Generalized \((k, \mu)\)-Space forms and Ricci solitons

D. L. Kiran Kumar, H. G. Nagaraja and Uppara Manjulamma

Abstract

In this paper, we study Ricci-semisymmetric and Ricci pseudo-symmetric generalized \((k, \mu)\)-space forms along with characterization of generalized \((k, \mu)\)-space forms satisfying the curvature conditions \(Q(g, S) = 0\) and \(Q(S, R) = 0\). Further, we study Ricci solitons in generalized \((k, \mu)\)-space forms and obtained some interesting results.

Keywords and 2010 Mathematics Subject Classification

Keywords: Generalized \((k, \mu)\)-Space form, Ricci-semisymmetric, Ricci pseudo-symmetric, Ricci solitons, shrinking, expanding, steady.

MSC: 53D10, 53D15

1. Introduction

In [1], the authors generalized the notion of Sasakian space form defined generalized Sasakian space form as a contact metric manifold \((M, \phi, \xi, \eta, g)\) whose curvature tensor \(R\) satisfies

\[
R(X, Y)Z = f_1 \{g(Y, Z)X - g(X, Z)Y\}
\]

\[
+ f_2 \{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\}
\]

\[
+ f_3 \{\eta(X)\eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\},
\]

for any vector fields \(X, Y, Z\), where \(f_1, f_2, f_3\) are smooth functions on \(M\). As a generalization of the notion of \((k, \mu)\)-space form, Carriazo et al [4] introduced generalized \((k, \mu)\)-space form as a contact metric manifold \((M, \phi, \xi, \eta, g)\) whose curvature tensor \(R\) satisfies

\[
R(X, Y)Z = f_1 \{g(Y, Z)X - g(X, Z)Y\}
\]

\[
+ f_2 \{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\}
\]

\[
+ f_3 \{\eta(X)\eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}
\]

\[
+ f_4 \{g(Y, Z)hX - g(X, Z)hY + g(hY, Z)X - g(hX, Z)Y\}
\]

\[
+ f_5 \{g(hX, Z)hX - g(hX, Z)hY + g(\phi hX, Z)\phi hY - g(\phi hY, Z)\phi hX\}
\]

\[
+ f_6 \{\eta(X)\eta(Y)\eta(Z)hX + g(hX, Z)\eta(Y)\xi - g(hY, Z)\eta(X)\xi\},
\]

where \(f_1, f_2, f_3, f_4, f_5, f_6\) are smooth functions on \(M\) and \(2h = L_\phi, L\) is the usual Lie derivative. They proved that the generalized Sasakian space form and the generalized \((k, \mu)\)-space form share some properties and identities in common. Further the authors established that the generalized \((k, \mu)\)-space forms reduce to generalized \((k, \mu)\) spaces for \(k = f_1 - f_3, \mu = f_4 - f_5\) and to \((k, \mu)\) spaces greater than or equal to 5 with \(k = -f_6\) and \(\mu = 1 - f_6\). \((k, \mu)\)-space form have been studied widely by several authors like [3, 13, 7, 19, 18, 21, 23] and various others.

Let \((M, g)\) be a Riemannian manifold with the Riemannian metric \(\nabla\). A tensor field \(F : \chi(M) \times \chi(M) \times \chi(M) \longrightarrow \chi(M)\) of type \((1, 3)\) is said to be curvature-like if it has the properties of \(R\). For example, the tensor \(R\) given by

\[
R(X, Y)Z = (X \wedge A Y)Z = A(Y, Z)X - A(X, Z)Y,
\]
where \(X, Y, Z \in \chi(M)\), \(\chi(M)\) is the set of all differentiable vector fields on \(M\), \(A\) is the symmetric \((0, 2)\)-tensor, \(R\) is the Riemannian curvature tensor of type \((1, 3)\) and \(\nabla\) is the Levi-Civita connection. For a \((0, k)\)-tensor field \(T, k \leq 1\), on \((M, g)\), we define the tensor \(R \cdot T\) and \(Q(g, T)\) by

