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Applications of double-framed soft ideals in BE-algebras

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Abstract: In this study, the notion of a double-framed soft ideal (DFS-ideal) in BE-algebras is introduced and discussed related properties by means of R-inclusive set and S-exclusive set. We presented DFS-int-uni ideal of two DFS-ideals in BE-algebras. We characterized the properties of the image and pre-image of DFS-ideals by homomorphism of BE-algebras.

Keywords: BE-Algebra, soft set, DFS-set, DFS-ideals, R-inclusive set, S-exclusive set homomarphism.

1 Introduction

Molodtsov [1], popularized the soft sets as a new soft tool in mathematics for dealing with uncertainties. Subsequently, this theory has been applied in many research areas such as data analysis, approximate reasoning and decision-making. Maji et al. [2], proposed some useful results in soft sets theory. In 2009 Ali et al. [6], presented some new operations in soft set theory. Nowadays, research in this area is advancing rapidly with remarkable applications. Soft sets have been applied in various algebraic structures, such as in group theory, rings theory and semi-rings theory etc. [3,4,5,17,18].

In 2007 Kim and Kim [7], presented concept of BE-algebras as a generalization of a dual BCK-algebra. They examined some properties of BE-algebras by utilizing upper sets in BE-algebras. In [8], Rezaei and Saeid investigated commutative ideals of BE–algebras and see [9]. Ahn and So [10], characterized BE-algebras by using ideals. Ahn, Kim, So [11], presented the notion of fuzzy BE-algebras, and Jun, Lee, Song [12], investigated the concept of fuzzy ideals in BE-algebras. Recently, Abdullah et al. [16] presented N-structures in implicative filters of BE-algebras. Applications of soft sets in BE-algebras, have been introduced by Jun and Ahn [13]. A. R. Hadipour [14] introduced double-framed soft BF-algebras and discussed related results. Moreover, see [15].

In this study, notion of a DFS-ideal in BE-algebras is introduced and discussed related properties by means of R-inclusive set, and S-exclusive set. We introduced DFS-int-uni ideal of two DFS-ideals in BE-algebras. We characterized the properties of the image and pre-image of DFS-ideals by homomorphism of BE-algebras.

2 Preliminaries

Definition 1. [7] *An algebra* $(X,\star,1)$ *of type* $(2,0)$ *is called a* BE*-algebra if the following axioms holds:*

(i) $a\star a=1$,

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(ii) $a \star 1 = 1$,

(iii) $1 \star a = a$,

(iv) $a_1 \star (a_2 \star a_3) = a_2 \star (a_1 \star a_3)$, *for all* $a_1, a_2, a_3 \in X$.

Example 1. Let $X = \{1, p, q, r, s\}$ be a set with table:

$\star$	1	p	q	r	s
1	1	p	q	r	s
p	1	1	p	s	s
q	1	p	1	s	s
r	1	1	p	1	p
s	1	1	1	1	1

Clearly, $(X, \star, 1)$ is a BE-algebra.

We consider the relation, $\preceq$ on $(X, \star, 1)$ by $a \preceq b$ if and only if $a \star b = 1$. In rest of paper, X is BE-algebra unlesselse particularized.

Definition 2. *A subset $L \neq \emptyset$ of X is called subalgebra of a BE-algebra X if it is closed under the operation "$\star$ ". Since $a \star a = 1$ for all $a \in X$, then clearly $1 \in X$.*

Proposition 1. *If $(X, \star, 1)$ is a BE-algebra, then $a_1 \star (a_2 \star a_1) = 1$ for any $a_1, a_2 \in X$.*

Definition 3. [10] *A BE-algebra $(X, \star, 1)$ is said to be transitive if it holds:*

$$(a_2 \star a_3) \preceq (a_1 \star a_2) \star (a_1 \star a_3), \ \forall a_1, a_2, a_3 \in X.$$

Definition 4. [7] *A BE-algebra $(X, \star, 1)$ with condition $a_1 \star (a_2 \star a_3) = (a_1 \star a_2) \star (a_1 \star a_3)$ $\forall a_1, a_2, a_3 \in X$ is called self distributive. If X is self distributive, then it is transitive. On the other hand, if X is transitive, then it is not self distributive in general.*

Definition 5. [10] *A subset $L \neq \emptyset$ of a BE-algebra X is called an ideal of X if it holds:*

(i) $y \star a \in L$ $\forall y \in X, a \in L$,

(ii) $((a_1 \star (a_2 \star y)) \star y) \in L$ $\forall y \in X, a_1, a_2 \in L$.

Lemma 1. [12] *A subset $L \neq \emptyset$ of a BE-algebra X is an ideal of X if and only if it holds*

(i) $1 \in L$,

(ii) $(\forall b_1, b_2 \in \Lambda, a \in L)$, $(b_1 \star (a \star b_2)) \in L \Longrightarrow (b_1 \star b_2) \in L$.

Let U be an universe, and E be a parameters set. Let $P(U)$ be a set of power set of U and L be non-empty subsets of E. Then we define following definitions.

Definition 6. [1] *A soft set (S-set) $\widetilde{\gamma}$ over U is defined as $\widetilde{\gamma} : E \to P(U)$ such that $\widetilde{\gamma}(a) = \emptyset$ if $a \notin L$. It can be represented by $\widetilde{\gamma} = \{(a, \widetilde{\gamma}(a)) \mid a \in E, \widetilde{\gamma}(a) \in P(U)\}$.*

Moreover, $\widetilde{\gamma}$ is also called an approximate function.

Definition 7. *Let $\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma} \in \widetilde{S(U)}$*

(i) *A soft set is called null soft set if* $\widetilde{\gamma}(a) = \emptyset$, *for* $a \in E$. *It denoted by* $\widetilde{\Phi}_L$.
(ii) *A soft set is called whole soft set if* $\widetilde{\gamma}(a) = U$, *for* $a \in E$. *It denoted by* $\widetilde{\mu}_L$.
(iii) *Soft set* $\widetilde{\gamma}_1$ *is called subset of* $\widetilde{\gamma}_2$, *denoted by* $\widetilde{\gamma}_1 \sqsubseteq \widetilde{\gamma}_2$ *and defined by* $\widetilde{\gamma}_1(a) \subseteq \widetilde{\gamma}_2(a)$ *for all* $a \in E$.
(iv) *Intersection of* $\widetilde{\gamma}_1$ *and* $\widetilde{\gamma}_2$ *denoted by* $\widetilde{\gamma}_1 \sqcap \widetilde{\gamma}_2$ *and defined by* $(\widetilde{\gamma}_1 \sqcap \widetilde{\gamma}_2)(a) = \widetilde{\gamma}_1(a) \cap \widetilde{\gamma}_2(a)$ *for all* $a \in E$.
(v) *Union of* $\widetilde{\gamma}_1$ *and* $\widetilde{\gamma}_2$ *denoted by* $\widetilde{\gamma}_1 \sqcup \widetilde{\gamma}_2$ *and defined by* $(\widetilde{\gamma}_1 \sqcup \widetilde{\gamma}_2)(a) = \widetilde{\gamma}_1(a) \cup \widetilde{\gamma}_2(a)$ *for all* $a \in E$.

Definition 8. *Let* $\widetilde{\gamma}$ *be a S-set in BE-algebra* X *over* U. *Then upper* $\widetilde{\alpha}$*-inculsion of* $\widetilde{\gamma}$ *is denoted and defined as*

$$\widetilde{\gamma}_{\widetilde{\alpha}}^{\supseteq} = \{a \in X \mid \widetilde{\gamma}(a) \supseteq \widetilde{\alpha} \text{ for } \widetilde{\alpha} \sqsubseteq U\}.$$

Definition 9. [15] *A double framed soft pair* $(\langle\widetilde{\gamma},\widetilde{\delta}\rangle, X)$ *is called double framed soft set (breifly DFS-set) of BE-algebra* X *over* U, *where* $\widetilde{\gamma} : X \longrightarrow P(U)$ *and* $\widetilde{\delta} : X \longrightarrow P(U)$.

