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Tensor,symmetric and exterior algebras Kähler modules

Necati Olgun, Mehmet Sahin and Vakkas Ulucay

Department Of Mathematics, Faculty of Arts and Sciences, University of Gaziantep, Gaziantep, Turkey

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Abstract: Let k be an algebraically closed field of characteristic zero, R an affine k-algebra and let $\Omega^{(q)}(R/k)$ denote its universal finite Kähler module of differentials over k. In this paper, we consider the tensor, exterior and symmetric algebras of Kähler modules introduced by H. Osborn [9]. We explore some interesting properties of the algebras of Kähler modules, which have not been considered before.

Keywords: Kähler module, projective module, regular ring, tensor, exterior and symmetric algebra.

1 Introduction

The concept of Kähler module of qth order was introduced by H. Osborn in 1967 [9]. Same notion has appeared in Heyneman et al. [11]. Johnson introduced differential module structures on certain modules of Kähler differentials [4]. Nakai developed the fundamental theories for the calculus of high order differentials[6]. Then many authors studied the properties of Kähler modules [2,7,8].

We will let R be a commutative algebra over an algebraically closed field k with characteristic zero.

When R is a k-algebra, $\Omega^{(q)}(R/k)$ denotes the module of q-th order Kähler differentials of R over k and $\delta^{(q)}_{R/k}$ or $\delta^{(q)}$ denotes the canonical q-th order k-derivation $R \to \Omega^{(q)}(R/k)$ of R. The pair $\{\Omega^{(q)}(R/k), \delta^{(q)}_{R/k}\}$ has the universal mapping property with respect to the q-th order k-derivations of R. $I_{R/k}$ or I_R denotes the kernel of the canonical mapping $R \otimes_k R \to R$ $(a \otimes b \to ab)$. $\Omega^{(q)}(R/k)$ is identified with I_R/I_R^{q+1}.

$\Omega^{(q)}(R/k)$ is a generated by the set $\{\delta^{(q)}(r): \ r \in R\}$. Hence if R is finitely generated k-algebra,then $\Omega^{(q)}(R/k)$ will be a finitely generated R-module.

In this paper we will study some interesting properties of tensor, exterior and symmetric algebras of Kähler modules. Our study presents some important and new results which are listed below:
Let R be an affine domain with dimension s. Then

(i) $\Omega^{(q)}(R/k)$ is a free R-module if and only if $\otimes^n(\Omega^{(q)}(R/k))$ is a free R-module.
(ii) $\Omega^{(q)}(R/k)$ is a free R-module if and only if $\wedge^n(\Omega^{(q)}(R/k))$ is a free R-module.
(iii) $\Omega^{(q)}(R/k)$ is a free R-module if and only if $S^n(\Omega^{(q)}(R/k))$ is a free R-module.

Let R be an affine local domain. Then the following conditions are equivalent.

(i) R is a regular local ring,
(ii) $\otimes^n(\Omega^{(1)}(R/k))$ is a free R-module,

* Corresponding author e-mail: olgun@gantep.edu.tr

(iii) $\wedge^n(\Omega^{(1)}(R/k))$ is a free R-module,
(iv) $S^n(\Omega^{(1)}(R/k))$ is a free R-module.

The fundamental definitions and properties of the tensor,exterior and symmetric algebras are presented in Sec.2. Our main results begin with Example 2.

2 Preliminary notes

In this section, we review some basic definitions and results of the algebras. Let M and P are R-modules. Recall that an R-bilinear map $\mu : M^n \to P$ is called alternating if $\mu(x_1,...,x_n) = 0$ whenever two successive arguments x_i, x_{i+1} are equal. From this condition one easily shows that transposing two successive arguments reverses the sign of $\mu(x_1,...,x_n) = 0$, from which it follows that any permutation of the arguments multiplies the value by the sign of that permutation; hence, from the original condition we see that the value is zero if any two terms are equal. We recall likewise that an R-bilinear map $M^n \to P$ is called symmetric if it is unchanged under transposing successive arguments; equivalently, if it is unchanged under all permutations of the arguments. Let us now define some universal maps.

