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On generalization of different type inequalities for (α,m)-convex functions via fractional integrals

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Abstract: In this paper, new identity for fractional integrals have been defined. By using of this identity, the authors obtained new general inequalities containing all of Hadamard, Ostrowski and Simpson type inequalities for functions whose derivatives in absolute value at certain power are (α,m)-convex via Riemann Liouville fractional integral.

Keywords: Hermite–Hadamard inequality, Riemann–Liouville fractional integral, (α,m)-convex function.

1 Introduction

Following inequalities are well known in the literature as Hermite-Hadamard inequality, Ostrowski inequality and Simpson inequality respectively:

Theorem 1. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function defined on the interval I of real numbers and $a,b\in I$ with $a<b$. The following double inequality holds*

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2}. \tag{1}$$

Theorem 2. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a mapping differentiable in I°, the interior of I, and let $a,b\in I^\circ$ with $a<b$. If $|f'(x)|\le M$ for all $x\in[a,b]$, then the following inequality holds*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\frac{M}{b-a}\left[\frac{(x-a)^2+(b-x)^2}{2}\right]$$

for all $x\in[a,b]$.

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_\infty=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{1}{2880}\left\|f^{(4)}\right\|_\infty(b-a)^4.$$

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In [16], G. Toader considered the class of m-convex functions: another intermediate between the usual convexity and starshaped convexity.

Definition 1. *The function* $f:[0,b]\to\mathbb{R}, b>0$, *is said to be* $m-$*convex, where* $m\in[0,1]$, *if we have*

$$f(tx+m(1-t)y)\leq tf(x)+m(1-t)f(y)$$

for all $x,y\in[0,b]$ *and* $t\in[0,1]$. *We say that* f *is* $m-$*concave if* $(-f)$ *is* $m-$*convex.*

The class of (α,m)-convex functions was first introduced In [7], and it is defined as follows:

Definition 2. *The function* $f:[0,b]\to\mathbb{R}$, $b>0$, *is said to be* (α,m)*-convex where* $(\alpha,m)\in[0,1]^2$, *if we have*

$$f\left(tx+m(1-t)y\right)\leq t^{\alpha}f(x)+m(1-t^{\alpha})f(y)$$

for all $x,y\in[0,b]$ *and* $t\in[0,1]$.

It can be easily that for $(\alpha,m)\in\{(0,0),(\alpha,0),(1,0),(1,m),(1,1),(\alpha,1)\}$ one obtains the following classes of functions: increasing, α-starshaped, starshaped, m-convex, convex, α-convex.

Denote by $K_m^{\alpha}(b)$ the set of all (α,m)-convex functions on $[0,b]$ for which $f(0)\leq 0$.

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 3. *Let* $f\in L[a,b]$. *The Riemann-Liouville integrals* $J_{a+}^{\theta}f$ *and* $J_{b-}^{\theta}f$ *of oder* $\theta>0$ *with* $a\geq 0$ *are defined by*

$$J_{a+}^{\theta}f(x)=\frac{1}{\Gamma(\theta)}\int_{a}^{x}(x-t)^{\theta-1}f(t)dt,\ x>a$$

and

$$J_{b-}^{\theta}f(x)=\frac{1}{\Gamma(\theta)}\int_{x}^{b}(t-x)^{\theta-1}f(t)dt,\ x<b$$

respectively, where $\Gamma(\theta)$ *is the Gamma function defined by* $\Gamma(\theta)=\int_{0}^{\infty}e^{-t}t^{\theta-1}dt$ *and* $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$.

In the case of $\theta=1$, the fractional integral reduces to the classical integral. Properties concerning this operator can be found [3,8,12].

In recent years, many athors have studied errors estimations for Hermite-Hadamard, Ostrowski and Simpson inequalities. For recent some results and generalizations concerning (α,m)-convex functions see [2,4,5,6,9,10,11,13,14,15].

In this paper, new identity for fractional integrals have been defined. By using of this identity, we obtained a generalization of Hadamard, Ostrowski and Simpson type inequalities for (α,m)-convex functions via Riemann Liouville fractional integral.

