subseteq A \cap_k B$.

Definition 10. *Let A and B be $(\in, \in \vee q_k)^*$-intuitionistic fuzzy ideals of R. The intuitionistic fuzzy subset $A +_k B$ of R is defined by*

$$A +_k B = (\mu_A +_k \mu_B, \lambda_A +_k \lambda_B)$$

where

$$(\mu_A +_k \mu_B)(x) = \bigvee_{x=y+z} [\mu_A(y) \wedge \mu_B(z)] \wedge \frac{1-k}{2},$$

$$(\lambda_A +_k \lambda_B)(x) = \bigwedge_{x=y+z} [\lambda_A(y) \vee \lambda_B(z)] \vee \frac{1-k}{2} \; for\, x \in R.$$

Proposition 1. *For $(\in, \in \vee q_k)^*$-intuitionistic fuzzy ideals A and B of R, $A +_k B$ is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy ideal of R.*

Proof. For any $x, x' \in R$,

$$\begin{aligned}
(\mu_A +_k \mu_B)(x) \wedge (\mu_A +_k \mu_B)(x') \wedge \frac{1-k}{2} &= \begin{bmatrix} [\bigvee_{x=y+z}[\mu_A(y) \wedge \mu_B(z)] \wedge \frac{1-k}{2}] \wedge \\ [\bigvee_{x'=y'+z'}[\mu_A(y') \wedge \mu_B(z')] \wedge \frac{1-k}{2}] \wedge \frac{1-k}{2} \end{bmatrix} \\
&= \left[\bigvee_{x=y+z} \bigvee_{x'=y'+z'} \begin{bmatrix} [[\mu_A(y) \wedge \mu_B(z)] \wedge \frac{1-k}{2}] \wedge \\ [[\mu_A(y') \wedge \mu_B(z')] \wedge \frac{1-k}{2}] \end{bmatrix} \right] \wedge \frac{1-k}{2} \\
&= \left[\bigvee_{x=y+z} \bigvee_{x'=y'+z'} \begin{bmatrix} [[\mu_A(y) \wedge \mu_A(y')] \wedge \frac{1-k}{2}] \wedge \\ [[\mu_B(z) \wedge \mu_B(z')] \wedge \frac{1-k}{2}] \end{bmatrix} \right] \wedge \frac{1-k}{2} \\
&\leq \bigvee_{x=y+z} \bigvee_{x'=y'+z'} [\mu_A(y+y') \wedge \mu_B(z+z')] \wedge \frac{1-k}{2} \\
&\leq (\mu_A +_k \mu_B)(x +_k x'),
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_A +_k \lambda_B)(x) \vee (\lambda_A +_k \lambda_B)(x') \vee \frac{1-k}{2} &= \begin{bmatrix} [\bigwedge_{x=y+z}[\lambda_A(y) \vee \lambda_B(z)] \vee \frac{1-k}{2}] \vee \\ [\bigwedge_{x'=y'+z'}[\lambda_A(y') \vee \lambda_B(z')] \vee \frac{1-k}{2}] \vee \frac{1-k}{2} \end{bmatrix} \\
&= \left[\bigwedge_{x=y+z} \bigwedge_{x'=y'+z'} \begin{bmatrix} [[\lambda_A(y) \vee \lambda_B(z)] \vee \frac{1-k}{2}] \vee \\ [[\lambda_A(y') \vee \lambda_B(z')] \vee \frac{1-k}{2}] \end{bmatrix} \right] \vee \frac{1-k}{2} \\
&= \left[\bigwedge_{x=y+z} \bigwedge_{x'=y'+z'} \begin{bmatrix} [[\lambda_A(y) \vee \lambda_A(y')] \vee \frac{1-k}{2}] \vee \\ [[\lambda_B(z) \vee \lambda_B(z')] \vee \frac{1-k}{2}] \end{bmatrix} \right] \vee \frac{1-k}{2} \\
&\geq \bigwedge_{x=y+z} \bigwedge_{x'=y'+z'} [\lambda_A(y+y') \vee \lambda_B(z+z')] \vee \frac{1-k}{2} \\
&\geq (\lambda_A +_k \lambda_B)(x +_k x').
\end{aligned}$$

