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Coefficient bounds for new subclasses of bi-univalent functions

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Abstract: In the present paper, introduction of new subclasses of bi-univalent functions in the open disk was defined. Moreover,by using Salagean operator,in these new subclasses for functions, upper bounds for the second and third coefficients were found. Presented results are a generalization of the results obtained by Srivastava et al.[12], Frasin and Aouf [7] and Çağlar et al.[5].

Keywords: Univalent functions, bi-univalent functions, coefficient bounds, coefficient estimates, salagean operator.

1 Introduction

We will denote the class of functions of the form as $\mathscr{A}$

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and provide the normalization condition $f(0) = f'(0) - 1 = 0$. Let $\mathscr{S}$ symbolize the subclass of functions in $\mathscr{A}$ which are univalent in $\mathbb{U}$ (for details, see [6]).

In 1983, Differential operator was established by Salagean [10] as $D^n : \mathscr{A} \to \mathscr{A}$ defined by

$$D^0 f(z) = f(z),$$
$$D^1 f(z) = Df(z) = zf'(z),$$

and

$$D^n f(z) = D(D^{n-1} f(z)), \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

We express that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad (n \in \mathbb{N}_0).$$

It is known that every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, (z \in \mathbb{U})$$

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and

$$f\left(f^{-1}(w)\right) = w, \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{2}$$

A function $f \in \mathscr{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. We denote by Σ the class of all bi-univalent functions in $\mathbb{U}$ stated by Taylor-Maclaurin series expansion (1).

For a brief history and examples of subclasses in the class Σ, see [12](see,for example [4,8,9,14]; see also [3,13]). Recently, Srivastava et al.[11-12], Frasin and Aouf [7], Altınkaya and Yalçın [1-2] and Çağlar et al.[5] have investigated estimate on the coefficients $|a_2|$ and $|a_3|$ for the various subclasses of the function class Σ.

The aim of this paper is to introduce two new subclasses of the function class Σ related with Salagean differential operator and find estimate on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ. We also generalize results of Srivastava et al.[12], Frasin and Aouf [7] and Çağlar et al.[5]. In order to prove our main results, we require the following lemma due to [6].

Lemma 1. *If $p \in \mathscr{P}$ then $|c_k| \leq 2$ for each k, where $\mathscr{P}$ is the family of functions p analytic in $\mathbb{U}$ for which $Re\{p(z)\} > 0, p(z) = 1 + c_1z + c_2z^2 + ...$ for $z \in \mathbb{U}$.*

2 Coefficient bounds for the function class $N_{\Sigma}^{n;\mu}(\alpha,\lambda)$

Definition 1. *A function $f(z)$ given by (1) is said to be in the class $N_{\Sigma}^{n;\mu}(\alpha,\lambda)$ if the following conditions are satisfied:*

$$f \in \Sigma \text{ and } \left| arg\left\{(1-\lambda)\left(\frac{D^n f(z)}{z}\right)^{\mu} + \lambda \frac{D^{n+1}f(z)}{z}\left(\frac{D^n f(z)}{z}\right)^{\mu-1}\right\}\right| < \frac{\alpha\pi}{2} \tag{3}$$

$$(0 < \alpha \leq 1; \lambda \geq 1; \mu \geq 0; n \in \mathbb{N}_0; z \in \mathbb{U})$$

and

$$\left| arg\left\{(1-\lambda)\left(\frac{D^n g(w)}{w}\right)^{\mu} + \lambda \frac{D^{n+1}g(w)}{w}\left(\frac{D^n g(w)}{w}\right)^{\mu-1}\right\}\right| < \frac{\alpha\pi}{2} \tag{4}$$

$$(0 < \alpha \leq 1; \lambda \geq 1; \mu \geq 0; n \in \mathbb{N}_0; w \in \mathbb{U})$$

where the function $g(w)$ is given by (2).

For functions in the class $N_{\Sigma}^{n;\mu}(\alpha,\lambda)$, we start by finding the estimates on the coefficients $|a_2|$ and $|a_3|$.