2 Main Results

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, the interior of I, throughout this section we will take

$$S_f(mx,\lambda,\theta,ma,mb) = (1-\lambda)m^{\theta-1}\left[\frac{(x-a)^{\theta}+(b-x)^{\theta}}{b-a}\right]f(mx) + \lambda m^{\theta-1}\left[\frac{(x-a)^{\theta}f(ma)+(b-x)^{\theta}f(mb)}{b-a}\right]$$
$$-\frac{\Gamma(\theta+1)}{m(b-a)}\left[J^{\theta}_{(mx)^-}f(ma)+J^{\theta}_{(mx)^+}f(mb)\right]$$

where $m \in (0,1]$, $ma, b \in I$ with $a < b$, $x \in [a,b]$, $\lambda \in [0,1]$, $\theta > 0$ and Γ is Euler Gamma function. In order to prove our main results we need the following identity.

Lemma 1. *,Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[ma,mb]$, where $m \in (0,1]$, $ma, b \in I$ with $a < b$. Then for all $x \in [a,b]$, $\lambda \in [0,1]$ and $\theta > 0$ we have:*

$$S_f(mx,\lambda,\theta,ma,mb) = \frac{m^{\theta}(x-a)^{\theta+1}}{b-a}\int_0^1\left(t^{\theta}-\lambda\right)f'(tmx+m(1-t)a)\,dt$$
$$+\frac{m^{\theta}(b-x)^{\theta+1}}{b-a}\int_0^1\left(\lambda-t^{\theta}\right)f'(tmx+m(1-t)b)\,dt.$$

A simple proof of the equality can be done by performing an integration by parts in the integrals from the right side and changing the variable. The details are left to the interested reader.

Theorem 4. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[ma,mb]$, where $m \in (0,1]$,$ma, b \in I^{\circ}$ with $a < b$. If $|f'|^q$ is (α,m)-convex on $[ma,b]$ for some fixed $q \geq 1$, $x \in [a,b]$, $\lambda \in [0,1]$ and $\theta > 0$ then the following inequality for fractional integrals holds*

$$\left|S_f(mx,\lambda,\theta,ma,mb)\right| \leq \frac{m^{\theta}A_1^{1-\frac{1}{q}}(\theta,\lambda)}{b-a}\left\{(x-a)^{\theta+1}\left(\left|f'(mx)\right|^qA_2(\alpha,\theta,\lambda)+m\left|f'(a)\right|^qA_3(\alpha,\theta,\lambda)\right)^{\frac{1}{q}}\right.$$
$$\left.+(b-x)^{\theta+1}\left(\left|f'(mx)\right|^qA_2(\alpha,\theta,\lambda)+m\left|f'(b)\right|^qA_3(\alpha,\theta,\lambda)\right)^{\frac{1}{q}}\right\} \tag{2}$$

where

$$A_1(\theta,\lambda) = \frac{2\theta\lambda^{1+\frac{1}{\theta}}+1}{\theta+1}-\lambda,$$
$$A_2(\alpha,\theta,\lambda) = \frac{2\theta\lambda^{1+\frac{1+\alpha}{\theta}}}{(\alpha+1)(\alpha+\theta+1)}+\frac{1}{\alpha+\theta+1}-\frac{\lambda}{\alpha+1},$$

$$A_3(\alpha,\theta,\lambda) = A_1(\theta,\lambda)-A_2(\alpha,\theta,\lambda)$$
$$=\frac{2\theta\lambda^{1+\frac{1}{\theta}}+1}{\theta+1}-\frac{2\theta\lambda^{1+\frac{1+\alpha}{\theta}}}{(\alpha+1)(\alpha+\theta+1)}-\frac{1}{\alpha+\theta+1}-\frac{\alpha\lambda}{\alpha+1}.$$

Proof. From Lemma 1, property of the modulus and the power-mean inequality we have

$$\begin{aligned}
&\left|S_f(mx,\lambda,\theta,ma,mb)\right| \\
&\le \frac{m^\theta (x-a)^{\theta+1}}{b-a}\left(\int_0^1 \left|t^\theta-\lambda\right| dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|t^\theta-\lambda\right|\left|f'(tmx+m(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{m^\theta (b-x)^{\theta+1}}{b-a}\left(\int_0^1 \left|t^\theta-\lambda\right| dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|t^\theta-\lambda\right|\left|f'(tmx+m(1-t)b)\right|^q dt\right)^{\frac{1}{q}}. \qquad (3)
\end{aligned}$$