\[
(R(X, Y) \cdot T)(X_1, X_2, \ldots, X_k) = -T(R(X, Y)X_1, X_2, \ldots, X_k) - T(X_1, R(X, Y)X_2, \ldots, X_k) \\
\ldots - T(X_1, X_2, \ldots, R(X, Y)X_k)
\]  

and

\[
Q(g, T)(X_1, X_2, \ldots, X_k, Y) = -T((X \wedge Y)X_1, X_2, \ldots, X_k) - T(X_1, (X \wedge Y)X_2, \ldots, X_k) \\
\ldots - T(X_1, X_2, \ldots, (X \wedge Y)X_k),
\]

respectively [24]. If the tensors \((R \cdot S)\) and \(Q(g, S)\) are linearly dependent, then \(M\) is called Ricci pseudo-symmetric [24]. Which is equivalent to

\[
(R \cdot S) = fQ(g, S),
\]

holding on the set \(U_S = \{x \in M : S \neq 0\ \text{at}\ x\}\), where \(f\) is some function on \(U_S\). Also if the tensors \(R \cdot R\) and \(Q(S, R)\) are linearly dependent, then \(M\) is said to be Ricci generalized pseudo-symmetric [24]. This is equivalent to

\[
R \cdot R = fQ(S, R).
\]

In [12], Kowaleczky studied semi-Riemannian manifolds satisfying \(Q(S, R) = 0\) and \(Q(g, S) = 0\), where \(S, R\) are the Ricci tensor and curvature tensor respectively. De et al. [6, 14] studied Ricci pseudo-symmetric and Ricci generalized pseudo-symmetric \(P\)-Sasakian manifolds and generalized \((k, \mu)\)-paracontact metric manifolds.

Ricci soliton, introduced by Hamilton [8] are natural generalizations of the Einstein metrics and is defined on a Riemannian manifold \((M, g)\). A Ricci soliton \((g, V, \lambda)\) defined on \((M, g)\) as

\[
(L_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0,
\]

where \(L_V\) denotes the Lie-derivative of Riemannian metric \(g\) along a vector field \(V\), \(\lambda\) be a constant and \(X, Y\) are arbitrary vector fields on \(M\). A Ricci soliton is said to shrinking or steady or expanding to the extent that \(\lambda\) is negative, zero or positive respectively. Ricci solitons have been considered broadly with regards to contact geometry; we may refer to [22, 5, 20, 9, 16, 17, 15, 11] and references therein.

The paper is organized as follows: The section 2 contains some basic results on almost contact geometry and generalized \((k, \mu)\)-space forms. Section 3 deals with the curvature conditions like Ricci-semisymmetric, Ricci pseudo-symmetric, \(Q(g, S) = 0\) and \(Q(S, R) = 0\) on generalized \((k, \mu)\)-space forms. Also we study Ricci solitons in generalized \((k, \mu)\)-space forms and obtained some interesting results.

### 2. Preliminaries

In this section, we recall some general definitions and fundamental equations are presented which will be utilized later. A \((2n + 1)\)-dimensional smooth manifold \(M\) is said to be contact if it has a global 1-form \(\eta\) such that \(\eta \wedge (d\eta)^n \neq 0\) on \(M\). Given a contact 1-form \(\eta\) there always exists a unique vector field \(\xi\) such that \((d\eta)(\xi, X) = 0\). Polarization of \(d\eta\) on the contact subbundle \(D\) (defined by \(D = 0\)), yields a Riemannian metric \(g\) and a \((1, 1)\)-tensor field \(\phi\) such that

\[
\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad g(X, \xi) = \eta(X), \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0,
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\]

\[
g(X, \phi Y) = d\eta(X, Y), \quad g(X, \phi Y) = -g(Y, \phi X),
\]