For the simplicity, we shall use the symbol $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ *for DFS-set* $(\langle\widetilde{\gamma},\widetilde{\delta}\rangle, X)$.

Definition 10. *Let* $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ *and* $(\widetilde{f}_X, \widetilde{g}_X)$ *two DFS-set of* X *over* U. *The DFS-int-uni set of* $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ *and* $(\widetilde{f}_X, \widetilde{g}_X)$ *over* U *is defined to be DFS-set* $(\widetilde{\gamma}_X \sqcap \widetilde{f}_X, \widetilde{\delta}_X \sqcup \widetilde{g}_X)$ *where* $(\widetilde{\gamma}_X \sqcap \widetilde{f}_X)(a) = \widetilde{\gamma}_X(a) \cap \widetilde{f}_X(a)$ *and* $(\widetilde{\gamma}_X \sqcup \widetilde{f}_X)(a) = \widetilde{\gamma}_X(x) \cup \widetilde{f}_X(a)$.

3 *DFS*-ideals in *BE*-algebras

3.1 Definition

A double-framed soft set $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ of a *BE*-algebra X over U is called a double-framed soft ideal (i.e, breifly *DFS*-ideal) in X, if it holds;
(DF_1) $\widetilde{\gamma}_X(x \star y) \supseteq \widetilde{\gamma}_X(y), \widetilde{\delta}_X(x \star y) \subseteq \widetilde{\delta}_X(y)$,
(DF_2) $\widetilde{\gamma}_X(x \star (y \star z) \star z) \supseteq \widetilde{\gamma}_X(x) \cap \widetilde{\gamma}_X(y), \widetilde{\delta}_X(x \star (y \star z) \star z) \subseteq \widetilde{\delta}_X(x) \cup \widetilde{\delta}_X(y)$.

3.2 Example

Consider *BE*-algebra $X = \{1, r_1, r_2, r_3, r_4, 0\}$ defined as follow:

$\star$	1	r_1	r_2	r_3	r_4	0
1	1	r_1	r_2	r_3	r_4	0
r_1	1	1	r_1	r_3	r_3	r_4
r_2	1	1	1	r_3	r_3	r_3
r_3	1	r_1	r_2	1	r_1	r_2
r_4	1	1	r_1	1	1	r_1
0	1	1	1	1	1	1.

We define *DFS*-set $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ in X over $U = \mathbb{Z}_4$ as:

x	1	r_1	r_2	r_3	r_4	0
$\widetilde{\gamma}_X(x)$	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$

x	1	r_1	r_2	r_3	r_4	0
$\widetilde{\delta}_X(x)$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$.

Clearly, $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ is a DFS-ideal in X over $U = \mathbb{Z}_4$.

Lemma 2. *Every DFS-ideal of X holds the following inequality: $\widetilde{\gamma}_X(1) \supseteq \widetilde{\gamma}_X(a)$ and $\widetilde{\delta}_X(1) \subseteq \widetilde{\delta}_X(a)$ $\forall a \in X$.*

Proof. Assume $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ is a DFS-ideal of X over U. Thus $\widetilde{\gamma}_X(a\star b) \supseteq \widetilde{\gamma}_X(b)$ and $\widetilde{\delta}_X(a\star b) \subseteq \widetilde{\delta}_X(b)$ for $a,b \in X$. Since $a \star a = 1$, then $\widetilde{\gamma}_X(1) = \widetilde{\gamma}_X(a\star a) \supseteq \widetilde{\gamma}_X(a)$ and $\widetilde{\delta}_X(1) = \widetilde{\delta}_X(a\star a) \subseteq \widetilde{\delta}_X(a)$.

Theorem 1. *The DFS-int-uni set of two DFS-ideals $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ and $(\widetilde{f}_X, \widetilde{g}_X)$ over U, is a DFS-ideal over U.*

Proof. Let $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ and $(\widetilde{f}_X, \widetilde{g}_X)$ be two ideals over U. For any $a_1, a_2 \in X$, we have $(\widetilde{\gamma}_X \sqcap \widetilde{f}_X)(a_1 \star a_2) = \widetilde{\gamma}_X(a_1\star a_2) \cap \widetilde{f}_X(a_1 \star a_2) \supseteq \widetilde{\gamma}_X(a_2) \cap \widetilde{f}_X(a_2) = (\widetilde{\gamma}_X \sqcap \widetilde{f}_X)(a_2)$, and $(\widetilde{\delta}_X \sqcup \widetilde{g}_X)(a_1\star a_2) = \widetilde{\delta}_X(a_1\star a_2) \cup \widetilde{g}_X(a_1\star a_2) \subseteq \widetilde{\delta}_X(a_2) \cup \widetilde{g}_X(a_2) = (\widetilde{\delta}_X \sqcup \widetilde{g}_X)(a_2)$. Now let $b_1, b_2 \in X$, we have

$$(\widetilde{\gamma}_X \sqcap \widetilde{f}_\Lambda)(b_1\star((b_2\star z)\star z)) = \widetilde{\gamma}_X(b_1\star((b_2\star z)\star z)) \cap \widetilde{f}_X(b_1\star((b_2\star z)\star z))$$
$$\supseteq (\widetilde{\gamma}_X(b_1) \cap \widetilde{\gamma}_X(b_2)) \cap \left(\widetilde{f}_X(b_1) \cap \widetilde{f}_X(b_2)\right) = (\widetilde{\gamma}_X \sqcap \widetilde{f}_X)(b_1) \cap (\widetilde{\gamma}_X \sqcap \widetilde{f}_X)(b_2), \text{ and}$$
$$(\widetilde{\delta}_X \sqcup \widetilde{g}_X)(b_1\star((b_2\star z)\star z)) = \widetilde{\delta}_X(b_1\star((b_2\star z)\star z)) \cup \widetilde{g}_X(b_1\star((b_2\star z)\star z))$$
$$\supseteq (\widetilde{\delta}_X(b_1) \cup \widetilde{\delta}_X(b_2)) \cup (\widetilde{g}_X(b_1) \cup \widetilde{g}_X(b_2)) = (\widetilde{\delta}_X \sqcup \widetilde{g}_X)(b_1) \cup (\widetilde{\delta}_X \sqcup \widetilde{g}_X)(b_2).$$

Hence DFS-int-uni set is a DFS-ideal over U.

Definition 11. *Let $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ be a DFS-set of X over U and R, $S \subseteq U$, the R- inclusive and S- exclusive sets of $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$, denoted by $(\widetilde{\gamma}^{\subseteq})_R$ and $(\widetilde{\delta}^{\supseteq})_S$ respectively and defined as follows:*

$$(\widetilde{\gamma}^{\subseteq})_R = \{x \in X \mid R \subseteq \widetilde{\gamma}_X(x)\} \text{ and } (\widetilde{\delta}^{\supseteq})_S = \left\{x \in X \mid S \supseteq \widetilde{\delta}_X(x)\right\} \text{ respectively.}$$

The set $DF_X(\widetilde{\gamma}^{\subseteq}, \widetilde{\delta}^{\supseteq})_{(R,S)} = \left\{x \in X \mid R \subseteq \widetilde{\gamma}_\Lambda(x), S \supseteq \widetilde{\delta}_\Lambda(x)\right\}$ is called a double-framed including set of $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$. Clearly, $DF_X(\widetilde{\gamma}^{\subseteq}, \widetilde{\delta}^{\supseteq})_{(R,S)} = (\widetilde{\gamma}^{\subseteq})_R \cap (\widetilde{\delta}^{\supseteq})_S$.