Since $|f'|^q$ is (α,m)-convex on $[ma,b]$, for all $t\in[0,1]$

$$\left|f'(tmx+m(1-t)a)\right|^q \le t^\alpha\left|f'(mx)\right|^q + m(1-t^\alpha)\left|f'(a)\right|^q$$

and

$$\left|f'(tmx+m(1-t)b)\right|^q \le t^\alpha\left|f'(mx)\right|^q + m(1-t^\alpha)\left|f'(b)\right|^q.$$

Hence by simple computation we get

$$\int_0^1 \left|t^\theta-\lambda\right| dt = \frac{2\theta\lambda^{1+\frac{1}{\theta}}+1}{\theta+1}-\lambda \qquad (4)$$

$$\begin{aligned}
\int_0^1 \left|t^\theta-\lambda\right|\left|f'(tmx+m(1-t)a)\right|^q dt &\le \int_0^1 \left|t^\theta-\lambda\right|\left(t^\alpha\left|f'(mx)\right|^q+m(1-t^\alpha)\left|f'(a)\right|^q\right) dt \\
&= \left|f'(mx)\right|^q A_2(\alpha,\theta,\lambda)+m\left|f'(a)\right|^q A_3(\alpha,\theta,\lambda), \qquad (5)
\end{aligned}$$

and similarly

$$\begin{aligned}
\int_0^1 \left|t^\theta-\lambda\right|\left|f'(tmx+m(1-t)b)\right|^q dt &\le \int_0^1 \left|t^\theta-\lambda\right| t^\alpha\left|f'(mx)\right|^q+m(1-t^\alpha)\left|f'(b)\right|^q dt \\
&= \left|f'(mx)\right|^q A_2(\alpha,\theta,\lambda)+m\left|f'(b)\right|^q A_3(\alpha,\theta,\lambda). \qquad (6)
\end{aligned}$$

If we use (4), (5) and (6) in (3), we obtain the desired result. This completes the proof.

Corollary 1. *In Theorem 4,*

(i) *If we take $\theta=1$, then inequality (2) reduced to the following inequality*

$$\begin{aligned}
\left|S_f(mx,\lambda,1,ma,mb)\right| &= \left|(1-\lambda)f(mx)+\lambda\left[\frac{(x-a)f(ma)+(b-x)f(mb)}{b-a}\right]-\frac{1}{m(b-a)}\int_{ma}^{mb} f(t)dt\right| \\
&\le \frac{mA_1^{1-\frac{1}{q}}(1,\lambda)}{b-a}\left\{(x-a)^2\left(\left|f'(mx)\right|^q A_2(\alpha,1,\lambda)+m\left|f'(a)\right|^q A_3(\alpha,1,\lambda)\right)^{\frac{1}{q}}\right. \\
&\left.+(b-x)^2\left(\left|f'(mx)\right|^q A_2(\alpha,1,\lambda)+m\left|f'(b)\right|^q A_3(\alpha,1,\lambda)\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

(ii) *If we take $q=1$, then inequality (2) reduced to the following inequality*

$$\left|S_f(mx,\lambda,\theta,ma,mb)\right| \le \frac{m^{\theta}}{b-a}\Big\{(x-a)^{\theta+1}\left(\left|f'(mx)\right|A_2(\alpha,\theta,\lambda)+m\left|f'(a)\right|A_3(\alpha,\theta,\lambda)\right)$$
$$+(b-x)^{\theta+1}\left(\left|f'(mx)\right|A_2(\alpha,\theta,\lambda)+m\left|f'(b)\right|A_3(\alpha,\theta,\lambda)\right)\Big\}$$

(iii) *If we take $x=\frac{a+b}{2}$, $\lambda=\frac{1}{3}$,then we get the following Simpson type inequality via fractional integrals*

$$\left|\frac{2^{\theta-1}}{m^{\theta-1}(b-a)^{\theta-1}}S_f\left(m\left(\frac{a+b}{2}\right),\frac{1}{3},\theta,ma,mb\right)\right|$$
$$=\left|\frac{1}{6}\left[f(ma)+4f\left(\frac{m(a+b)}{2}\right)+f(mb)\right]-\frac{\Gamma(\theta+1)2^{\theta-1}}{m^{\theta}(b-a)^{\theta}}\left[J^{\theta}_{\left(\frac{m(a+b)}{2}\right)^-}f(ma)+J^{\theta}_{\left(\frac{m(a+b)}{2}\right)^+}f(mb)\right]\right|$$
$$\le\frac{m(b-a)}{4}A_1^{1-\frac{1}{q}}\left(\theta,\frac{1}{3}\right)\left\{\left(\left|f'\left(\frac{m(a+b)}{2}\right)\right|^{q}A_2\left(\alpha,\theta,\frac{1}{3}\right)+m\left|f'(a)\right|^{q}A_3\left(\alpha,\theta,\frac{1}{3}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f'\left(\frac{m(a+b)}{2}\right)\right|^{q}A_2\left(\alpha,\theta,\frac{1}{3}\right)+m\left|f'(b)\right|^{q}A_3\left(\alpha,\theta,\frac{1}{3}\right)\right)^{\frac{1}{q}}\right\}.$$