for all vector fields \(X, Y\) on \(M\). In a contact metric manifold, we characterize a \((1, 1)\) tensor field \(h\) by \(h = \frac{1}{2}L_\xi \phi\), where \(L\) signifies the Lie differentiation. At this point \(h\) is symmetric and satisfies \(h\phi = -\phi h\). Likewise we have \(Tr \cdot \tilde{h} = Tr \cdot \phi h = 0\).
and \( h\xi = 0 \).
Moreover, if \( \nabla \) signifies the Riemannian connection of \( g \), then the following relation holds:

\[
\nabla_X \xi = -\phi X - \phi hX.
\]

(12)

In a \((k, \mu)\)-contact metric manifold the following relations hold [2] [10]:

\[
h^2 = (k-1)\phi^2, \quad k \leq 1,
\]

(13)

\[
(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX),
\]

(14)

\[
(\nabla_X h)Y = [(1-k)g(X, \phi Y) - g(X, \phi hY)]\xi - \eta(Y)[(1-k)\phi X + \phi hX] - \mu \eta(X)\phi hY.
\]

(15)

Also in a \((2n+1)\)-dimensional generalized \((k, \mu)\)-space form, the following relations hold.

\[
R(X, Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y \}
+ (f_4 - f_6)\{\eta(Y)hX - \eta(X)hY \},
\]

(16)

\[
QX = \{2nf_1 + 3f_2 - f_3\}X + \{(2n-1)f_4 - f_6\}hX - \{3f_2 + (2n-1)f_3\} \eta(X)\xi,
\]

(17)

\[
S(X, Y) = \{2nf_1 + 3f_2 - f_3\}g(X, Y) + \{(2n-1)f_4 - f_6\}g(hX, Y) - \{3f_2 + (2n-1)f_3\} \eta(X)\eta(Y),
\]

(18)

\[
S(X, \xi) = 2n(f_1 - f_3)\eta(X),
\]

(19)

\[
r = 2n\{(2n+1)f_1 + 3f_2 - f_3\},
\]

(20)

where \( Q \) is the Ricci operator, \( S \) is the Ricci tensor and \( r \) is the scalar curvature of \( M(f_1, \ldots, f_6) \).

### 3. Generalized \((k, \mu)\)-Space forms and Ricci solitons

A generalized \((k, \mu)\)-space form is said to be Ricci-semisymmetric if its Ricci tensor \( S \) satisfies the condition \( R \cdot S = 0 \). Then we have

\[
S(R(X, Y)U, V) + S(U, R(X, Y)V) = 0.
\]

(21)

Taking \( X = U = \xi \) in the equation (21), we get

\[
S(R(\xi, Y)\xi, V) + S(\xi, R(\xi, Y)V) = 0.
\]

(22)

Using (16) and (19) in (22), we obtain

\[
(f_1 - f_3)\{2n(f_1 - f_3)g(Y, V) - S(Y, V)\}
+ (f_4 - f_6)\{2n(f_1 - f_3)g(hY, V) - S(hY, V)\} = 0.
\]

(23)

Replacing \( Y \) by \( hY \) in (23) and using (13), we get

\[
(f_1 - f_3)\{2n(f_1 - f_3)g(hY, V) - S(hY, V)\}
- (k-1)(f_4 - f_6)\{2n(f_1 - f_3)g(Y, V) - S(Y, V)\} = 0.
\]

(24)

Eliminating \( g(hY, V) \) and \( S(hY, V) \) from (23) and (24), we get

\[
\{(k-1)(f_4 - f_6)^2 + (f_1 - f_3)^2\}\{2n(f_1 - f_3)g(Y, V) - S(Y, V)\} = 0.
\]

(25)

Now for \( k = 1 \), either \( f_1 = f_3 \) or \( S(Y, V) = 2n(f_1 - f_3)g(Y, V) \).