Theorem 1. *Let the function $f(z)$ given by (1) be in the class $N_{\Sigma}^{n;\mu}(\alpha,\lambda)$ $(0 < \alpha \leq 1; \lambda \geq 1; \mu \geq 0; n \in \mathbb{N}_0; z \in \mathbb{U})$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2^{2n}\left((\mu+\lambda)^2 - \alpha(\lambda(2+\lambda)+\mu)\right) + 2\alpha.3^n(\mu+2\lambda)}} \tag{5}$$

and

$$|a_3| \leq \frac{2\alpha}{3^n(\mu+2\lambda)} + \frac{4\alpha^2}{2^n(\mu+\lambda)^2}. \tag{6}$$

Proof. It can be written that the inequalities (3) and (4) are equivalent to

$$(1-\lambda)\left(\frac{D^n f(z)}{z}\right)^{\mu} + \lambda \frac{D^{n+1} f(z)}{z}\left(\frac{D^n f(z)}{z}\right)^{\mu-1} = [p(z)]^{\alpha} \tag{7}$$

and

$$(1-\lambda)\left(\frac{D^n g(w)}{w}\right)^{\mu} + \lambda \frac{D^{n+1} g(w)}{w}\left(\frac{D^n g(w)}{w}\right)^{\mu-1} = [q(w)]^{\alpha} \tag{8}$$

where $p(z)$ and $q(w)$ in $\mathscr{P}$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{9}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots. \tag{10}$$

Now, equating the coefficients in (7) and (8), we obtain

$$2^n(\mu+\lambda)a_2 = \alpha p_1 \tag{11}$$

$$2^{2n-1}(\mu-1)(\mu+2\lambda)a_2^2 + 3^n(\mu+2\lambda)a_3 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2 \tag{12}$$

$$-2^n(\mu+\lambda)a_2 = \alpha q_1 \tag{13}$$

and

$$2^{2n-1}(\mu-1)(\mu+2\lambda)a_2^2 + 3^n(\mu+2\lambda)\left(2a_2^2 - a_3\right) = \alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2. \tag{14}$$

From (11) and (13), we get

$$p_1 = -q_1 \tag{15}$$

and

$$2^{2n+1}(\mu+\lambda)^2 a_2^2 = \alpha^2(p_1^2 + q_1^2). \tag{16}$$

Also from (12), (14) and (16), we find that

$$\begin{aligned}\left[2^{2n}(\mu-1)(\mu+2\lambda) + 2.3^n(\mu+2\lambda)\right]a_2^2 &= \alpha(p_2+q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2) \\ &= \alpha(p_2+q_2) + \frac{\alpha(\alpha-1)}{2}\frac{2^{2n+1}(\mu+\lambda)^2 a_2^2}{\alpha^2}.\end{aligned}$$

Therefore, we have

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{2^{2n}\left((\mu+\lambda)^2 - \alpha(\lambda(2+\lambda)+\mu)\right) + 2\alpha.3^n(\mu+2\lambda)}. \tag{17}$$

If we can apply Lemma 1 for the coefficients p_2 and q_2, we have

$$|a_2| \leq \frac{2\alpha}{\sqrt{2^{2n}\left((\mu+\lambda)^2-\alpha(\lambda(2+\lambda)+\mu)\right)+2\alpha.3^n(\mu+2\lambda)}}.$$

This gives the desired estimate for $|a_2|$ as asserted (5).

Next, in order to find the bound on $|a_3|$, by subtracting (14) from (12), we get

$$2.3^n(\mu+2\lambda)a_3-2.3^n(\mu+2\lambda)a_2^2=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2),$$

$$a_3=\frac{\alpha(p_2-q_2)}{2.3^n(\mu+2\lambda)}+\frac{\alpha^2(p_1^2+q_1^2)}{2^{2n+1}(\mu+\lambda)^2}. \tag{18}$$

We apply Lemma 1 one more time for the coefficients p_1, p_2, q_1 and q_2, we obtain

$$|a_3| \leq \frac{2\alpha}{3^n(\mu+2\lambda)}+\frac{4\alpha^2}{2^{2n}(\mu+\lambda)^2}.$$

This complete the proof of the Theorem 1.