(iv) *If we take $x=\frac{a+b}{2}$, $\lambda=0$,then we get the following midpoint type inequality via fractional integrals*

$$\frac{2^{\theta-1}}{m^{\theta-1}(b-a)^{\theta-1}}\left|S_f\left(m\left(\frac{a+b}{2}\right),0,\theta,ma,mb\right)\right|$$
$$=\left|f\left(\frac{m(a+b)}{2}\right)-\frac{\Gamma(\theta+1)2^{\theta-1}}{m^{\theta}(b-a)^{\theta}}\left[J^{\theta}_{\left(\frac{m(a+b)}{2}\right)^-}f(ma)+J^{\theta}_{\left(\frac{m(a+b)}{2}\right)^+}f(mb)\right]\right|$$
$$\le\frac{m(b-a)}{4}\left(\frac{1}{\theta+1}\right)\left(\frac{1}{\alpha+\theta+1}\right)^{\frac{1}{q}}\left\{\left[(\theta+1)\left|f'\left(\frac{m(a+b)}{2}\right)\right|^{q}+\alpha m\left|f'(a)\right|^{q}\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[(\theta+1)\left|f'\left(\frac{m(a+b)}{2}\right)\right|^{q}+\alpha m\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}\right\}.$$

(v) *If we take $\lambda=1$,then we get the following generalized trapezoid type inequality via fractional integrals*

$$m^{1-\theta}\left|S_f(mx,1,\alpha,ma,mb)\right|$$
$$=\left|\frac{(x-a)^{\theta}f(ma)+(b-x)^{\theta}f(mb)}{b-a}-\frac{\Gamma(\theta+1)}{m^{\theta}(b-a)}\left[J^{\theta}_{mx^-}f(ma)+J^{\theta}_{mx^+}f(mb)\right]\right|$$
$$\le\frac{m}{b-a}\left(\frac{\theta}{\theta+1}\right)^{1-\frac{1}{q}}\left(\frac{1}{(\alpha+1)(\theta+1)(\alpha+\theta+1)}\right)^{\frac{1}{q}}$$
$$\times\left\{(x-a)^{\theta+1}\left[\theta(\theta+1)\left|f'(mx)\right|^{q}+m\alpha\theta(\alpha+\theta+2)\left|f'(a)\right|\right]^{\frac{1}{q}}\right.$$
$$\left.+(b-x)^{\theta+1}\left[\theta(\theta+1)\left|f'(mx)\right|^{q}+m\alpha\theta(\alpha+\theta+2)\left|f'(b)\right|\right]^{\frac{1}{q}}\right\}.$$

(vi) *If* $|f'(u)| \leq M$ *for all* $u \in [ma,b]$ *and* $\lambda = 0$, *then we get the following Ostrowski type inequality via fractional integrals*

$$\left|\left[\frac{(x-a)^{\theta}+(b-x)^{\theta}}{b-a}\right] f(mx)-\frac{\Gamma(\theta+1)}{m^{\theta}(b-a)}\left[J_{mx-}^{\theta} f(ma)+J_{mx+}^{\theta} f(mb)\right]\right|$$
$$\leq mM\left(\frac{1}{\theta+1}\right)\left(\frac{\alpha m+\theta+1}{\alpha+\theta+1}\right)^{\frac{1}{q}}\left[\frac{(x-a)^{\theta+1}+(b-x)^{\theta+1}}{b-a}\right],$$

for each $x \in [a,b]$.