Again,

$$(\mu_A +_k \mu_B)(x) \wedge \frac{1-k}{2} = \left[\bigvee_{x=y+z} [\mu_A(y) \wedge \mu_B(z)] \wedge \frac{1-k}{2}\right] \wedge \frac{1-k}{2}$$
$$= \left[\bigvee_{x=y+z} [(\mu_A(y) \wedge \frac{1-k}{2}) \wedge (\mu_B(z) \wedge \frac{1-k}{2})]\right]$$
$$\leq \left[\bigvee_{x=y+z} [\mu_A(ya) \wedge \mu_B(za)]\right] \wedge \frac{1-k}{2}$$
$$\leq \left[\bigvee_{xa=y'+z'} [\mu_A(y') \wedge \mu_B(z')]\right] \wedge \frac{1-k}{2} = (\mu_A +_k \mu_B)(xa),$$

and

$$(\lambda_A +_k \lambda_B)(x) \vee \frac{1-k}{2} = \left[\bigwedge_{x=y+z} [\lambda_A(y) \vee \lambda_B(z)] \vee \frac{1-k}{2}\right] \vee \frac{1-k}{2}$$
$$= \left[\bigwedge_{x=y+z} [(\lambda_A(y) \vee \frac{1-k}{2}) \vee (\lambda_B(z)) \vee \frac{1-k}{2}]\right] \vee \frac{1-k}{2}$$
$$\geq \left[\bigwedge_{x=y+z} (\lambda_A(ya) \vee \lambda_B(za))\right] \vee \frac{1-k}{2}$$
$$\geq \left[\bigwedge_{xa=y'+z'} (\lambda_A(y') \vee \lambda_B(z'))\right] \vee \frac{1-k}{2} = (\lambda_A +_k \lambda_B)(xa).$$

Similarly we can prove

$$(\mu_A +_k \mu_B)(x) \wedge \frac{1-k}{2} \leq (\mu_A +_k \mu_B)(ax) \text{ and } (\lambda_A +_k \lambda_B)(x) \vee \frac{1-k}{2} \geq (\lambda_A +_k \lambda_B)(ax).$$

Hence $A +_k B$ is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy ideal of R.

Definition 11.[18] *If $S \subseteq R$, then intuitionistic characteristic function of S is denoted by $C_S = \left(\chi_S, \chi_S^c\right)$ and is defined by*

$\chi_S(x) = \begin{cases} 1 & if\ x \in S \\ 0 & if\ x \notin S \end{cases}$ *and* $\chi_S^c(x) = \begin{cases} 0 & if\ x \in S \\ 1 & if\ x \notin S \end{cases}$ *In particular, we let $\overline{1} = (\chi_R, \chi_R^c)$ be the intuitionistic fuzzy set in R.*

Lemma 2. *A non-empty subset L of a hemiring R is a left ideal of R if and only if the intuitionistic characteristic function $C_L = (\chi_L, \chi_L^c)$ is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left ideal of R.*

Proof. Let L be a left ideal of R, then obviously C_L is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left ideal of R.

Conversely assume that C_L is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left ideal of R. Let $x, y \in L$. Then $\chi_L(x) = 1$, $\chi_L^c(x) = 0$, and $\chi_L(y) = 1$, $\chi_L^c(y) = 0$ so $x(1,0), y(1,0) \in C_L$. Since C_L is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left ideal, so $(\chi_L)(x+y) \geq \min\left\{\chi_L(x), \chi_L(y), \frac{1-k}{2}\right\}$ and $(\chi_L^c)(x+y) \leq \max\left\{\chi_L^c(x), \chi_L^c(y), \frac{1-k}{2}\right\}$
i.e $(\chi_L)(x+y) = 1$ and $(\chi_L^c)(x+y) = 0$. Thus $x+y \in L$.

Let $y \in L$ and $x \in R$. Then $\chi_L(y) = 1$, and $\chi_L(y) = 0$ so $y(1,0) \in C_L$. Since C_L is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left ideal, so $(\chi_L)(xy) \geq \min\left\{\chi_L(y), \frac{1-k}{2}\right\}$ and $(\chi_L^c)(xy) \leq \max\left\{\chi_L^c(y), \frac{1-k}{2}\right\}$. i.e. $(\chi_L)(xy) = 1$ and $(\chi_L^c)(xy) = 0$. Hence $xy \in L$. Thus L is a left ideal of R

Lemma 3. *A non-empty subset L of a hemiring R is a left ideal of R if and only if the intuitionistic fuzzy set $(C_L)_k = \left(\chi_L \wedge \frac{1-k}{2}, \chi_L^c \vee \frac{1-k}{2}\right)$ is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left ideal of R.*

Proof. Straightforward.

Lemma 4. *Let A and B be non-empty subsets of a hemiring R. Then the following hold:*
(1) $C_A \cap_k C_B = (C_{A\cap B})_k$
(2) $C_A \cdot_k C_B = (C_{A\cdot B})_k$.