If we take $\mu=1$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let f(z) given by (1) be in the class* $N_{\Sigma}^{n,\mu}(\alpha,\lambda)$, $0<\alpha\leq 1, \lambda \geq 1$ *and* $n \in \mathbb{N}_0$. *Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2^{2n}\left((1+\lambda)^2-\alpha(\lambda(2+\lambda)+1)\right)+2\alpha.3^n(1+2\lambda)}}$$

and

$$|a_3| \leq \frac{2\alpha}{3^n(1+2\lambda)}+\frac{4\alpha^2}{2^{2n}(1+\lambda)^2}.$$

*Remark.*For $n=0$ in Corollary 1, provides an improvement of the following estimates obtained by Frasin and Aouf [7].

If we take $\lambda=\mu=1$ in Theorem 1, we have the following corollary.

Corollary 2. *Let f(z) given by (1) be in the class* $N_{\Sigma}^{n,\mu}(\alpha,\lambda)$, $0<\alpha\leq 1$ *and* $n \in \mathbb{N}_0$. *Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2^{2n+2}(1-\alpha)+2\alpha.3^{n+1}}}$$

and

$$|a_3| \leq \frac{2\alpha}{3^{n+1}}+\frac{\alpha^2}{2^{2n}}.$$

Remark. For $n=0$ in Corollary 3, provides an improvement of the following estimates obtained by Srivastava et al. [12].

Remark. For $n=0$, Theorem 1 reduces to a result in [5].

3 Coefficient bounds for the function class $N_{\Sigma}^{n,\mu}(\beta,\lambda)$

Definition 2. *A function $f(z)$ given by (1) is said to be in the class $N_{\Sigma}^{n,\mu}(\beta,\lambda)$ if the following conditions are satisfied:*

$$f\in\Sigma \text{ and } Re\left\{(1-\lambda)\left(\frac{D^n f(z)}{z}\right)^{\mu}+\lambda\frac{D^{n+1}f(z)}{z}\left(\frac{D^n f(z)}{z}\right)^{\mu-1}\right\}>\beta \tag{19}$$

$$(0\le\beta<1;\lambda\ge1;\mu\ge0;n\in\mathbb{N}_0;z\in\mathbb{U})$$

and

$$Re\left\{(1-\lambda)\left(\frac{D^n g(w)}{w}\right)^{\mu}+\lambda\frac{D^{n+1}g(w)}{w}\left(\frac{D^n g(w)}{w}\right)^{\mu-1}\right\}>\beta \tag{20}$$

$$(0\le\beta<1;\lambda\ge1;\mu\ge0;n\in\mathbb{N}_0;w\in\mathbb{U})$$

where the function $g(w)$ is given by (2).

Theorem 2. *Let the function $f(z)$ given by (1) be in the class $N_{\Sigma}^{n,\mu}(\beta,\lambda)$ $(0\le\beta<1;\lambda\ge1;\mu\ge0;n\in\mathbb{N}_0;z\in\mathbb{U})$, then*

$$|a_2|\le\sqrt{\frac{2(1-\beta)}{2^{2n-1}(\mu-1)(\mu+2\lambda)+3^n(\mu+2\lambda)}} \tag{21}$$

and

$$|a_3|\le\left(\frac{1-\beta}{2^n(\mu+\lambda)}\right)^2+\frac{2(1-\beta)}{3^n(\mu+2\lambda)}. \tag{22}$$

Proof. It follows from (19) and (20) that there exists $p(z)\in\mathscr{P}$ and $q(z)\in\mathscr{P}$ such that

$$(1-\lambda)\left(\frac{D^n f(z)}{z}\right)^{\mu}+\lambda\frac{D^{n+1}f(z)}{z}\left(\frac{D^n f(z)}{z}\right)^{\mu-1}=\beta+(1-\beta)p(z) \tag{23}$$

and

$$(1-\lambda)\left(\frac{D^n g(w)}{w}\right)^{\mu}+\lambda\frac{D^{n+1}g(w)}{w}\left(\frac{D^n g(w)}{w}\right)^{\mu-1}=\beta+(1-\beta)q(w) \tag{24}$$

where $p(z)$ and $q(w)$ have the forms (9)and (10),respectively. Equating coefficients in (23) and (24) yields

$$2^n(\mu+\lambda)a_2=(1-\beta)p_1, \tag{25}$$

$$2^{2n-1}(\mu-1)(\mu+2\lambda)a_2^2+3^n(\mu+2\lambda)a_3=(1-\beta)p_2, \tag{26}$$

$$-2^n(\mu+\lambda)a_2=(1-\beta)q_1, \tag{27}$$

and

$$2^{2n-1}(\mu-1)(\mu+2\lambda)a_2^2+3^n(\mu+2\lambda)\left(2a^2-a_3\right)=(1-\beta)q_2. \tag{28}$$