Theorem 5. *Let* $f: I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[ma,mb]$, *where* $m \in (0,1]$, $ma,b \in I^{\circ}$ *with* $a<b$. *If* $|f'|^{q}$ *is* (α,m)-*convex on* $[ma,b]$ *for some fixed* $q>1$, $x \in [a,b]$, $\lambda \in [0,1]$ *and* $\theta>0$ *then the following inequality for fractional integrals holds*

$$\left|S_{f}(mx,\lambda,\theta,ma,mb)\right| \leq \frac{m^{\theta}A_{4}(\theta,\lambda,p)^{\frac{1}{p}}}{b-a}\left\{(x-a)^{\theta+1}\left(\frac{|f'(mx)|^{q}+\alpha m|f'(a)|^{q}}{\alpha+1}\right)^{\frac{1}{q}}\right.$$
$$\left.+(b-x)^{\theta+1}\left(\frac{|f'(mx)|^{q}+\alpha m|f'(b)|^{q}}{\alpha+1}\right)^{\frac{1}{q}}\right\} \tag{7}$$

where

$$A_{4}(\theta,\lambda,p)=\begin{cases}\frac{1}{\theta p+1}, & \lambda=0\\ \left\{\frac{\lambda^{\frac{\theta p+1}{\theta}}}{\theta}\beta\left(\frac{1}{\theta},p+1\right)+\frac{(1-\lambda)^{p+1}}{\theta(p+1)}\right. & \\ \left.\times{}_{2}F_{1}\left(1-\frac{1}{\theta},1;p+2;1-\lambda\right)\right\}, & 0<\lambda<1\\ \frac{1}{\theta}\beta\left(p+1,\frac{1}{\theta}\right), & \lambda=1\end{cases},$$

β *is Euler Beta function defined by*

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_{0}^{1} t^{x-1}(1-t)^{y-1}dt,\ \ x,y>0,$$

${}_{2}F_{1}$ *is hypergeometric function defined by*

$${}_{2}F_{1}(a,b;c;z)=\frac{1}{\beta(b,c-b)}\int_{0}^{1} t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt,\ c>b>0,\ |z|<1 \text{ (see [1])},$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1, property of the modulus and using the Hölder inequality we have

$$\left|S_{f}(mx,\lambda,\theta,ma,mb)\right| \leq \frac{m^{\theta}(x-a)^{\theta+1}}{b-a}\left(\int_{0}^{1}\left|t^{\theta}-\lambda\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|f'(tmx+m(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}$$
$$+\frac{m^{\theta}(b-x)^{\theta+1}}{b-a}\left(\int_{0}^{1}\left|t^{\theta}-\lambda\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|f'(tmx+m(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}. \tag{8}$$

Since $|f'|^q$ is (α,m)-convex on $[ma,b]$, we get

$$\int_0^1 \left|f'(tmx+m(1-t)a)\right|^q dt \le \int_0^1 t^\alpha \left|f'(mx)\right|^q + m(1-t^\alpha)\left|f'(a)\right|^q dt = \frac{|f'(mx)|^q+\alpha m|f'(a)|^q}{\alpha+1}, \tag{9}$$

$$\int_0^1 \left|f'(tmx+m(1-t)b)\right|^q dt \le \int_0^1 t^\alpha \left|f'(mx)\right|^q + m(1-t^\alpha)\left|f'(b)\right|^q dt = \frac{|f'(mx)|^q+\alpha m|f'(b)|^q}{\alpha+1}, \tag{10}$$

and by simple computation

$$\int_0^1 \left|t^\theta-\lambda\right|^p dt = \begin{cases} \frac{1}{\theta p+1}, & \lambda=0 \\ \frac{\lambda^{\frac{\theta p+1}{\theta}}}{\theta}\beta\left(\frac{1}{\theta},p+1\right)+\frac{(1-\lambda)^{p+1}}{\theta(p+1)} \\ \times {}_2F_1\left(1-\frac{1}{\theta},1;p+2;1-\lambda\right)\}, & 0<\lambda<1 \\ \frac{1}{\theta}\beta\left(p+1,\frac{1}{\theta}\right), & \lambda=1 \end{cases} \tag{11}$$

Hence, If we use (9)-(11) in (8), we obtain the desired result. This completes the proof.