On the other hand for \( k < 1 \), either \( S(Y, V) = 2n(f_1 - f_3)g(Y, V) \) or

\[
(f_1 - f_3)^2 = (1-k)(f_4 - f_6)^2.
\]

(26)

Then from (26), we have \( f_1 = f_3 \) implies \( f_4 = f_6 \).

Thus from the above discussions we state the following:
Theorem 1. If a \((2n+1)\)-dimensional generalized \((k, \mu)\)-space form \(M(f_1, \ldots, f_6)\) with \(f_1 \neq f_3\) is Ricci-semisymmetric, then space form is an Einstein manifold.

Suppose the generalized \((k, \mu)\)-space form satisfying the curvature condition \(Q(S, R) = 0\). Then we have

\[
(X \wedge SY \cdot R)(U, V)W = 0. \tag{27}
\]

Using (7) in (27), we obtain

\[
\]
\[
\]
\[
- S(Y, R(U, V)W)X + S(X, W)R(U, V)Y = 0. \tag{28}
\]

Replacing \(X = U = \xi\) in (28), we get

\[
S(Y, R(\xi, V)W)\xi - S(\xi, R(\xi, V)W)Y - S(\xi, R(V, \xi)W)
\]
\[
+ S(\xi, R(\xi, Y)W) - S(\xi, R(\xi, V)W) + S(\xi, R(\xi, Y)W)
\]
\[
- S(\xi, W)R(\xi, V)\xi + S(\xi, W)R(\xi, V)Y = 0. \tag{29}
\]

Using (16) and (19) in (29), we obtain

\[
- (f_1 - f_3)\eta(W)S(Y, V)\xi - (f_4 - f_6)\eta(W)S(Y, hV)\xi
\]
\[
- 2n(f_1 - f_3)^2g(V, W)Y - 2n(f_1 - f_3)(f_4 - f_6)g(V, hW)Y
\]
\[
+ 2n(f_1 - f_3)R(Y, V)W + 2n(f_1 - f_3)^2g(V, W)\eta(V)Y
\]
\[
+ 2n(f_1 - f_3)(f_4 - f_6)g(V, hW)\eta(V)\xi - (f_1 - f_3)S(Y, W)\eta(V)\xi
\]
\[
- 2n(f_1 - f_3)\eta(V)\eta(W)hY + (f_1 - f_3)S(Y, W)V
\]
\[
+ (f_4 - f_6)S(Y, W)hV + 2n(f_1 - f_3)^2g(V, Y)\eta(W)\xi
\]
\[
+ 2n(f_1 - f_3)(f_4 - f_6)g(V, hY)\eta(W)\xi = 0. \tag{30}
\]

Taking inner product with \(Z\), we obtain

\[
- (f_1 - f_3)\eta(W)S(Y, V)\eta(Z) - (f_4 - f_6)\eta(W)S(Y, hV)\eta(Z)
\]
\[
- 2n(f_1 - f_3)^2g(V, W)g(Y, Z) - 2n(f_1 - f_3)(f_4 - f_6)g(V, hW)g(Y, Z)
\]
\[
+ 2n(f_1 - f_3)R(Y, V)WZ + 2n(f_1 - f_3)^2g(Y, W)\eta(V)\eta(Z)
\]
\[
+ 2n(f_1 - f_3)(f_4 - f_6)g(Y, hW)\eta(V)\eta(Z) - (f_1 - f_3)S(Y, W)g(Y, V)\eta(Z)
\]
\[
- 2n(f_1 - f_3)\eta(V)\eta(W)g(hY, Z) + (f_1 - f_3)S(Y, W)g(V, Z)
\]
\[
+ (f_4 - f_6)S(Y, W)g(hV, Z) + 2n(f_1 - f_3)^2g(V, Y)\eta(W)\eta(Z)
\]
\[
+ 2n(f_1 - f_3)(f_4 - f_6)g(V, hY)\eta(W)\eta(Z) = 0. \tag{31}
\]