Theorem 2. *Let $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ be a DFS-set of X over U. Then $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ is a DFS-ideal of X over U if and only if R- inclusive set of $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$, $(\widetilde{\gamma}^{\subseteq})_R \neq \emptyset$ with $R \in I_m(\widetilde{\gamma}_X)$ and S- exclusive set of $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$, $(\widetilde{\delta}^{\supseteq})_S \neq \emptyset$ with $S \in I_m(\widetilde{\delta}_X)$ are ideals of X.*

Proof. Assume that $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ is a DFS-ideal of X over U, $(\widetilde{\gamma}^{\subseteq})_R \neq \emptyset$ with $R \in I_m(\widetilde{\gamma}_X)$ and $(\widetilde{\delta}^{\supseteq})_S \neq \emptyset$ with $S \in I_m(\widetilde{\delta}_X)$. Let $a,b \in X$ such that $b \in (\widetilde{\gamma}^{\subseteq})_R$ and $b \in (\widetilde{\delta}^{\supseteq})_S$. Then $\widetilde{\gamma}_X(b) \supseteq R$ and $\widetilde{\delta}_X(b) \supseteq S$. Since $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ is a DFS-ideal of X over U, then $\widetilde{\gamma}_X(a\star b) \supseteq \widetilde{\gamma}_X(b) \supseteq R$ and $\widetilde{\delta}_X(a\star b) \subseteq \widetilde{\delta}_X(b) \subseteq S$. Hence $a\star b \in (\widetilde{\gamma}^{\subseteq})_R$ and $a \star b \in (\widetilde{\delta}^{\supseteq})_S$.

Now let $x,y,z \in X$ such that $x,y \in (\widetilde{\gamma}^{\subseteq})_R$ and $x,y \in (\widetilde{\delta}^{\supseteq})_S$. Then $\widetilde{\gamma}_X(x) \supseteq R, \widetilde{\gamma}_X(y) \supseteq R$ and $\widetilde{\delta}_X(x) \subseteq S, \widetilde{\delta}_X(y) \subseteq S$. Since $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ is a DFS-ideal of X over U, then $\widetilde{\gamma}_X(x\star(y\star z)\star z) \supseteq \widetilde{\gamma}_X(x) \cap \widetilde{\gamma}_X(y) \supseteq R$ and $\widetilde{\delta}_X(x\star(y\star z)\star z) \subseteq \widetilde{\delta}_X(x) \cup \widetilde{\delta}_X(y) \subseteq S$, so that $x\star(y\star z)\star z \in (\widetilde{\gamma}^{\subseteq})_R$ and $x\star(y\star z)\star z \in (\widetilde{\delta}^{\supseteq})_S$. Thus R- inclusive set $(\widetilde{\gamma}^{\subseteq})_R$ and S- exclusive set $(\widetilde{\delta}^{\supseteq})_S$ of $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ are ideals of X.

Conversely, assume that R- inclusive set $(\widetilde{\gamma}^{\subseteq})_R$ and S- exclusive set $(\widetilde{\delta}^{\supseteq})_S$ of $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ are ideals of X. If $\widetilde{\gamma}_X(a\star b) \subset \widetilde{\gamma}_X(b)$ and $\widetilde{\delta}_X(a\star b) \supset \widetilde{\delta}_X(b)$ for $a,b \in X$, then $\widetilde{\gamma}_X(a\star b) \subset R_O \subset \widetilde{\gamma}_\Lambda(b)$ and $\widetilde{\delta}_X(a\star b) \supset S_O \supset \widetilde{\delta}_X(b)$. Thus $a\star b \notin (\widetilde{\gamma}^{\subseteq})_{R_o}, b \in (\widetilde{\gamma}^{\subseteq})_{R_o}$ and $a \star b \notin (\widetilde{\delta}^{\supseteq})_{S_o}, b \in (\widetilde{\delta}^{\supseteq})_{S_o}$, which is a contradiction. Let $\widetilde{\gamma}_X(x\star(y\star z)\star z) \subset \widetilde{\gamma}_X(x) \cap \widetilde{\gamma}_X(y)$ and $\widetilde{\delta}_X(x\star(y\star z)\star z) \supset \widetilde{\delta}_X(x) \cup \widetilde{\delta}_X(y)$ for $x,y,z \in X$, $\widetilde{\gamma}_X(x\star(y\star z)\star z) \subset R'_O \subset \widetilde{\gamma}_X(x) \cap \widetilde{\gamma}_X(y)$ and $\widetilde{\delta}_X(x\star(y\star z)\star z) \supset S'_O \supset \widetilde{\delta}_X(x) \cup \widetilde{\delta}_X(y)$. Thus $x \star (y\star z) \star z \notin (\widetilde{\gamma}^{\subseteq})_{R'_o}$, $x \in (\widetilde{\gamma}^{\subseteq})_{R'_o}, y \in (\widetilde{\gamma}^{\subseteq})_{R'_o}$ and $x\star(y\star z)\star z \notin (\widetilde{\delta}^{\supseteq})_{S'_o}, x \in (\widetilde{\delta}^{\supseteq})_{S'_o}, y \in (\widetilde{\delta}^{\supseteq})_{S'_o}$ which is a contradiction. Therefore, $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ is a DFS-ideal of X over U.

Corollary 1. *If $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ be a DFS-ideal of X over U, then the double-framed including set of $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ is an ideal X over U.*

Definition 12. *For DFS-set $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ of X over U, let $(\widetilde{\gamma}^*_X, \widetilde{\delta}^*_X)$ be a DFS-ideal of X over U defined by*

$\widetilde{\gamma}^*_X : X \longrightarrow P(U)$, $\widetilde{\delta}^*_X : X \longrightarrow P(U)$ *such that*

$$\widetilde{\gamma}^*_X(x) = \begin{cases} \widetilde{\gamma}_X(x) & \text{if } x \in (\widetilde{\gamma}^{\subseteq})_R, \\ \widetilde{\varsigma}_X & \text{if Otherwise,} \end{cases} \quad \widetilde{\delta}^*_X(x) = \begin{cases} \widetilde{\delta}_X(x) & \text{if } x \in (\widetilde{\delta}^{\supseteq})_S, \\ \widetilde{\tau}_X & \text{if Otherwise,} \end{cases}$$

where $\widetilde{\gamma}_X, \widetilde{\delta}_X, \widetilde{\varsigma}$ and $\widetilde{\tau}$ are subset of U, with $\widetilde{\varsigma} \subsetneq \widetilde{\gamma}_X$ and $\widetilde{\tau} \supsetneq \widetilde{\delta}_X$.

Theorem 3. *If $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ be a DFS-ideal of X over U, then so is $(\widetilde{\gamma}^*_X, \widetilde{\delta}^*_X)$.*

Proof. Assume $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ be a DFS-ideal of X over U. Then $(\widetilde{\gamma}^{\subseteq})_R$ and $(\widetilde{\delta}^{\supseteq})_S$ are ideals of X for all R and S are subset of U with $R \in I_m(\widetilde{\gamma}_X)$ and $S \in I_m(\widetilde{\delta}_X)$. Let $a, x \in X$. If $a \in (\widetilde{\gamma}^{\subseteq})_R$, and $a \in (\widetilde{\delta}^{\supseteq})_S$ then $x \star a \in (\widetilde{\gamma}^{\subseteq})_R$, and $x \star a \in (\widetilde{\delta}^{\supseteq})_S$. Then $\widetilde{\gamma}^*_X(x \star a) = \widetilde{\gamma}_X(x \star a) \supseteq \widetilde{\gamma}_X(a) = \widetilde{\gamma}^*_X(a)$ and $\widetilde{\delta}^*_X(x \star a) = \widetilde{\delta}_X(x \star a) \subseteq \widetilde{\delta}_X(a) = \widetilde{\delta}^*_X(a)$.

If $a \notin (\widetilde{\gamma}^{\subseteq})_R$ and $a \notin (\widetilde{\delta}^{\supseteq})_S$ then $\widetilde{\gamma}^*_X(a) = \widetilde{\varsigma}_X$ and $\widetilde{\delta}^*_X(a) = \widetilde{\tau}_X$. Then $\widetilde{\gamma}^*_X(x \star a) \supseteq \widetilde{\varsigma}_X = \widetilde{\gamma}^*_X(a)$ and $\widetilde{\delta}^*_X(x \star a) \subseteq \widetilde{\tau}_X = \widetilde{\delta}^*_X(a)$.