Definition 1. [3] *Let M be an R-module. By $\bigotimes^n M$ we shall denote the R-module with a universal R-bilinear map of M^n into it, written $(x_1,...,x_n) \to x_1 \otimes ... \otimes x_n$ This module is called the n-fold tensor power of M.*

By $\bigwedge^n M$ we shall denote the R-module with a universal alternating R-bilinear map of M^n into it, written $(x_1,...,x_n) \to x_1 \wedge ... \wedge x_n$. This module is called the n-fold exterior power of M.

By $S^n M$ we shall denote the R-module with a universal symmetric R-bilinear map of M^n into it, written $(x_1,...,x_n) \to x_1...x_n$. This module is called the n-fold symmetric power of M.

Each of these modules can clearly be constructed directly by generators and relations. $\bigotimes^n M$ can also be constructed as $M \otimes_R (M \otimes_R (...))$ (with n M' s), while $\bigwedge^n M$ and $S^n M$ can be constructed as factor-modules of $\bigotimes^n M$ by the submodules generated by all elements of the forms

$$x_1 \otimes ... \otimes y \otimes y \otimes ... \otimes x_n,$$

respectively

$$x_1 \otimes ... \otimes y \otimes z \otimes ... \otimes x_n - x_1 \otimes ... \otimes z \otimes y \otimes ... \otimes x_n.$$

(where each term is $x_1 \otimes ... \otimes x_n$ modified only in two successive places).

Let $\bigotimes^1 M$, $\bigwedge^1 M$ and $S^1 M$ are all identified with M, while $\bigotimes^0 M$, $\bigwedge^0 M$ and $S^0 M$ are all identified with R.

The universal properties defining our three constructions can be conveniently restated as follows:
An R-module homomorphism with domain $\bigotimes^n M$ is uniquely determined by specifying it on the elements $x_1 \otimes ... \otimes x_n$ for all $x_1,...,x_n \in M$, in such a way that the value is an R-bilinear function of $x_1,...,x_n$.

An R-module homomorphism with domain $\bigwedge^n M$ is uniquely determined by specifying it on the elements $x_1 \wedge ... \wedge x_n$ for all $x_1,...,x_n \in M$, so that the result is an alternating R-bilinear function of $x_1,...,x_n$.

An R-module homomorphism with domain $S^n M$ is uniquely determined by specifying it on the elements $x_1...x_n$ for all $x_1,...,x_n \in M$, so that the value is a symmetric R-bilinear function of $x_1,...,x_n$.

Each of $\bigotimes^n$, $\bigwedge^n$ and S^n may be made into a functor. Given $f : M \to N$, one defines, for example, $\wedge^n f : \wedge^n M \to \wedge^n N$ as the unique module homomorphism taking $x_1 \wedge ... \wedge x_n$ to $f(x_1) \wedge ... \wedge f(x_n)$ for all $x_1, ..., x_n \in M$. This homomorphism exists because the latter expression in $x_1, ..., x_n$ is alternating and R- bilinear.

Lemma 1. [3] *Let M be a R-module and m, n nonnegative integers. Then there exist a unique R-bilinear map $\bigotimes^m M \times \bigotimes^n M \to \bigotimes^{m+n} M$ which carries each pair $(x_1 \otimes ... \otimes x_m, x_{m+1} \otimes ... \otimes x_{m+n})$ to $x_1 \otimes ... \otimes x_{m+n}$, a unique R-bilinear map $\bigwedge^m M \times \bigwedge^n M \to \bigwedge^{m+n} M$ which carries each pair $(x_1 \wedge ... \wedge x_m, x_{m+1} \wedge ... \wedge x_{m+n})$ to $x_1 \wedge ... \wedge x_{m+n}$, and a unique R-bilinear map $S^m M \times S^n M \to S^{m+n} M$ which carries each pair $(x_1 ... x_m, x_{m+1} ... x_{m+n})$ to $x_1 ... x_{m+n}$.*

Theorem 1. [3] *The multiplications on $\bigotimes M$, $\bigwedge M$ and SM defined above are structures of (associative unital) an R-algebra.*

The algebra $\bigotimes M = \bigoplus_n (\bigotimes^n M)$, called the tensor algebra on M, is universal among R-algebras given with R-module homomorphisms of M into them.