Let \(\{e_i\}, i = 1, 2, 3, \ldots, (2n + 1)\) be a local orthonormal basis in the tangent space \(T_pM\) at each point \(p \in M\). Taking \(V = W = e_i\) in (31) and summing over \(i = 1, 2, 3, \ldots, (2n + 1)\), then we have

\[
(2n + 1)(f_1 - f_3)\{2n(f_1 - f_3)g(Y, Z) - S(Y, Z)\}
\]
\[
+ (f_4 - f_6)\{2n(f_1 - f_3)g(hY, Z) - S(hY, Z)\} = 0. \tag{32}
\]

Replacing \(Y\) by \(hY\) in (32) and using (13), we get

\[
(2n + 1)(f_1 - f_3)\{2n(f_1 - f_3)g(hY, Z) - S(hY, Z)\}
\]
\[
- (k - 1)(f_4 - f_6)\{2n(f_1 - f_3)g(Y, Z) - S(Y, Z)\} = 0. \tag{33}
\]

Multiplying (32) by \((2n + 1)(f_1 - f_3)\) and (33) by \((f_4 - f_6)\) and subtracting from (32) to (33), we get

\[
\{(k - 1)(f_4 - f_6)^2 + (2n + 1)^2(f_1 - f_3)^2\}\{2n(f_1 - f_3)g(Y, Z) - S(Y, Z)\} = 0. \tag{34}
\]
Now for $k = 1$, either $f_1 = f_3$ or $S(Y,V) = 2n(f_1 - f_3)g(Y,V)$. On the other hand for $k < 1$, either $S(Y,V) = 2n(f_1 - f_3)g(Y,V)$ or

$$(2n+1)^2 (f_1 - f_3)^2 = (1-k)(f_4 - f_6)^2.$$  

Then from (35), we have $f_1 = f_3$ implies $f_4 = f_6$.

Thus we can state the following:

**Theorem 2.** If a $(2n+1)$-dimensional generalized $(k,\mu)$-space form $M_{\{f_1,\ldots,f_6\}}$ with $f_1 \neq f_3$ satisfying the condition $Q(S,R) = 0$, then the space form is an Einstein manifold.

Suppose, we consider Ricci pseudo-symmetric generalized $(k,\mu)$-space form $M_{\{f_1,\ldots,f_6\}}$, that is, the manifold satisfying the curvature condition $R \cdot S = fQ(g,S)$, then we have from (6)

$$(R(X,Y) \cdot S)(U,V) = fQ(g,S)(X,Y;U,V),$$

which is equivalent to

$$(R(X,Y) \cdot S)(U,V) = f((X \wedge Y \cdot S)(U,V)).$$  

Using (6) in (37), we get

$$-
S(R(X,Y)U,V) - S(U,R(X,Y)V)
= f\{-g(Y,U)S(X,V) + g(X,U)S(Y,V) - g(Y,V)S(U,X) + g(X,V)S(U,Y)\}.$$  

Replacing $X = U = \xi$ in (38), we obtain

$$S(R(\xi,Y)\xi,V) + S(\xi,R(\xi,Y)V)
= f\{g(Y,\xi)S(\xi,V) - g(\xi,\xi)S(Y,V) + g(Y,V)S(\xi,\xi) - g(\xi,V)S(\xi,Y)\}.$$  

Using (9), (16) and (19) in (39), we get

$$(f_3 - f)\{2n(f_1 - f_3)g(Y,V) - S(Y,V)\}
+ (f_4 - f_6)\{2n(f_1 - f_3)g(hY,V) - S(hY,V)\} = 0.$$  

Replacing $Y$ by $hY$ in (40) and using (13), we get

$$(f_3 - f)\{2n(f_1 - f_3)g(hY,V) - S(hY,V)\}
- (k-1)(f_4 - f_6)\{2n(f_1 - f_3)g(Y,V) - S(Y,V)\} = 0.$$  