Proof. Straightforward.

Theorem 15. *For a hemiring R, the following conditions are equivalent:*

(i) *R is hemiregular.*
(ii) *$A \cap_k B = A \cdot_k B$ for every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right ideal A and every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left ideal B of R.*

Proof. Let A be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right ideal and B be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left ideal of R and $x \in R$. Then there exists $a \in R$, such that $x = xax$. Now

$$(\mu_A \cdot_k \mu_B)(x) = \left\{ \bigvee_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigwedge_{1\leq i\leq p} [\mu_A(y_i) \wedge \mu_B(z_i)] \right] \wedge \frac{1-k}{2} \right\} \geq \left[\mu_A(xa) \wedge \mu_B(x) \wedge \frac{1-k}{2}\right]$$
$$\geq \left[\mu_A(x) \wedge \mu_B(x) \wedge \frac{1-k}{2}\right] = (\mu_A \wedge_k \mu_B)(x)$$

and

$$(\lambda_A \cdot_k \lambda_B)(x) = \left\{ \bigwedge_{x=\sum_{i=1}^{p} y_i z_i} \left[\bigvee_{1\leq i\leq p} [\lambda_A(y_i) \vee \lambda_B(z_i)] \right] \vee \frac{1-k}{2} \right\} \leq \left[\lambda_A(xa) \vee \lambda_B(x) \vee \frac{1-k}{2}\right]$$
$$\leq \left[\lambda_A(x) \vee \lambda_B(x) \vee \frac{1-k}{2}\right] = (\lambda_A \vee_k \lambda_B)(x).$$

Thus $A \cap_k B \subseteq A \cdot_k B$.

By Theorem 14 $A \cdot_k B \subseteq A \cap_k B$. Hence $A \cdot_k B = A \cap_k B$.

$(ii) \implies (i)$ Let A and B be right ideal and left ideal of R respectively. Then C_A is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right ideal and C_B is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left ideal of R, by assumption

$$C_A \cdot_k C_B = C_A \cap_k C_B \implies (C_A \cdot C_B)_k = (C_A \cap C_B)_k \implies (C_{AB})_k = (C_{A\cap B})_k \implies AB = A \cap B.$$

Thus by Theorem 1 R is regular.

Theorem 16. *The following assertions for a hemiring R with identity are equivalent:*

(1) *R is fully idempotent.*

(2) *Each* $(\in, \in \vee q_k)^*$*-intuitionistic fuzzy ideal of* R *is idempotent.* (*an* $(\in, \in \vee q_k)^*$*-intuitionistic fuzzy ideal* A *of* R *is called idempotent if* $A \cdot_k A = A_k$.)

(3) *for each pair of* $(\in, \in \vee q_k)^*$*-intuitionistic fuzzy ideals* A *and* B *of* R, $A \cap_k B = A \cdot_k B$.

(4) *If* R *is assumed to be commutative, then the above assertions are equivalent to* R *is regular.*

Proof. $(1) \implies (2)$. Let $A = (\mu_A, \lambda_A)$ be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy ideal of R. For any $x \in R$, by Theorem 14 $A \cdot_k A \subseteq A_k$.

Since each ideal of R is idempotent, therefore, $(x) = (x)^2$ for each $x \in R$. Since $x \in (x)$ it follows that $x \in (x)^2 = RxRRxR$. Hence $x = \sum_{i=1}^{q} a_i x a_i' b_i x b_i'$ and $q \in N$. Now,

$$\left(\mu_A \wedge \frac{1-k}{2}\right)(x) = \mu_A(x) \wedge \mu_A(x) \wedge \frac{1-k}{2} = \left[\mu_A(x) \wedge \frac{1-k}{2}\right] \wedge \left[\mu_A(x) \wedge \frac{1-k}{2}\right] \wedge \frac{1-k}{2}$$
$$\leq \mu_A(a_i x a_i') \wedge \mu_A(b_i x b_i') \wedge \frac{1-k}{2}, \; (1 \leq i \leq q).$$