From (25) and (27), we get

$$p_1=-q_1, \tag{29}$$

$$2^{2n+1}(\mu+\lambda)^2 a_2^2 = (1-\beta)^2(p_1^2+q_1^2). \tag{30}$$

Also from (26)and (28), we find that

$$\left[2^{2n}(\mu-1)(\mu+2\lambda)+2.3^n(\mu+2\lambda)\right]a_2^2 = (1-\beta)(p_2+q_2).$$

Thus, we have

$$|a_2^2| \le \frac{(1-\beta)(|p_2|+|q_2|)}{[2^{2n}(\mu-1)(\mu+2\lambda)+2.3^n(\mu+2\lambda)]}$$

$$|a_2^2| \le \frac{2(1-\beta)}{2^{2n-1}(\mu-1)(\mu+2\lambda)+3^n(\mu+2\lambda)}$$

which is the bound on $|a_2|$ as given in the (21).

Next, in order to find the bound on $|a_3|$, by subtracting (28) from (26), we get

$$2.3^n(\mu+2\lambda)a_3 - 2.3^n(\mu+2\lambda)a_2^2 = (1-\beta)(p_2-q_2)$$

or equivalently

$$a_3 = a_2^2 + \frac{(1-\beta)(p_2-q_2)}{2.3^n(\mu+2\lambda)}.$$

Upon substituting the value of a_2^2 from (30), we have

$$a_3 = \frac{(1-\beta)^2(p_1^2+q_1^2)}{2^{2n+1}(\mu+\lambda)^2} + \frac{(1-\beta)(p_2-q_2)}{2.3^n(\mu+2\lambda)}.$$

Applying Lemma 1, once again for the coefficients p_1, p_2, q_1 and q_2, we obtain

$$|a_3| \le \left(\frac{1-\beta}{2^n(\mu+\lambda)}\right)^2 + \frac{2(1-\beta)}{3^n(\mu+2\lambda)}.$$

which is the bound on $|a_3|$ as asserted in (22).

If we take $\mu = 1$ in Theorem 2, we obtain the following corollary.

Corollary 3. *Let f(z) given by (1) be in the class* $N_\Sigma^{n,\mu}(\beta,\lambda)$, $0 \le \beta < 1, \lambda \ge 1$ *and* $n \in \mathbb{N}_0$. *Then*

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{3^n(1+2\lambda)}}$$

and

$$|a_3| \le \left(\frac{1-\beta}{2^{n-1}(1+\lambda)}\right)^2 + \frac{2(1-\beta)}{3^n(1+2\lambda)}.$$

Remark. For $n = 0$ in Corollary 3, provides an improvement of the following estimates obtained by Frasin and Aouf [7].

If we take $\lambda = \mu = 1$ in Theorem 2, we have the following corollary.

Corollary 4. *Let f(z) given by (1) be in the class* $N_{\Sigma}^{n,\mu}(\beta,\lambda)$, $0 \le \beta < 1$ *and* $n \in \mathbb{N}_0$. *Then*

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{3^{n+1}}}$$

and

$$|a_3| \le \left(\frac{1-\beta}{2^n}\right)^2 + \frac{2(1-\beta)}{3^{n+1}}.$$

Remark. For $n = 0$ in Corollary 3, provides an improvement of the following estimates obtained by Srivastava et al. [12].

Remark. For $n = 0$, Theorem 2 reduces to a result in [5].

4 Conclusion

In our present study, we have considered new subclasses $N_{\Sigma}^{n,\mu}(\alpha,\lambda)$ and $N_{\Sigma}^{n,\mu}(\beta,\lambda)$ of bi-univalent functions in the open disk $\mathbb{U}$. We have investigated estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to this classes. we have shown already that the results and corollaries presented in this paper would generalize and improve some recent works of Srivastava et al.[12], Frasin and Aouf [7] and Çağlar et al.[5].

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