Now let $a, b, x \in X$. If $a \in (\widetilde{\gamma}^{\subseteq})_R$, $b \in (\widetilde{\gamma}^{\subseteq})_R$ and $a \in (\widetilde{\delta}^{\supseteq})_S$, $b \in (\widetilde{\delta}^{\supseteq})_S$ then $(a \star ((b \star x) \star x)) \in (\widetilde{\gamma}^{\subseteq})_R$, and $(a \star ((b \star x) \star x)) \in (\widetilde{\delta}^{\supseteq})_S$. Then $\widetilde{\gamma}^*_X(a \star ((b \star x) \star x)) = \widetilde{\gamma}_X((a \star ((b \star x) \star x))) \supseteq \widetilde{\gamma}_X(a) \cap \widetilde{\gamma}_X(b) = \widetilde{\gamma}^*_X(a) \cap \widetilde{\gamma}^*_X(b)$ and $\widetilde{\delta}^*_X(a \star ((b \star x) \star x)) = \widetilde{\delta}_X(a \star ((b \star x) \star x)) \subseteq \widetilde{\delta}_X(a) \cup \widetilde{\delta}_X(b) = \widetilde{\delta}^*_X(a) \cup \widetilde{\delta}^*_X(b)$.

If $a \notin (\widetilde{\gamma}^{\subseteq})_R$ or $a \notin (\widetilde{\gamma}^{\subseteq})_R$ and $a \notin (\widetilde{\delta}^{\supseteq})_S$ or $b \notin (\widetilde{\delta}^{\supseteq})_S$ then $\widetilde{\gamma}^*_X(a) = \widetilde{\varsigma}_X$ or $\widetilde{\gamma}^*_X(b) = \widetilde{\varsigma}_X$ and $\widetilde{\delta}^*_X(a) = \widetilde{\tau}_X$ or $\widetilde{\delta}^*_X(b) = \widetilde{\tau}_X$. Then $\widetilde{\gamma}^*_X(a \star ((b \star x) \star x)) \supseteq \widetilde{\varsigma}_X = \widetilde{\gamma}^*_X(a) \cap \widetilde{\gamma}^*_X(b)$ and $\widetilde{\delta}^*_X(a \star ((b \star x) \star x)) \subseteq \widetilde{\tau}_X = \widetilde{\delta}^*_X(a) \cup \widetilde{\delta}^*_X(b)$.

Hence $(\widetilde{\gamma}^*_X, \widetilde{\delta}^*_X)$ is a DFS-ideal of X over U.

Now we consider an example to show that the converse of Theorem 3, is not true in general.

Example 2. Consider BE-algebra X presented in Example 3.2. Also consider DFS-set $(\widetilde{\gamma}_X, \widetilde{\delta}_X)$ of X described as follows:

$$\widetilde{\gamma}_X(x) = \begin{cases} \{0,1,2,3\}, & \text{if } x \in \{1\}, \\ \{0,1,2\}, & \text{if } x \in \{r_1, r_2\}, \\ \{1,2\}, & \text{if } x = r_3, \\ \{1\}, & \text{if } x \in \{r_4, 0\}, \end{cases}$$

$$\widetilde{\delta}_X(x) = \begin{cases} \{0\}, & \text{if } x \in \{1\}, \\ \{0,1\}, & \text{if } x \in \{r_1, r_2\}. \\ \{0,1,2\}, & \text{if } x = r_3, \\ \{0,1,2,3\}, & \text{if } x \in \{r_4, 0\}, \end{cases}$$

Note that

$$(\widetilde{\gamma}^{\subseteq})_R = \{1, r_1, r_2\} = (\widetilde{\delta}^{\supseteq})_S 4$$

with $R=\{0,1,2\}$ and $S=\{0,1\}$. Let $(\widetilde{\gamma}_X^*,\widetilde{\delta}_X^*)$ be a DFS-set over U defined by

$$\widetilde{\gamma}_X^*(x)=\begin{cases}\widetilde{\gamma}_X, & \text{if } x\in(\widetilde{\gamma}^{\subseteq})_R,\\ \emptyset, & \text{if Otherwise,}\end{cases}$$

$$\widetilde{\delta}_X^*(x)=\begin{cases}\widetilde{\delta}_X, & \text{if, } x\in(\widetilde{\delta}^{\supseteq})_S,\\ U, & \text{if, Otherwise,}\end{cases}.$$

It is routine to verify that $(\widetilde{\gamma}_X^*,\widetilde{\delta}_X^*)$ is a DFS-ideals over U. But $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ is not DFS-ideal of X over U.

Proposition 2. *If $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ is a DFS-ideal of X over U, then $\widetilde{\gamma}_X((x\star y)\star y)\supseteq\widetilde{\gamma}_X(x)$, $\widetilde{\delta}_X((x\star y)\star y)\subseteq\widetilde{\delta}_X(x)$.*

Proof. Let $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ is a DFS-ideal of X over U. Since $a=1\star a$ for all $a\in X$ and by Lemma 2, $\widetilde{\gamma}_X(1)\supseteq\widetilde{\gamma}_X(x)$ and $\widetilde{\delta}_X(1)\subseteq\widetilde{\delta}_X(x)$, then $\widetilde{\gamma}_X((x\star y)\star y)=\widetilde{\gamma}_X((x\star(1\star y))\star y)\supseteq\widetilde{\gamma}_X(x)\cap\widetilde{\gamma}_X(1)=\widetilde{\gamma}_X(x)$ and $\widetilde{\delta}_X((x\star y)\star y)=\widetilde{\delta}_X((x\star(1\star y))\star y)\subseteq\widetilde{\delta}_X(x)\cup\widetilde{\delta}_X(1)=\widetilde{\delta}_X(x)$ for $x,y\in X$.

Corollary 2. *Every DFS-ideal $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ of X is order preserving, that is, $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ holds: $a\preceq b\Longrightarrow\widetilde{\gamma}_X(a)\subseteq\widetilde{\gamma}_X(b)$ and $\widetilde{\delta}_X(a)\supseteq\widetilde{\delta}_X(b)$ $\forall a,b\in X$.*

Proof. Let $a\preceq b$ for $a,b\in X$. Thus $a\star b=1$, and by 2, $\widetilde{\gamma}_X(b)=\widetilde{\gamma}_X(1\star b)=\widetilde{\gamma}_X((a\star b)\star b)\supseteq\widetilde{\gamma}_X(a)$ and $\widetilde{\delta}_X(b)=\widetilde{\delta}_X(1\star b)=\widetilde{\delta}_X((a\star b)\star b)\subseteq\widetilde{\delta}_X(a)$.

Proposition 3. *Let $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ be a DFS-set of X which satisfies:*

(i) $\widetilde{\gamma}_X(1)\supseteq\widetilde{\gamma}_X(a)$ *and* $\widetilde{\delta}_X(1)\subseteq\widetilde{\delta}_X(a)$ $\forall a\in X$.
(ii) $\widetilde{\gamma}_X(a\star z)\supseteq\widetilde{\gamma}_X(a\star(b\star z))\cap\widetilde{\gamma}_X(b)$, *and* $\widetilde{\delta}_X(a\star z)\subseteq\widetilde{\delta}_X(a\star(b\star z))\cup\widetilde{\delta}_X(b)$.

Then $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ is ordered preserving.

Proof. Let $a,b\in\Lambda$ be such that $a\preceq b$. Then $a\star b=1$, we have $\widetilde{\gamma}_\Lambda(b)=\widetilde{\gamma}_\Lambda(1\star b)\supseteq\widetilde{\gamma}_\Lambda(1\star(a\star b))\cap\widetilde{\gamma}_\Lambda(a)=\widetilde{\gamma}_\Lambda(a)$, and $\widetilde{\delta}_\Lambda(b)=\widetilde{\delta}_\Lambda(1\star b)\subseteq\widetilde{\delta}_\Lambda(1\star(a\star b))\cup\widetilde{\delta}_\Lambda(a)=\widetilde{\delta}_\Lambda(a)$, by Definition 1 and Lemma 2. Thus $(\widetilde{\gamma}_\Lambda,\widetilde{\delta}_\Lambda)$ is ordered preserving.