The algebra $\bigwedge M = \bigoplus_n (\bigwedge M)$, called the exterior algebra on M, is universal among R-algebras given with R-module homomorphisms of M into them such that the images of all elements of M have zero square.

The algebra $SM = \bigoplus_n (S^n M)$, called the symmetric algebra on M, is universal among R-algebras given with R-module homomorphisms of M into them such that the images of elements of M commute with one another, and is also universal among all commutative R-algebras given with R-module homomorphisms of M into them.

Theorem 2. [1] *Let M be a free R-module on a basis X. Then: $\bigotimes M$ is the free associative R-algebra $\bigotimes M$, equivalently, the semigroup algebra on the free semigroup $< X >$. It has the set of all products $x_1 \otimes ... \otimes x_n$ (or, using ordinary multiplicative notation, as is common when this ring is regarded as a free algebra or a semigroup algebra,the set of products $x_1 ... x_n$) an an R-module basis for $x_1, ..., x_n \in X$. Thus, if X is a finite set $\{x_1, ..., x_r\}$, then for each n, $dim_R(\bigotimes^n M) = r^n$.*

$\bigwedge M$ may be presented by the generating set X and the relations $x \wedge x = 0$, $x \wedge y + y \wedge x = 0$ $(x, y \in X$). If a total ordering "$\leq$" is chosen on X, then a R-module basis for $\bigwedge M$ is given by those products $x_1 \wedge ... \wedge x_n$ with $x_1 < ... < x_n \in X$.

In particular, if X is a finite set $\{x_1, ..., x_r\}$, then a basis is given by those products $x_{i_1} \wedge ... \wedge x_{i_n}$ with $1 \leq i_1 < ... < i_n \leq r$, hence for each n, $dim_R(\bigwedge^n M) = \binom{r}{n}$.

(SM) may be presented by the generating set X, and relations $xy = yx$ $(x, y \in X$), and is the (commutative) polynomial algebra $R[X]$, equivalently, the free commutative R-algebra on X, equivalently, the semigroup algebra on the free commutative semigroup on X. If a total ordering "$\leq$" is chosen on X, then an R-module basis for SM is given by those products $x_1 ... x_n$ with $x_1 \leq ... \leq x_n \in X$. If X is a finite set $\{x_1 ... x_r\}$, then the elements of this basis can be written $x_1^{i_1} ... x_r^{i_r}$ with $i_1, ..., i_r \geq 0$, and for each n, $dim_R(S^n M) = \binom{r+n-1}{r-1}$.

Proposition 1. [1] *Let T be an R-algebra and let M be an R-module. Then*

$$S^n(M) \otimes_R T \simeq S^n(M \otimes_R T).$$

Proposition 2. [3] *Let M and N be R-modules. Then $\bigwedge(M \bigoplus N) \cong \bigoplus_{m+n=p} (\bigwedge^m M) \bigotimes (\bigwedge^n N)$, via the map $x_1 \wedge ... \wedge x_n, y_1 \wedge ... \wedge y_n \leftarrow (x_1 \wedge ... \wedge x_n) \otimes (y_1 \wedge ... \wedge y_n)$, and likewise $S^p(M \bigoplus N) \cong \bigoplus_{m+n=p} (S^m M) \otimes (S^n N)$, via the map $x_1 ... x_n, y_1 ... y_n \leftarrow (x_1 ... x_n) \otimes (y_1 ... y_n)$.*

You can find previous results and other results about the algebras in [5],[12] and [13].

3 The tensor, exterior and symmetric algebras of Kähler modules

Let R be an affine k-algebra of dimension $s \geq 1$ and let $\Omega^{(q)}(R/k)$ denote the module of q-th order Kähler differentials of R over k. Then the rank of $\Omega^{(q)}(R/k)$ is $\binom{q+s}{s} - 1$.