Therefore,

$$\left(\mu_A \wedge \frac{1-k}{2}\right)(x) \leq \bigwedge_{1 \leq i \leq q} [\mu_A(a_i x a_i') \wedge \mu_A(b_i x b_i')] \wedge \frac{1-k}{2}$$
$$\leq \bigvee_{x=\sum_{i=1}^{q} a_i x a_i' b_i x b_i'} \left[\bigwedge_{1 \leq i \leq q} [\mu_A(a_i x a_i') \wedge \mu_A(b_i x b_i')]\right] \wedge \frac{1-k}{2}$$
$$\leq \bigvee_{x=\sum_{j=1}^{r} a_j b_j} \left[\bigwedge_{1 \leq j \leq r} [\mu_A(a_j) \wedge \mu_A(b_j)]\right] \wedge \frac{1-k}{2} = (\mu_A \cdot_k \mu_A)(x)$$

and

$$\left(\lambda_A \vee \frac{1-k}{2}\right)(x) = \lambda_A(x) \vee \lambda_A(x) \vee \frac{1-k}{2}$$
$$= \left[\lambda_A(x) \vee \frac{1-k}{2}\right] \vee \left[\lambda_A(x) \vee \frac{1-k}{2}\right] \vee \frac{1-k}{2}$$
$$\geq \lambda_A(a_i x a_i') \vee \lambda_A(b_i x b_i') \vee \frac{1-k}{2}, \; (1 \leq i \leq q).$$

Therefore,

$$\left(\lambda_A \vee \frac{1-k}{2}\right)(x) \geq \bigvee_{1 \leq i \leq q} [\lambda_A(a_i x a_i') \vee \lambda_A(b_i x b_i')] \vee \frac{1-k}{2}$$
$$\geq \bigwedge_{x=\sum_{i=1}^{q} a_i x a_i' b_i x b_i'} \left[\bigvee_{1 \leq i \leq q} [\lambda_A(a_i x a_i') \vee \lambda_A(b_i x b_i')]\right] \vee \frac{1-k}{2}$$
$$\geq \bigwedge_{x=\sum_{j=1}^{r} a_j b_j} \left[\bigvee_{1 \leq j \leq r} [\lambda_A(a_j) \vee \lambda_A(b_j)]\right] \vee \frac{1-k}{2} = (\lambda_A \cdot_k \lambda_A)(x).$$

Thus $A \cdot_k A = A_k$.

$(2) \implies (1)$. Let I be an ideal of R. Then C_I, the intuitionistic characteristic function of I, is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy ideal of R. Hence,$C_I \cdot_k C_I = (C_I \cdot C_I)_k = (C_{I^2})_k = (C_I)_k$. It follows that $I^2 = I$.

$(1) \implies (3)$. Let A and B be $(\in, \in \vee q_k)^*$ -intuitionistic fuzzy ideals of R.

By Theorem 14 $A \cdot_k B \subseteq A \cap_k B$. Again since R is fully idempotent, $(x) = (x)^2$, for any $x \in R$. Hence, as argued in the first part of the proof of this theorem, we have

$$\begin{aligned}(\mu_A \wedge_k \mu_B)(x) &= (\mu_A)(x) \wedge (\mu_B)(x) \wedge \frac{1-k}{2} \\ &\leq \bigvee_{x=\sum_{1=1}^{p} a_i b_i} \left[\bigwedge_{1 \leq i \leq r} [\mu_A(a_i) \wedge \mu_B(b_i)] \right] \wedge \frac{1-k}{2} = (\mu_A \cdot_k \mu_B)(x)\end{aligned}$$

and

$$\begin{aligned}(\lambda_A \vee_k \lambda_B)(x) &= \lambda_A(x) \vee \lambda_B(x) \vee \frac{1-k}{2} \\ &\geq \bigwedge_{x=\sum_{i=1}^{p} a_i b_i} \left[\bigvee_{1 \leq i \leq r} [\lambda_A(a_i) \vee \lambda_B(b_i)] \right] \vee \frac{1-k}{2} = (\lambda_A \cdot_k \lambda_B)(x).\end{aligned}$$

Thus $A \cdot_k B = A \cap_k B$.

$(3) \implies (1)$. Let A and B be any pair of $(\in, \in \vee q_k)^*$-intuitionistic fuzzy ideals of R. We have $A \cdot_k B = A \cap_k B$. Take $A = B$. Thus $A \cdot_k A = A \cap_k A = A_k$, where A is any $(\in, \in \vee q_k)^*$-intuitionistic fuzzy ideal of R. Hence, $(3) \implies (2)$. Since we already proved that (1) and (2) are equivalent,hence $(3) \implies (1)$ and so $(1) \Leftrightarrow (3)$. This establishes $(1) \Leftrightarrow (2) \Leftrightarrow (3)$. Finally, If A is commutative then it is easy to verify that $(1) \Leftrightarrow (4)$.