Theorem 4. *Let X be a transitive BE-algebra. A DFS-set $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ is a DFS-ideal of X over U if and only if it holds following conditions:*

(i) $\widetilde{\gamma}_X(1)\supseteq\widetilde{\gamma}_X(x)$ *and* $\widetilde{\delta}_X(1)\subseteq\widetilde{\delta}_X(x)$ $\forall x\in X$.
(ii) $\widetilde{\gamma}_X(x\star z)\supseteq\widetilde{\gamma}_X(x\star(y\star z))\cap\widetilde{\gamma}_X(y)$, *and* $\widetilde{\delta}_X(x\star z)\subseteq\widetilde{\delta}_X(x\star(y\star z))\cup\widetilde{\delta}_X(y)$.

Proof. Assume $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ is a DFS-ideal of X over U. By Lemma 2, we have $\widetilde{\gamma}_X(1)\supseteq\widetilde{\gamma}_X(x)$ and $\widetilde{\delta}_X(1)\subseteq\widetilde{\delta}_X(x)$ $\forall x\in X$. Since X is transitive, then $(y\star z)\star z\preceq(x\star(y\star z))\star(x\star z)$ so that $((y\star z)\star z)\star((x\star(y\star z))\star(x\star z))=1$ for all $x,y,z\in X$. Also $a=1\star a$ for all $a\in X$, and by Proposition 2, $\widetilde{\gamma}_X((x\star y)\star y)\supseteq\widetilde{\gamma}_X(x)$, $\widetilde{\delta}_X((x\star y)\star y)\subseteq\widetilde{\delta}_X(x)$. Thus
$\widetilde{\gamma}_X(x\star z)=\widetilde{\gamma}_X(1\star(x\star z))=\widetilde{\gamma}_X((((y\star z)\star z)\star((x\star(y\star z))\star(x\star z)))\star(x\star z))\supseteq\widetilde{\gamma}_X((y\star z)\star z)\cap\widetilde{\gamma}_X(x\star(y\star z))$
$\supseteq\widetilde{\gamma}_X((x\star y)\star z)\cap\widetilde{\gamma}_X(y)$
and
$\widetilde{\delta}_X(x\star z)=\widetilde{\delta}_X(1\star(x\star z))=\widetilde{\delta}_X((((y\star z)\star z)\star((x\star(y\star z))\star(x\star z)))\star(x\star z))$
$\subseteq\widetilde{\delta}_X((y\star z)\star z)\cup\widetilde{\delta}_X(x\star(y\star z))\subseteq\widetilde{\delta}_X((x\star y)\star z)\cup\widetilde{\delta}_X(y)$

Conversely, suppose that DFS-set $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ holds (i) and (ii). Then

$\widetilde{\gamma}_X(x\star y)\supseteq\widetilde{\gamma}_X(x\star(y\star y))\cap\widetilde{\gamma}_X(y)=\widetilde{\gamma}_X(y\star 1)\cap\widetilde{\gamma}_X(y)=\widetilde{\gamma}_X(1)\cap\widetilde{\gamma}_X(y)=\widetilde{\gamma}_X(y)$
and
$\widetilde{\delta}_X(x\star y)\subseteq\widetilde{\delta}_X(x\star(y\star y))\cup\widetilde{\delta}_X(y)=\widetilde{\delta}_X(y\star 1)\cup\widetilde{\delta}_X(y)=\widetilde{\delta}_X(1)\cup\widetilde{\delta}_X(y)=\widetilde{\delta}_X(y)$, for all $x,y\in X$.
Now
$\widetilde{\gamma}_X((x\star y)\star y)\supseteq\widetilde{\gamma}_X((x\star y)\star(x\star y))\cap\widetilde{\gamma}_X(x)=\widetilde{\gamma}_X(1)\cap\widetilde{\gamma}_X(x)=\widetilde{\gamma}_X(x)$
and
$\widetilde{\delta}_X((x\star y)\star y)\subseteq\widetilde{\delta}_X((x\star y)\star(x\star y))\cup\widetilde{\delta}_X(x)=\widetilde{\delta}_X(1)\cup\widetilde{\delta}_X(x)=\widetilde{\delta}_X(x)$, for all $x,y\in X$.

Since $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ is ordered preserving by Proposition 3, therefore, $(y\star z)\star z\preceq(x\star(y\star z))\star(x\star z)$ implies $\widetilde{\gamma}_X((y\star z)\star z)\subseteq\widetilde{\gamma}_X((x\star(y\star z))\star(x\star z))$ and $\widetilde{\delta}_X((y\star z)\star z)\supseteq\widetilde{\delta}_X((x\star(y\star z))\star(x\star z))$. Thus

$\widetilde{\gamma}_X(x\star(y\star z)\star z)\supseteq\widetilde{\gamma}_X((x\star(y\star z))\star(x\star z))\cap\widetilde{\gamma}_X(x)\supseteq\widetilde{\gamma}_X((y\star z)\star z)\cap\widetilde{\gamma}_X(x)\supseteq\widetilde{\gamma}_X(y)\cap\widetilde{\gamma}_X(x)$,
and
$\widetilde{\delta}_X(x\star(y\star z)\star z)\subseteq\widetilde{\delta}_X((x\star(y\star z))\star(x\star z))\cup\widetilde{\delta}_X(x)\subseteq\widetilde{\delta}_X((y\star z)\star z)\cup\widetilde{\delta}_X(x)\subseteq\widetilde{\delta}_X(y)\cup\widetilde{\delta}_X(x)$, $\forall\, x,y,z\in X$. Therefore, $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ is a DFS-ideal of X.

Corollary 3. *Let X be a self distributive BE-algebra. A DFS-set $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ is a DFS-ideal of X over U if and only if it holds following conditions:*

(i) $\widetilde{\gamma}_X(1)\supseteq\widetilde{\gamma}_X(a)$ *and* $\widetilde{\delta}_X(1)\subseteq\widetilde{\delta}_X(a)\ \forall a\in X$.
(ii) $\widetilde{\gamma}_X(a\star z)\supseteq\widetilde{\gamma}_X(a\star(b\star z))\cap\widetilde{\gamma}_X(b)$, *and* $\widetilde{\delta}_X(a\star z)\subseteq\widetilde{\delta}_X(a\star(b\star z))\cup\widetilde{\delta}_X(b)$.

Definition 13. *For $p,q\in X$, we define DFS-set $(\widetilde{\gamma}_p^q,\widetilde{\delta}_p^q)$ in X over U by*
$$\widetilde{\gamma}_q^p(x)=\begin{cases}\sigma_1 & p\star(q\star x)=1,\\ \sigma_2 & \text{Otherwise}\end{cases},$$
$$\widetilde{\delta}_q^p(x)=\begin{cases}\sigma_1' & p\star(q\star x)=1,\\ \sigma_2' & \text{Otherwise}\end{cases}\ \text{for all } x\in X \text{ and } \sigma_1,\sigma_2,\sigma_1',\sigma_2'\subseteq U \text{ with } \sigma_1\supseteq\sigma_2,\ \sigma_2'\supseteq\sigma_1'.$$

Example 3. Consider BE-algebra $X=\{1,r_1,r_2,r_3,r_4,0\}$ defined in Example 1. Then $(\widetilde{\gamma}_1^{r_1},\widetilde{\delta}_1^{r_1})$ is a DFS-ideal of X over U. Since
$\widetilde{\gamma}_1^{r_1}((r_1\star(r_1\star r_2))\star r_2)=\widetilde{\gamma}_1^{r_1}((r_1\star r_1)\star r_2)=\widetilde{\gamma}_1^{r_1}(1\star r_2)=\widetilde{\gamma}_1^{r_1}(r_2)=\sigma_2\subseteq\sigma_1=\widetilde{\gamma}_1^{r_1}(r_1)\cap\widetilde{\gamma}_1^{r_1}(r_1)$,
and
$\widetilde{\delta}_1^{r_1}((r_1\star(r_1\star r_2))\star r_2)=\widetilde{\delta}_1^{r_1}((r_1\star r_1)\star r_2)=\widetilde{\delta}_1^{r_1}(1\star r_2)=\widetilde{\delta}_1^{r_1}(b)=\sigma_2'\supseteq\sigma_1'=\widetilde{\delta}_1^{r_1}(r_1)\cup\widetilde{\delta}_1^{r_1}(r_1)$. Therefore, $(\widetilde{\gamma}_1^{r_1},\widetilde{\delta}_1^{r_1})$ is not a DFS-ideal of X over U.