Corollary 2. *In Theorem 5,*

1. *If we take $\theta=1$, then inequality (7) reduced to the following inequality*

$$\left|(1-\lambda)f(mx)+\lambda\left[\frac{(x-a)f(ma)+(b-x)f(mb)}{b-a}\right]-\frac{1}{m(b-a)}\int_{ma}^{mb} f(t)dt\right|$$
$$\le \frac{mA_4(1,\lambda,p)^{\frac{1}{p}}}{b-a}\left\{(x-a)^2\left(\frac{|f'(mx)|^q+\alpha m|f'(a)|^q}{\alpha+1}\right)^{\frac{1}{q}}+(b-x)^2\left(\frac{|f'(mx)|^q+\alpha m|f'(b)|^q}{\alpha+1}\right)^{\frac{1}{q}}\right\}.$$

2. *If we take $x=\frac{a+b}{2}$, $\lambda=\frac{1}{3}$,then we get the following Simpson type inequality via fractional integrals*

$$\left|\frac{1}{6}\left[f(ma)+4f\left(\frac{m(a+b)}{2}\right)+f(mb)\right]-\frac{\Gamma(\theta+1)2^{\theta-1}}{m^\theta(b-a)^\theta}\left[J^\theta_{\left(\frac{m(a+b)}{2}\right)^-}f(ma)+J^\theta_{\left(\frac{m(a+b)}{2}\right)^+}f(mb)\right]\right|$$
$$\le \frac{m(b-a)}{4}A_4^{\frac{1}{p}}\left(\theta,\frac{1}{3},p\right)\left\{\left(\frac{\left|f'\left(\frac{m(a+b)}{2}\right)\right|^q+\alpha m|f'(a)|^q}{\alpha+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{m(a+b)}{2}\right)\right|^q+\alpha m|f'(b)|^q}{\alpha+1}\right)^{\frac{1}{q}}\right\}.$$

3. *If we take $x=\frac{a+b}{2}$, $\lambda=0$,then we get the following midpoint type inequality via fractional integrals*

$$\left|f\left(\frac{m(a+b)}{2}\right)-\frac{\Gamma(\theta+1)2^{\theta-1}}{m^\theta(b-a)^\theta}\left[J^\theta_{\left(\frac{m(a+b)}{2}\right)^-}f(ma)+J^\theta_{\left(\frac{m(a+b)}{2}\right)^+}f(mb)\right]\right|$$
$$\le \frac{m(b-a)}{4}\left(\frac{1}{\theta p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{\left|f'\left(\frac{m(a+b)}{2}\right)\right|^q+\alpha m|f'(a)|^q}{\alpha+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{m(a+b)}{2}\right)\right|^q+\alpha m|f'(b)|^q}{\alpha+1}\right)^{\frac{1}{q}}\right\}.$$

4. *If we take* $\lambda=1$,*then we get the following generalized trapezoid type inequality via fractional integrals*

$$\left|\frac{(x-a)^{\theta}f(ma)+(b-x)^{\theta}f(mb)}{b-a}-\frac{\Gamma(\theta+1)}{m^{\theta}(b-a)}\left[J^{\theta}_{mx^{-}}f(ma)+J^{\theta}_{mx^{+}}f(mb)\right]\right|$$
$$\leq\frac{m\left(\frac{1}{\theta}\beta\left(p+1,\frac{1}{\theta}\right)\right)^{\frac{1}{p}}}{b-a}\left\{(x-a)^{\theta+1}\left(\frac{|f'(mx)|^{q}+\alpha m|f'(a)|^{q}}{\alpha+1}\right)^{\frac{1}{q}}+(b-x)^{\theta+1}\left(\frac{|f'(mx)|^{q}+\alpha m|f'(b)|^{q}}{\alpha+1}\right)^{\frac{1}{q}}\right\}.$$

5. *If* $|f'(u)|\leq M$ *for all* $u\in[ma,b]$ *and* $\lambda=0$, *then we get the following Ostrowski type inequality via fractional integrals*

$$\left|\left[\frac{(x-a)^{\theta}+(b-x)^{\theta}}{b-a}\right]f(mx)-\frac{\Gamma(\theta+1)}{m^{\theta}(b-a)}\left[J^{\theta}_{mx^{-}}f(ma)+J^{\theta}_{mx^{+}}f(mb)\right]\right|$$
$$\leq mM\left(\frac{1}{\theta p+1}\right)^{\frac{1}{p}}\left(\frac{1+\alpha m}{\alpha+1}\right)^{\frac{1}{q}}\left[\frac{(x-a)^{\theta+1}+(b-x)^{\theta+1}}{b-a}\right],$$

for each $x\in[a,b]$.

3 conclusion

The paper deals with general integral inequalities containing all of Hadamard, Ostrowski and Simpson type inequalities for (α,m)-convex functions via fractional integrals. Firstly, some theorems on general integral inequalities are given. Later, several results of this general integral inequalities are mentioned.

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