Theorem 17. *For a hemiring R with 1, the following conditions are equivalent.*

(1) *R is right weakly regular hemiring.*
(2) *All $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right ideals of R are idempotent.*
(3) *$A \cdot_k B = A \cap_k B$ for $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right ideal A and all $(\in, \in \vee q_k)^*$-intuitionistic fuzzy two-sided ideals B of R.*

Proof. $(1) \implies (2)$ Let A be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right ideal of R. Then we have $A \cdot_k A \subseteq A_k$.

For the reverse inclusion, let $x \in R$. Since R is right weakly regular, so there exist $a_i, b_i \in R$ such that $x = \sum_{i=i}^{q} x a_i x b_i$. Now we have

$$\begin{aligned}\left(\mu_A \wedge \frac{1-k}{2}\right)(x) &= \mu_A(x) \wedge \mu_A(x) \wedge \frac{1-k}{2} \\ &= \left[\mu_A(x) \wedge \frac{1-k}{2}\right] \wedge \left[\mu_A(x) \wedge \frac{1-k}{2}\right] \wedge \frac{1-k}{2} \\ &\leq \mu_A(xa_i) \wedge \mu_A(xb_i) \wedge \frac{1-k}{2}, \ (1 \leq i \leq q).\end{aligned}$$

Therefore,

$$\left(\mu_A \wedge \frac{1-k}{2}\right)(x) \leq \bigwedge_{1\leq i\leq q} [\mu_A(xa_i) \wedge \mu_A(xb_i)] \wedge \frac{1-k}{2}$$
$$\leq \bigvee_{x=\sum_{i=1}^{q} xa_i xb_i} \left[\bigwedge_{1\leq i\leq q} [\mu_A(xa_i) \wedge \mu_A(xb_i)]\right] \wedge \frac{1-k}{2}$$
$$\leq \bigvee_{x=\sum_{j=1}^{r} a_j b_j} \left[\bigwedge_{1\leq j\leq r} [\mu_A(a_j) \wedge \mu_A(b_j)]\right] \wedge \frac{1-k}{2} = (\mu_A \cdot_k \mu_A)(x).$$

and

$$\left(\lambda_A \vee \frac{1-k}{2}\right)(x) = \lambda_A(x) \vee \lambda_A(x) \vee \frac{1-k}{2}$$
$$= \left[\lambda_A(x) \vee \frac{1-k}{2}\right] \vee \left[\lambda_A(x) \vee \frac{1-k}{2}\right] \vee \frac{1-k}{2}$$
$$\geq \lambda_A(xa_i) \vee \lambda_A(xb_i) \vee \frac{1-k}{2},\ (1 \leq i \leq q).$$

Therefore,

$$\left(\lambda_A \vee \frac{1-k}{2}\right)(x) \geq \bigvee_{1\leq i\leq q} [\lambda_A(xa_i) \vee \lambda_A(xb_i)] \vee \frac{1-k}{2}$$
$$\geq \bigwedge_{x=\sum_{i=1}^{q} xa_i xb_i} \left[\bigvee_{1\leq i\leq q} [\lambda_A(xa_i) \vee \lambda_A(xb_i)]\right] \vee \frac{1-k}{2}$$
$$\geq \bigwedge_{x=\sum_{j=1}^{r} a_j b_j} \left[\bigvee_{1\leq j\leq r} [\lambda_A(a_j) \vee \lambda_A(b_j)]\right] \vee \frac{1-k}{2} = (\lambda_A \cdot_k \lambda_A)(x).$$

Thus $A \cdot_k A = A_k$

(2) $\implies$ (3) Let A and B be $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right ideal and $(\in, \in \vee q_k)^*$-intuitionistic fuzzy two-sided ideal of R respectively. Then $A \cap_k B$ is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right ideal of R. By Theorem 14 $A \cdot_k B \subseteq A \cap_k B$. By hypothesis,

$$(A \cap_k B) = (A \cap_k B) \cdot_k (A \cap_k B) \subseteq A \cdot_k B$$

Hence $A \cdot_k B = A \cap_k B$.

(3) $\implies$ (1) Let B be a right ideal of R and A be two sided-ideal of R. Then the intuitionistic characteristic function C_A and C_B are $(\in, \in \vee q_k)^*$-intuitionistic fuzzy two-sided ideal and $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right ideal of R, respectively. Hence by hypothesis

$$C_B \cdot_k C_A = C_B \cap_k C_A \implies (C_{B\cdot A})_k = (C_{A\cap B})_k \implies B \cdot A = B \cap A.$$

Thus by Theorem 2, R is right weakly regular hemiring.

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