Theorem 5. *If X is self distributive, then the DFS-set $(\widetilde{\gamma}_p^q,\widetilde{\delta}_p^q)$ is a DFS-ideal of X over U for all $p,q\in X$.*

Proof. Let $p,q\in X$. For every $x,y\in X$, if $p\star(q\star y)\neq 1$, then $\widetilde{\gamma}_p^q(y)=\sigma_2\subseteq\widetilde{\gamma}_p^q(x\star y)$ and $\widetilde{\delta}_p^q(y)=\sigma_2'\supseteq\widetilde{\delta}_p^q(x\star y)$. Suppose $p\star(q\star y)=1$, then

$$p\star(q\star(x\star y))=p\star((q\star x)\star(q\star y))=(p\star(q\star x))\star(p\star(q\star y))=(p\star(q\star x))\star 1=1.$$

Thus $\widetilde{\gamma}_p^q(y)=\sigma_2=\widetilde{\gamma}_p^q(x\star y)$ and $\widetilde{\delta}_p^q(y)=\sigma_2'=\widetilde{\delta}_p^q(x\star y)$. Hence $\widetilde{\gamma}_p^q(y)\subseteq\widetilde{\gamma}_p^q(x\star y)$ and $\widetilde{\delta}_p^q(y)\supseteq\widetilde{\delta}_p^q(x\star y)$.
Now, for $x,y,z\in X$, if $p\star(q\star x)\neq 1$ and $p\star(q\star y)\neq 1$, then $\widetilde{\gamma}_p^q(x)=\sigma_2$, $\widetilde{\gamma}_p^q(y)=\sigma_2$ and $\widetilde{\delta}_p^q(x)=\sigma_2',\widetilde{\delta}_p^q(y)=\sigma_2'$. Then $\widetilde{\gamma}_p^q((x\star(y\star z))\star z)\supseteq\sigma_2=\widetilde{\gamma}_p^q(x)\cap\widetilde{\gamma}_p^q(y)$ and $\widetilde{\delta}_p^q((x\star(y\star z))\star z)\subseteq\sigma_2'=\widetilde{\delta}_p^q(x)\cup\widetilde{\delta}_a^b(y)$.
If $p\star(q\star x)=1$ and $p\star(q\star y)=1$, then $p\star(q\star((x\star(y\star z))\star z))=1$. Thus $\widetilde{\gamma}_p^q((x\star(y\star z))\star z)=\sigma_2=\widetilde{\gamma}_p^q(x)\cap\widetilde{\gamma}_a^b(y)$ and $\widetilde{\delta}_p^q((x\star(y\star z))\star z)=\sigma_2'=\widetilde{\delta}_p^q(x)\cup\widetilde{\delta}_p^q(y)$.

Hence $\widetilde{\gamma}_p^q((x\star(y\star z))\star z)\supseteq\widetilde{\gamma}_p^q(x)\cap\widetilde{\gamma}_a^b(y)$ and $\widetilde{\delta}_p^q((x\star(y\star z))\star z)\subseteq\widetilde{\delta}_p^q(x)\cup\widetilde{\delta}_p^q(y)$ for $x,y,z\in X$. Therefore, DFS-set $(\widetilde{\gamma}_p^q,\widetilde{\delta}_p^q)$ is a DFS-ideal of X over U.

Definition 14.[10] *Let $p,q\in X$. The upper set of p,q is defined as follows:*
$\overline{K}(p,q)=\{x\in X\mid p\star(q\star x)=1\}$. It is clear that, $1,p,q\in L(p,q)$ for all $p,q\in X$. Note that $\overline{K}(p,q)$ is not an ideal of X in general.

Theorem 6. *Let $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ be a DFS-set in X and $p,q,x\in X$. Then $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ is a DFS-ideals in X if and only if $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ satisfies*

(i) $p,q\in\overline{K}(p,q)\Longrightarrow\overline{K}(p,q)\subseteq(\widetilde{\gamma}^{\subseteq})_R$,
(ii) $p,q\in\overline{K}(p,q)\Longrightarrow\overline{K}(p,q)\subseteq(\widetilde{\delta}^{\supseteq})_S$,

where $R\subseteq U$ and $S\subseteq U$.

Proof. Assume that $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ is a DFS-ideals in X and $p,q\in\overline{K}(p,q)$. Then $\widetilde{\gamma}_X(p)\supseteq R,\widetilde{\delta}_X(p)\subseteq S$ and $\widetilde{\gamma}_X(q)\supseteq R,\widetilde{\delta}_X(q)\subseteq S$. For $x\in\overline{K}(p,q)$ we have $p\star(q\star x)=1$. Thus

$\widetilde{\gamma}_X(x)=\widetilde{\gamma}_X(1\star x)=\widetilde{\gamma}_X((p\star(q\star x))\star x)\supseteq\widetilde{\gamma}_X(p)\cap\widetilde{\gamma}_X(q)\supseteq R$ and $\widetilde{\delta}_X(x)=\widetilde{\delta}_X(1\star x)=\widetilde{\delta}_X((p\star(q\star x))\star x)\subseteq\widetilde{\delta}_X(p)\cup\widetilde{\delta}_X(q)\subseteq S$.

Conversely, assume that $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ holds (i) and (ii). Since $1\in\overline{K}(p,q)\subseteq(\widetilde{\gamma}^{\subseteq})_R$ and $1\in\overline{K}(p,q)\subseteq(\widetilde{\delta}^{\supseteq})_S$ for $p,q\in X$. Let $x,y,z\in X$ be such that $x\star(y\star z)\in(\widetilde{\gamma}^{\subseteq})_R$, $x\star(y\star z)\in(\widetilde{\delta}^{\supseteq})_S$ and $y\in(\widetilde{\delta}^{\supseteq})_S$, $y\in(\widetilde{\gamma}^{\subseteq})_R$. Since $(x\star(y\star z))\star(y\star(x\star z))=(x\star(y\star z))\star(x\star(y\star z))=1$ then $x\star z\in\overline{K}(x\star(y\star z),y)\subseteq(\widetilde{\gamma}^{\subseteq})_R$ and $x\star z\in\overline{K}(x\star(y\star z),y)\subseteq(\widetilde{\delta}^{\supseteq})_S$. By Lemma 1, that $(\widetilde{\gamma}^{\subseteq})_R$ and $(\widetilde{\delta}^{\supseteq})_S$ are ideals of X. Therefore, by Theorem 2, $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ is a DFS-ideals in X.

Corollary 4. *If $(\widetilde{\gamma}_X,\widetilde{\delta}_X)$ is a DFS-ideals in X, then for $R\subseteq U$ and $S\subseteq U$, we have*

(i) $(\widetilde{\gamma}^{\subseteq})_R\neq\emptyset\Longrightarrow(\widetilde{\gamma}^{\subseteq})_R=\bigcup_{p,q\in(\widetilde{\gamma}^{\subseteq})_R}\overline{K}(p,q)$,
(ii) $(\widetilde{\delta}^{\supseteq})_S\neq\emptyset\Longrightarrow(\widetilde{\delta}^{\supseteq})_S=\bigcup_{p,q\in(\widetilde{\delta}^{\supseteq})_S}\overline{K}(p,q)$.