Multiplying (40) by $(f_3 - f)$ and (41) by $(f_4 - f_6)$ and subtracting from (40) to (41), we obtain

$$(k-1)(f_4 - f_6)^2 + (f_1 - f_3 - f)^2 \{2n(f_1 - f_3)g(Y,V) - S(Y,V)\} = 0.$$  

Now for $k = 1$, either $f = f_3$ or $S(Y,V) = 2n(f_1 - f_3)g(Y,V)$. On the other hand for $k < 1$, either $S(Y,V) = 2n(f_1 - f_3)g(Y,V)$ or

$$(f_3 - f)^2 = (1-k)(f_4 - f_6)^2.$$  

Then from (35), we have $f = f_3$ implies $f_4 = f_6$.

Thus we can state the following:

**Theorem 3.** A generalized $(k,\mu)$-space form $M_{\{f_1,\ldots,f_6\}}$ with $f_1 \neq f_3$ is Ricci pseudo-symmetric, then the space form is an Einstein manifold.

Suppose the generalized $(k,\mu)$-space form satisfying the curvature condition $Q(g,S) = 0$. Then we have

$$(X \wedge Y \cdot S)(U,V) = 0.$$  

Using (3) and (6) in (44), we get

$$-g(Y,U)S(X,V) + g(X,U)S(Y,V) - g(Y,V)S(U,X) + g(X,V)S(U,Y) = 0.$$  

Taking $X = U = \xi$ in (45) and using (19) and (9), we obtain

$$S(Y,V) = 2n(f_1 - f_3)g(Y,V).$$  

Thus we can state the following:
**Theorem 4.** If a \((2n + 1)\)-dimensional generalized \((k, \mu)\)-space form \(M(f_1, \ldots, f_6)\) satisfying the condition \(Q(g, S) = 0\), then the space form is either Ricci flat or an Einstein manifold.

**Definition 5.** A vector field \(V\) is said to be confirmal Killing vector field if it satisfies \(L_V g = \rho g\), for some function \(\rho\).

If the manifold admitting a Ricci solitons \((g, V, \lambda)\) is an Einstein manifold then the vector field \(V\) is confirmal Killing. Now by substituting (46) in (8), we get

\[
(L_V g)(X, Y) = \rho g(X, Y).
\]

Where \(\rho = -2\{2n(f_1 - f_3) + \lambda\}\), i.e. \(V\) is confirmal Killing.

This leads to the following:

**Theorem 6.** Let \((g, V, \lambda)\) be a Ricci soliton in generalized \((k, \mu)\)-space form \(M(f_1, \ldots, f_6)\). The potential vector field \(V\) is confirmal Killing if and only if the space form is Ricci-semisymmetric.

**Proposition 7.** Let \((g, V, \lambda)\) be a Ricci soliton in generalized \((k, \mu)\)-space form \(M(f_1, \ldots, f_6)\) with \(f_1 \neq f_3\) and \(f_4 \neq f_6\). The potential vector field \(V\) is confirmal Killing if and only if \(Q(g, \xi) = 0\), holds in \(M\).

**Proposition 8.** Let \((g, V, \lambda)\) be a Ricci soliton in generalized \((k, \mu)\)-space form \(M(f_1, \ldots, f_6)\) with \(f \neq f_1 - f_3\) and \(f_4 \neq f_6\). The potential vector field \(V\) is confirmal Killing if and only if the space form is Ricci pseudo-semisymmetric.

**Proposition 9.** Let \((g, V, \lambda)\) be a Ricci soliton in generalized \((k, \mu)\)-space form \(M(f_1, \ldots, f_6)\) with \(f_1 \neq f_3\) and \(f_4 \neq f_6\). The potential vector field \(V\) is confirmal Killing if and only if \(Q(S, R) = 0\), holds in \(M\).