Example 1. If $R = k[x,y]$ is a polynomial ring, then $\Omega^{(1)}(R/k)$ is a free R-module of rank 2 with the basis $\{\delta^{(1)}(x), \delta^{(1)}(y)\}$.

Proposition 3. *([6pp.17]) Let $R = k[x_1, ..., x_s]$ be a polynomial algebra of dimension s. Then $\Omega^{(q)}(R/k)$ is a free R-module of rank $\binom{q+s}{s} - 1$ with basis $\{\delta^{(q)}(x_1^{i_1} .....x_s^{i_s}) : i_1 + ... + i_s \leq q\}$.*

Proposition 4. *Let R be an affine k-algebra. If R is a regular ring, then $\Omega^{(q)}(R/k)$ is a projective R-module.*

Theorem 3. *Let R be an affine k-algebra. If R is a regular ring, then $\Omega^{(q)}(R/k)$ is a projective R-module.*

Proof. This is immediate from the decomposition $J^q(R/k) \cong \Omega^{(q)}(R/k) \oplus R$ and from proposition 3.

Theorem 4. *Let R be an affine k-algebra. R is regular ring if and only if $\Omega^{(1)}(R/k)$ is a projective R-module.*

Example 2. Let $R = k[x,y]$ is a polynomial algebra. Then

(i) $\bigotimes^2(\Omega^{(1)}(R/k))$ is a free R-module of rank 4 with the basis

$$\{\delta^{(1)}(x) \otimes \delta^1(x), \delta^{(1)}(x) \otimes \delta^{(1)}(y), \delta^{(1)}(y) \otimes \delta^{(1)}(x), \delta^{(1)}(y) \otimes \delta^{(1)}(y)\}.$$

(ii) $\bigwedge^2(\Omega^{(1)}(R/k))$ is a free R-module of rank 1 with the basis $\{\delta^{(1)}(x) \otimes \delta^{(1)}(y)\}$.

(iii) $S^2(\Omega^{(1)}(R/k))$ is a free R-module of rank 3 with the basis $\{\delta^{(1)}(x) \otimes \delta^1(x), \delta^{(1)}(x) \otimes \delta^{(1)}(y), \delta^{(1)}(y) \otimes \delta^{(1)}(y)\}$.

Proposition 5. *Let $R = k[x_1, ..., x_s]$ be a local k-algebra of dimension 1. Then R is a regular ring if and only if $\bigwedge^2(\Omega^{(1)}(R/k))$ is zero.*

Proof. Assume that R is a regular local k-algebra of dimension 1. Then by Theorem 3.4, $\Omega^{(1)}(R/k)$ is a free R-module of rank 1. This implies that $\bigwedge^2(\Omega^{(q)}(R/k))$ is zero by Theorem 4.

Conversely, assume that $\bigwedge^2(\Omega^{(q)}(R/k))$ is zero. Let m be the maximal ideal of R. Then by Proposition 2.5, we have $\bigwedge^2(\Omega^{(1)}(R/k)) \otimes_R R/m$ is isomorphic to $\bigwedge^2(\Omega^{(1)}(R/k) \otimes_R R/m) \cong \bigwedge^2(\frac{\Omega^{(1)}(R/k)}{m\Omega^{(1)}(R/k)}) = 0$. Since $\bigwedge^2(\frac{\Omega^{(1)}(R/k)}{m\Omega^{(1)}(R/k)})$ is a vector space over R/m, it follows that either $\bigwedge^2(\frac{\Omega^{(1)}(R/k)}{m\Omega^{(1)}(R/k)}) = 0$ or $dim_{R/m}\bigwedge^2(\frac{\Omega^{(1)}(R/k)}{m\Omega^{(1)}(R/k)}) = 1$.

If we have $\bigwedge^2(\frac{\Omega^{(1)}(R/k)}{m\Omega^{(1)}(R/k)}) = 0$ then $\Omega^{(1)}(R/k) = m\Omega^{(1)}(R/k)$ and so by Nakayama's Lemma, $\Omega^{(1)}(R/k) = 0$, which is a contradiction. So, the rank of $\Omega^{(1)}(R/k)$ is equal to the number of minimal generators of $\Omega^{(1)}(R/k)$. That is $\Omega^{(1)}(R/k)$ is a free R-module of rank 1. By Theorem 4, R is a regular ring.