Proof. Let $R\subseteq U$ and $(\widetilde{\gamma}^{\subseteq})_R\neq\emptyset$. Since $(\widetilde{\gamma}^{\subseteq})_R\subseteq\bigcup_{p,q\in(\widetilde{\gamma}^{\subseteq})_R}\overline{K}(p,q)\subseteq\bigcup_{p,q\in(\widetilde{\gamma}^{\subseteq})_R}\overline{K}(p,q)$. Now let $x\in\bigcup_{p,q\in(\widetilde{\delta}^{\supseteq})_S}\overline{K}(p,q)$, there exist $b',b''\in(\widetilde{\gamma}^{\subseteq})_R$ such that $x\in\overline{K}(b',b'')\subseteq(\widetilde{\gamma}^{\subseteq})_R$. Thus $\bigcup_{p,q\in(\widetilde{\gamma}^{\subseteq})_R}\overline{K}(p,q)\subseteq(\widetilde{\gamma}^{\subseteq})_R$. Hence (i) is proved. Similarly, we can prove (ii).

4 The image and pre-image of DFS-ideals

In this section, we introduced image and pre-image of DFS-ideals and discussed some theorems.

Definition 15. *Let $h:X_1\longrightarrow X_2$ be a function of BE-algebras and $(\widetilde{\gamma}_{X_1},\widetilde{\delta}_{X_1})$ and $(\widetilde{\gamma}_{X_2},\widetilde{\delta}_{X_2})$ be two DFS-sets over U. The DFS-set $(h^{-1}(\widetilde{\gamma}_{\Lambda_2}),h^{-1}(\widetilde{\delta}_{\Lambda_2}))$ such that*

$$h^{-1}(\widetilde{\gamma}_{X_2})=\left\{(a,h^{-1}(\widetilde{\gamma}_{X_2})(a)):a\in X_1,h^{-1}(\widetilde{\gamma}_{X_2})(a)\in P(U)\right\},$$

and $$h^{-1}(\widetilde{\delta}_{X_2})=\left\{(a,h^{-1}(\widetilde{\delta}_{X_2})(a)):a\in X_1,h^{-1}(\widetilde{\delta}_{X_2})(a)\in P(U)\right\},$$

where $h^{-1}(\widetilde{\gamma}_{X_2})(a)=\widetilde{\gamma}_{X_2}(h(a))$ and $h^{-1}(\widetilde{\delta}_{X_2})(a)=\widetilde{\delta}_{X_2}(h(a))$, is called DFS-pre-image of $(\widetilde{\gamma}_{X_2},\widetilde{\delta}_{X_2})$ under h.

Definition 16. *Let $h : X_1 \longrightarrow X_2$ be a function of BE-algebras and $(\widetilde{\gamma}_{X_1}, \widetilde{\delta}_{X_1})$ and $(\widetilde{\gamma}_{X_2}, \widetilde{\delta}_{X_2})$ be two DFS-sets over U. The DFS-set $(h(\widetilde{\gamma}_{X_1}), h(\widetilde{\delta}_{X_1}))$ such that $h(\widetilde{\gamma}_{X_1}) = \{(b, h(\widetilde{\gamma}_{X_1})(b)) : b \in X_2, h(\widetilde{\gamma}_{X_1})(b) \in P(U)\}$, and $h(\widetilde{\delta}_{X_1}) = \left\{(b, h(\widetilde{\delta}_{X_2})(b)) : b \in X_1, h(\widetilde{\delta}_{X_1})(b) \in P(U)\right\}$, where $h(\widetilde{\gamma}_{X_1})(b) = \begin{cases} \bigcup_{a \in h^{-1}(b)} \widetilde{\gamma}_{X_1}(a) & \text{if } h^{-1}(b) \neq \emptyset, \\ \emptyset & \text{if Otherwise,} \end{cases}$ and*

$$h(\widetilde{\delta}_{X_1})(b) = \begin{cases} \bigcap_{a \in h^{-1}(b)} \widetilde{\delta}_{X_1}(a) & \text{if } h^{-1}(b) \neq \emptyset, \\ U & \text{if Otherwise,} \end{cases}$$

is called DFS-image of $(\widetilde{\gamma}_{X_1}, \widetilde{\delta}_{X_1})$ under h.

Proposition 4. *Let $h : X_1 \longrightarrow X_2$ be a function of BE-algebras X_1 and X_2 and $(\widetilde{\gamma}_{X_1}, \widetilde{\delta}_{X_1})$ be a DFS-sets over U. Then $\widetilde{\gamma}_{X_1} \widetilde{\subseteq} h^{-1}(h(\widetilde{\gamma}_{X_1}))$ and $\widetilde{\delta}_{X_1} \widetilde{\supseteq} h^{-1}(h(\widetilde{\delta}_{X_1}))$.*

Proof. Since $h^{-1}(h(x)) \neq \emptyset$ for all $x \in X_1$. Thus, $\widetilde{\gamma}_{X_1}(x) \subseteq \bigcup_{a \in h^{-1}(\lambda(x))} \widetilde{\gamma}_{X_1}(x) = h(\widetilde{\gamma}_{X_1}(h(x))) = h^{-1}(h(\widetilde{\gamma}_{X_1}))(x)$

$\widetilde{\delta}_{X_1}(x) \supseteq \bigcap_{a \in h^{-1}(\lambda(x))} \widetilde{\delta}_{X_1}(x) = h(\widetilde{\delta}_{X_1}(h(x))) = h^{-1}(h(\widetilde{\delta}_{X_1}))(x)$. Hence $\widetilde{\gamma}_{X_1} \widetilde{\subseteq} h^{-1}(h(\widetilde{\gamma}_{X_1}))$ and $\widetilde{\delta}_{X_1} \widetilde{\supseteq} h^{-1}(h(\widetilde{\delta}_{X_1}))$.

Theorem 7. *Let $h : X_1 \longrightarrow X_2$ be a homomorphism of BE-algebras and $(\widetilde{\gamma}_{X_2}, \widetilde{\delta}_{X_2})$ be a DFS-set of X_2 over U. If $(\widetilde{\gamma}_{X_2}, \widetilde{\delta}_{X_2})$ is a DFS-ideal of X_2 over U. Then the DFS-pre-image $(h^{-1}(\widetilde{\gamma}_{X_2}), h^{-1}(\widetilde{\delta}_{X_2}))$ of $(\widetilde{\gamma}_{X_2}, \widetilde{\delta}_{X_2})$ under h is also a DFS-ideal of X_1 over U.*

Proof. Let $a, b \in X_1$ and $(\widetilde{\gamma}_{X_2}, \widetilde{\delta}_{X_2})$ is a DFS-ideal of X_2 over U. Then $h^{-1}(\widetilde{\gamma}_{X_2})(b) = \widetilde{\gamma}_{X_2}(h(b)) \subseteq \widetilde{\gamma}_{X_2}(h(a) \star h(b)) = \widetilde{\gamma}_{X_2}(h(a \star b)) = h^{-1}(\widetilde{\gamma}_{X_2})(a \star b)$,
and
$h^{-1}(\widetilde{\delta}_{X_2})(b) = \widetilde{\delta}_{X_2}(h(b)) \supseteq \widetilde{\delta}_{X_2}(h(a) \star h(b)) = \widetilde{\delta}_{X_2}(h(a \star b)) = h^{-1}(\widetilde{\delta}_{X_2})(a \star b)$. Now for $x, y, z \in X_2$ over U, we have
$h^{-1}(\widetilde{\gamma}_{X_2})(x) \cap h^{-1}(\widetilde{\gamma}_{X_2})(y) = \widetilde{\gamma}_{X_2}(h(x)) \cap \widetilde{\gamma}_{X_2}(h(y))$
$\subseteq \widetilde{\gamma}_{X_2}(h(x) \star (h(y) \star h(z)) \star h(z)) = \widetilde{\gamma}_{X_2}(h(x \star (y \star z) \star z)) = h^{-1}(\widetilde{\gamma}_{X_2})(x \star (y \star z) \star z)$,
and
$h^{-1}(\widetilde{\delta}_{X_2})(x) \cup h^{-1}\left(\widetilde{\delta}_{X_2}\right)(y) = \widetilde{\delta}_{X_2}(h(x)) \cup \widetilde{\gamma}_{X_2}(h(y))$
$\supseteq \widetilde{\delta}_{X_2}(h(x) \star (h(y) \star h(z)) \star h(z)) = \widetilde{\delta}_{X_2}(h(x \star (y \star z) \star z)) = h^{-1}(\widetilde{\delta}_{X_2})(x \star (y \star z) \star z)$
Thus $(h^{-1}(\widetilde{\gamma}_{X_2}), h^{-1}(\widetilde{\delta}_{X_2}))$ is a DFS-ideal of X_1 over U.