Suppose that a generalized \((k, \mu)\)-space form \(M(f_1, \ldots, f_6)\), admits a Ricci soliton \((g, V, \lambda)\), then from (8), we have

\[
g(\nabla_X V, Y) + g(X, \nabla_V Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0.
\]

Replacing \(X = Y = \xi\) in (48), we get

\[
2g(\nabla_\xi V, \xi) + 2S(\xi, \xi) + 2\lambda = 0.
\]

If \(V \perp \xi\), it provides \(\eta(\nabla_X V) = g(\phi X + \phi hX, V)\). Hence \(\eta(\nabla_\xi V) = 0\). Therefore on using (19) in (49), we obtain

\[
\lambda = -2n(f_1 - f_3).
\]

Hence we can state the following:

**Theorem 10.** A generalized \((k, \mu)\)-space form \(M(f_1, \ldots, f_6)\) admitting a Ricci soliton \((g, V, \lambda)\), where the potential vector field \(V\) is orthogonal to \(\xi\) is shrinking if \(f_1 > f_3\), expanding if \(f_1 < f_3\) or steady if \(f_1 = f_3\).

**Definition 11.** A vector field \(V\) is called torse forming vector field if it satisfies \(\nabla_X V = fX + \gamma(X)V\), where \(f\) is a smooth function and \(\gamma\) is a 1-form.

From (48) and using (18), we can write

\[
\nabla_X V = -\{2nf_1 + 3f_2 - f_3 + \lambda\}X - \{(2n - 1)f_4 - f_6\}\phi X + \{3f_2 + (2n - 1)f_3\}\gamma(X)\xi.
\]

If \((2n - 1)f_4 = f_6\), then the vector field \(\nabla = b\phi \xi\) is torse forming, where \(f = -\{2nf_1 + 3f_2 - f_3 + \lambda\}, \gamma(X)\) is 1-form and \(b = 3f_2 + (2n - 1)f_3\).

Thus we state the following:

**Theorem 12.** A generalized \((k, \mu)\)-space form \(M(f_1, \ldots, f_6)\) admitting a Ricci soliton \((g, V, \lambda)\), where the vector field \(V\) is collinear with \(\xi\). Then the vector field \(V\) is torse forming.

If the vector field \(V\) is torse forming vector field, then equation (48) becomes

\[
2fg(X, Y) + \gamma(X)g(V, Y) + \gamma(Y)g(V, X) + 2S(X, Y) + 2\lambda g(X, Y) = 0.
\]

Taking \(Y = \xi\) in (52), we get

\[
\{2f + 4n(f_1 - f_3) + 2\lambda\}\eta(X) + \gamma(\xi)g(V, X) + \gamma(X)\eta(V) = 0.
\]
Replacing $X$ by $\xi$ in (53), we obtain
\[
\lambda = -\{\eta(V)\gamma(\xi) + f + 2n(f_1 - f_3)\}.
\] (54)

If $f = -\eta(V)\gamma(\xi)$, then from (54), we get
\[
\lambda = 2n(f_3 - f_1).
\] (55)

Thus we can state the following:

**Theorem 13.** If $(g, V, \lambda)$ is a Ricci soliton in a generalized $(k, \mu)$-space form $M(f_1, \ldots, f_6)$ and $V$ is torse forming with $f = -\eta(V)\gamma(\xi)$, then the Ricci soliton is shrinking if $f_1 > f_3$, expanding if $f_1 < f_3$ or steady if $f_1 = f_3$.

### 4. Conclusions

Generalized $(k, \mu)$-Space forms generalize the notion of $(k, \mu)$-Space forms and generalized Sasakian space forms. Some semi-symmetry, Ricci pseudo symmetry on generalized $(k, \mu)$-Space form leads to the Einstein condition. Further the potential vector field of a Ricci soliton in a generalized $(k, \mu)$-Space form reduces to torse forming or conformal Killing under certain conditions.

### References