Proposition 6 does not hold for the 2-fold tensor power of $\Omega^{(1)}(R/k)$ and the 2-fold symmetric power of $\Omega^{(1)}(R/k)$. For instance, if we have $R = k[x]$ is a polynomial algebra of dimension 1, then $\bigotimes^2(\Omega^{(1)}(R/k))$ and $S^2(\Omega^{(1)}(R/k))$ are never zero. $\bigotimes^2(\Omega^{(1)}(R/k))$ and $S^2(\Omega^{(1)}(R/k))$ are free R-modules of ranks 1.

Corollary 1. *Let $R = k[x_1, ..., x_s]$ be a polynomial algebra of dimension s. Then*

(i) *$\bigotimes^2(\Omega^{(q)}(R/k))$ is a free R-module of rank t^2.*

(ii) *$\bigwedge^2(\Omega^{(q)}(R/k))$ is a free R-module of rank* $\binom{t}{2}$.

(iii) *$S^2(\Omega^{(q)}(R/k))$ is a free R-module of rank* $\binom{t+1}{t-1}$ *where* $t = \binom{q+s}{s} - 1$.

Proof. It is clear by Theorem 3, Theorem 4 and Proposition 2.

Proposition 6. *If R is a regular local ring dimension of s, then*

(i) *$\bigotimes^n(\Omega^{(q)}(R/k))$ is a free R-module of rank t^n.*

(ii) *$\bigwedge^n(\Omega^{(q)}(R/k))$ is a free R-module of rank* $\binom{t}{n}$.

(iii) *$S^n(\Omega^{(q)}(R/k))$ is a free R-module of rank* $\binom{t+n-1}{t-1}$ *where* $t = \binom{q+s}{s} - 1$.

Proof. It is clear that by Theorem 3.3, Theorem 2.4. and Proposition 3.2.

Corollary 2. *If R is a regular ring, then*

(i) *$\bigotimes^n(\Omega^{(q)}(R/k))$ is a projective R-module.*
(ii) *$\bigwedge^n(\Omega^{(q)}(R/k))$ is a projective R-module.*
(iii) *$S^2(\Omega^{(q)}(R/k))$ is a projective R-module.*

Proposition 7. *Let R be an affine domain with dimension s. Then*

(i) *$\Omega^{(q)}(R/k)$ is a free R-module if and only if $\bigotimes^2(\Omega^{(q)}(R/k))$ is a free R-module.*
(ii) *$\Omega^{(q)}(R/k)$ is a free R-module if and only if $\bigwedge^2(\Omega^{(q)}(R/k))$ is a free R-module.*
(iii) *$\Omega^{(q)}(R/k)$ is a free R-module if and only if $S^2(\Omega^{(q)}(R/k))$ is a free R-module.*

Proof. We prove the only (iii). Others can be proved similarly. Without loss of generality, we may assume that R is a local domain of dimension s. Suppose that $\Omega^{(q)}(R/k)$ is free R-module. By Theorem 4, $S^2(\Omega^{(q)}(R/k))$ is a free R-module.

Conversely, suppose that $S^2(\Omega^{(q)}(R/k))$ is a free R-module. If $dimR = s$, then the rank of $\Omega^{(q)}(R/k)$ is $\binom{q+s}{s} - 1$. Let $\binom{q+s}{s} - 1 = t$. Then the rank of $S^2(\Omega^{(q)}(R/k))$ is $\binom{t+1}{t-1}$. Let m be the maximal ideal of R. Then $S^2(\Omega^{(q)}(R/k)) \otimes_R R/m$ is an R/m vector space of dimension $\binom{t+1}{t-1}$. $S^2(\Omega^{(q)}(R/k)) \otimes_R R/m$ is isomorphic to $S^2(\frac{\Omega^{(q)}(R/k)}{m\Omega^{(q)}(R/k)})$. Then $S^2(\frac{\Omega^{(q)}(R/k)}{m\Omega^{(q)}(R/k)})$is an R/m vector space of dimension $\binom{t+1}{t-1}$ if and only if $\frac{\Omega^{(q)}(R/k)}{m\Omega^{(q)}(R/k)}$ is an R/m vector space of dimension t. Hence $\frac{\Omega^{(q)}(R/k)}{m\Omega^{(q)}(R/k)}$ is an R/m vector space of dimension t if and only if the number of minimal generators of $\Omega^{(q)}(R/k)$ is t. The rank of $\Omega^{(q)}(R/k)$ was t. Therefore we show that $\Omega^{(q)}(R/k)$ is a free R-module as required.