Theorem 8. *Let $h : X_1 \longrightarrow X_2$ be a homomorphism of BE-algebras and $(\widetilde{\gamma}_{X_1}, \widetilde{\delta}_{X_1})$ be a DFS-set of X_1 over U. If $(\widetilde{\gamma}_{X_1}, \widetilde{\delta}_{X_1})$ is a DFS-ideal of X_1 over U and h is injective, then the DFS-image $(h(\widetilde{\gamma}_{X_1}), h(\widetilde{\delta}_{X_1}))$ of $(\widetilde{\gamma}_{X_1}, \widetilde{\delta}_{X_1})$ under h is also a DFS-ideal of X_2 over U.*

Proof. Since λ is injective, by Theorem 2, it is sufficient to show that $h((\widetilde{\gamma}^{\subseteq})_R)$ and $h((\widetilde{\delta}^{\supseteq})_S)$ are ideals of X_2 over U. Let $R, S \subseteq U$ such that $(\widetilde{\gamma}^{\subseteq})_R \neq \emptyset$, $(\widetilde{\delta}^{\supseteq})_S \neq \emptyset$. Then for $b \in h((\widetilde{\gamma}^{\subseteq})_R)$ and $b \in h((\widetilde{\delta}^{\supseteq})_S)$, we can obtain

$$h(\widetilde{\gamma}_{X_1})(b) = \bigcup_{x \in h^{-1}(b)} \widetilde{\gamma}_{X_1}(x) \supseteq R, \; h(\widetilde{\delta}_{X_1})(b) = \bigcap_{x \in h^{-1}(b)} \widetilde{\delta}_{X_1}(x) \subseteq S.$$

This means that there exists $x_o \in h^{-1}(b)$ such that $\widetilde{\gamma}_{X_1}(x_o) \supseteq R$ and $\widetilde{\delta}_{X_1}(x_o) \subseteq S$. Since h is injective and $(\widetilde{\gamma}_{X_1}, \widetilde{\delta}_{X_1})$ is a DFS-ideal of X_1 over U. Then for $a \in X_2$ there exist $x \in X_1$ such that $h(x) = a$, $\widetilde{\gamma}_{X_1}(x \star x_o) \supseteq \widetilde{\gamma}_{X_1}(x_o)$ and $\widetilde{\delta}_{X_1}(x \star x_o) \subseteq \widetilde{\delta}_{X_1}(x_o)$. Then

$$h(\widetilde{\gamma}_{X_1})(a \star b) = \bigcup_{x \in h^{-1}(a \star b)} \widetilde{\gamma}_{X_1}(x) \supseteq \widetilde{\gamma}_{X_1}(x \star x_o) \supseteq \widetilde{\gamma}_{X_1}(x_o) \supseteq R,$$

$$h(\widetilde{\delta}_{X_1})(a \star b) = \bigcap_{x \in h^{-1}(a \star b)} \widetilde{\delta}_{X_1}(x) \subseteq \widetilde{\delta}_{X_1}(x \star x_o) \subseteq \widetilde{\delta}_{X_1}(x_o) \subseteq S.$$

Thus, $a \star b \in h((\widetilde{\gamma}^{\subseteq})_R)$ and $a \star b \in h((\widetilde{\delta}^{\supseteq})_S)$.

Now for $a_1, b_1 \in h((\widetilde{\gamma}^{\subseteq})_R)$ and $a_1, b_1 \in h((\widetilde{\delta}^{\supseteq})_S)$, we have $h(\widetilde{\gamma}_{X_1})(a_1) = \bigcup_{x \in h^{-1}(a_1)} \widetilde{\gamma}_{X_1}(x) \supseteq R$, $h(\widetilde{\gamma}_{X_1})(b_1) = \bigcup_{x \in h^{-1}(b_1)} \widetilde{\gamma}_{X_1}(x) \supseteq R$ and $h(\widetilde{\delta}_{X_1})(a_1) = \bigcap_{x \in h^{-1}(a_1)} \widetilde{\delta}_{X_1}(x) \subseteq S$, $h(\widetilde{\delta}_{X_1})(b_1) = \bigcap_{x \in h^{-1}(b_1)} \widetilde{\delta}_{X_1}(x) \subseteq S$.

This means that there exists $x_o \in h^{-1}(a_1)$, $y_o \in h^{-1}(b_1)$ such that $\widetilde{\gamma}_{X_1}(x_o) \supseteq R$, $\widetilde{\gamma}_{X_1}(y_o) \supseteq R$ and $\widetilde{\delta}_{X_1}(x_o) \subseteq S$, $\widetilde{\delta}_{X_1}(y_o) \subseteq S$. Since h is injective and $(\widetilde{\gamma}_{X_1}, \widetilde{\delta}_{X_1})$ is a DFS-ideal of X_1 over U. Then for $d \in X_2$, there exist $x \in X_1$ such that $h(x) = d$, $\widetilde{\gamma}_{X_1}(x_o \star (y_o \star x) \star x) \supseteq \widetilde{\gamma}_{X_1}(x_o) \cap \widetilde{\gamma}_{X_1}(y_o)$, and $\widetilde{\delta}_{X_1}(x_o \star (y_o \star x) \star x) \subseteq \widetilde{\delta}_{X_1}(x_o) \cup \widetilde{\delta}_{X_1}(y_o)$. Thus

$$h(\widetilde{\gamma}_{X_1})(a_1 \star (b_1 \star d) \star d) = \bigcup_{x \in h^{-1}(a_1 \star (b_1 \star d) \star d)} \widetilde{\gamma}_{X_1}(x) \supseteq \widetilde{\gamma}_{X_1}(x_o \star (y_o \star x) \star x) \supseteq \widetilde{\gamma}_{X_1}(x_o) \cap \widetilde{\gamma}_{X_1}(y_o) \supseteq R,$$

$$h(\widetilde{\delta}_{X_1})(a_1 \star (b_1 \star d) \star d) = \bigcap_{x \in h^{-1}(a_1 \star (b_1 \star d) \star d)} \widetilde{\delta}_{X_1}(x) \subseteq \widetilde{\delta}_{X_1}(x_o \star (y_o \star x) \star x) \subseteq \widetilde{\delta}_{X_1}(x_o) \cup \widetilde{\delta}_{X_1}(y_o) \subseteq S.$$

Hence the DFS-image $(h(\widetilde{\gamma}_{X_1}), h(\widetilde{\delta}_{X_1}))$ of $(\widetilde{\gamma}_{X_1}, \widetilde{\delta}_{X_1})$ under h is a DFS-ideal of X_2 over U.

5 Conclusion

In above study, we investigated DFS-ideals in BE-algebras and discussed some basic results of DFS-ideals. Our approach provide new insights into BE-algebras using properties of the images and pre-images of DFS-ideals by homomorphism. Hopefully, these notions and essential results may lead to significant and new results in related fields. In future, we will study rough soft BE-algebras and DFS-ideals in different algebraic structures.

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