Corollary 3. *Let R be an affine domain with dimension s. Then*

(i) *$\Omega^{(q)}(R/k)$ is a projective R-module if and only if $\bigotimes^2(\Omega^{(q)}(R/k))$ is a projective R-module.*
(ii) *$\Omega^{(q)}(R/k)$ is a projective R-module if and only if $\bigwedge^2(\Omega^{(q)}(R/k))$ is a projective R-module.*
(iii) *$\Omega^{(q)}(R/k)$ is a projective R-module if and only if $S^2(\Omega^{(q)}(R/k))$ is a projective R-module.*

Theorem 5. *Let R be an affine local domain. Then the following conditions are equivalent:*

(i) *R is a regular local ring.*
(ii) $\bigotimes^2(\Omega^{(1)}(R/k))$ *is a free R-module.*
(iii) $\bigwedge^2(\Omega^{(1)}(R/k))$ *is a free R-module.*
(iv) $S^2(\Omega^{(1)}(R/k))$ *is a free R-module.*

Proof. This is immediate from Theorem 5 and Proposition 7.

Hence, we obtain the following corollary.

Corollary 4. *Let R be an affine domain. Then the following conditions are equivalent.*

(i) *R is a regular ring.*
(ii) $\bigotimes^2(\Omega^{(1)}(R/k))$ *is a projective R-module.*
(iii) $\bigwedge^2(\Omega^{(1)}(R/k))$ *is a projective R-module.*
(iv) $S^2(\Omega^{(1)}(R/k))$ *is a projective R-module.*

4 Conclusion

The main purpose of this paper is to introduce the tensor, exterior and symmetric algebras of K¨ahler modules. We were investigated some interesting properties of the algebras of K¨ahler modules.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Eisenbud D., Commutative Algebra with a View Toward Algbraic Geometry, Springer Verlag, New York (1995).
[2] Erdogan,A., Differential Operators and Their Universal Modules, Phd. Thesis, University of Leeds,(1993).
[3] George M.Bergman, Tensor algebras, exterior algebras and symmetric algebras, http://math.berkeley.edu/ gbergman, (1997).
[4] Johnson J. Kähler Differential and differential algebra, The Annals of Mathematics, Second series, Vol.89, No 1, 92–98,(1969).
[5] McConnell, J.C. and Rabson, J.C. , Noncommutative Noetherian Rings, Wiley-Interscience,(1987).
[6] Nakai, Y., High Order Derevations, Osaka J. Math.7,1–27,(1970)
[7] Olgun, N., The Universal Differential Modules of Finitely Generated Algebras, Ph.D Thesis, Hacettepe University, (2005).
[8] Olgun, N. and Erdogan, A. , Some Results On Universal Modules, Int. Mat. For. 1(15), 707-712, (2006).
[9] Osborn, H., Modules of Differentials I, Math. Ann.170, 221-244, (1967).
[10] Sweedler, M.E., Groups of Simple Algebras, Publ. Math. No:44. (1975).
[11] Sweedler ,M.E. and Heyneman, R.G.,Affine Hopf Algebras, J.algebra 13,192–241,(1969).
[12] Bourbaki, N., Elements of Mathematics, Algebra I.Chapters1–3, Springer-Verlag, (1989).
[13] Mac Lane, S. and Birkhoff, G., Algebra, AMS Chelsea, (1999).