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Weingarten Map of the Hypersurface in Euclidean 4-Space and its Applications

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Abstract
In this paper, by taking into account the beginning of the hypersurface theory in Euclidean space $\mathbb{E}^4$, a practical method for the matrix of the Weingarten map (or the shape operator) of an oriented hypersurface M^3 in $\mathbb{E}^4$ is obtained. By taking this efficient method, it is possible to study of the hypersurface theory in $\mathbb{E}^4$ which is analog the surface theory in $\mathbb{E}^3$. Furthermore, the Gaussian curvature, Mean curvature, fundamental forms and Dupin indicatrix of M^3 is introduced.

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1. Introduction

Let $x=\sum_{i=1}^{4} x_i e_i,\ y=\sum_{i=1}^{4} y_i e_i,\ z=\sum_{i=1}^{4} z_i e_i$ be three vectors in $\mathbb{R}^4$, equipped with the standard inner product given by

$$\langle x,y\rangle = x_1y_1+x_2y_2+x_3y_3+x_4y_4,$$

where $\{e_1,e_2,e_3,e_4\}$ is the standard basis of $\mathbb{R}^4$. The norm of a vector $x\in\mathbb{R}^4$ is given by $\|x\|=\sqrt{\langle x,x\rangle}$. The vector product (or the ternary product or cross product) of the vectors $x,y,z\in\mathbb{R}^4$ is defined by

$$x\otimes y\otimes z = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}. \tag{1}$$

Some properties of the vector product are given as follows: (for the vector product in $\mathbb{R}^4$, see [1, 2, 5]

i.

$$\left\{ \begin{array}{l} e_1\otimes e_2\otimes e_3 = -e_4 \\ e_2\otimes e_3\otimes e_4 = \ e_1 \\ e_3\otimes e_4\otimes e_1 = -e_2 \\ e_4\otimes e_1\otimes e_2 = \ e_3 \\ e_3\otimes e_2\otimes e_1 = \ e_4 \end{array} \right\}$$

ii.

$$\|x\otimes y\otimes z\|^2 = \begin{vmatrix} \langle x,x\rangle & \langle x,y\rangle & \langle x,z\rangle \\ \langle y,x\rangle & \langle y,y\rangle & \langle y,z\rangle \\ \langle z,x\rangle & \langle z,y\rangle & \langle z,z\rangle \end{vmatrix} \tag{2}$$

iii. $\langle x \otimes y \otimes z, t \rangle = \det(x, y, z, t)$.

Let M^3 be an oriented hypersurface in 4-dimensional Euclidean space $\mathbb{E}^4$. Let examine the implicit and parametric equations of M^3. Firstly; the implicit equation of M^3 can be defined by

$$M^3 = \left\{ X \in E^4 | f : U \subset E^4 \overset{diff.}{\to} \mathbb{R},\ f(X) = const.\ \overrightarrow{\nabla f}|_P \neq 0,\ P \in M^3 \right\} \tag{3}$$

where $\overrightarrow{\nabla f}|_P$ is the gradient vector of M^3. The unit normal vector field of M^3 is defined by $N = \frac{\overrightarrow{\nabla f}}{\left\|\overrightarrow{\nabla f}\right\|}$.

The Weingarten map (or the shape operator) of M^3 is defined by

$$S : \chi\left(M^3\right) \to \chi\left(M^3\right),\ S(X) = D_X N,$$

where D is the connection of $\mathbb{E}^4$ and $\chi\left(M^3\right)$ is the space of vector fields of M^3. Then the Gauss curvature K and mean curvature H of M^3 are given by $K = \det S$ and $H = \frac{1}{3} TrS$, respectively. Also, the $q-th$ fundamental forms of M^3 are given by [3],

$$I^q(X, Y) = \left\langle S^{q-1}(X), Y \right\rangle,\ \forall X, Y \in \chi\left(M^3\right).$$

Secondly, to examine parametric form of the hypersurface M^3 given by the implicit equation in the equation (3), let consider

$$\begin{aligned} &\phi : U \subset \mathbb{R}^3 \to \mathbb{E}^4 \\ &\quad (u, v, w) \to \phi(u, v, w) = (\varphi_1(u, v, w), \varphi_2(u, v, w), \varphi_3(u, v, w), \varphi_4(u, v, w)) \end{aligned}$$

where $(u, v, w) \in R \subset \mathbb{R}^3$ and $\varphi_i, 1 \leq i \leq 4$ are the real functions defined on R.
$M^3 = \phi(R) \subset \mathbb{E}^4$ is a hypersurface if only if the frame field $\{\phi_u, \phi_v, \phi_w\}$ of M^3 is linearly independent system. It can be also seen by taking the Jacobian matrix $[\phi]_* = \begin{bmatrix} \phi_u & \phi_v & \phi_w \end{bmatrix}$ of the differential map of ϕ. It is clear that if rank $[\phi]_* = 3$, then the vector system $\{\phi_u, \phi_v, \phi_w\}$ is linearly independent. Furthermore, ϕ_u, ϕ_v, ϕ_w are the tangent vectors of the parameter curves $\alpha(u) = \phi(u, v_0, w_0)$, $\beta(v) = \phi(u_0, v, w_0)$ and $\gamma(w) = \phi(u_0, v_0, w)$, respectively. Then the unit normal vector field of M^3 is defined by

$$N = \frac{\phi_u \otimes \phi_v \otimes \phi_w}{\|\phi_u \otimes \phi_v \otimes \phi_w\|} \tag{4}$$

and it has the following properties:

$$\langle N, \phi_u \rangle = \langle N, \phi_v \rangle = \langle N, \phi_w \rangle = 0. \tag{5}$$

By using the Weingarten operator the below equalities can be written

$$\begin{aligned} S(\phi_u) &= D_{\phi_u} N = \frac{\partial N}{\partial u} \\ S(\phi_v) &= D_{\phi_v} N = \frac{\partial N}{\partial v} \\ S(\phi_w) &= D_{\phi_w} N = \frac{\partial N}{\partial w}. \end{aligned}$$

2. The matrix of the Weingarten map of hypersurface M^3 in $\mathbb{E}^4$

In this original section, a practical method for the matrix of the Weingarten map of hypersurface M^3 in $\mathbb{E}^4$ is introduced.

Let M^3 be an oriented hypersurface with the parametric equation $\phi(u, v, w)$. Then $\{\phi_u, \phi_v, \phi_w\}$ is linearly independent and we also can write

$$\begin{aligned} S(\phi_u) &= a_{11}\,\phi_u + a_{21}\,\phi_v + a_{31}\,\phi_w \\ S(\phi_v) &= a_{12}\,\phi_u + a_{22}\,\phi_v + a_{32}\,\phi_w \\ S(\phi_w) &= a_{13}\,\phi_u + a_{23}\,\phi_v + a_{33}\,\phi_w \end{aligned} \tag{6}$$

and the Weingarten matrix is given by

$$S=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where $a_{ij}\in\mathbb{R}, 1\leq i,j\leq 3$. Using the equation (6), we have the following systems of linear equations:

$$\begin{cases} \langle S(\phi_u),\phi_u\rangle = a_{11}\phi_{11}+a_{21}\phi_{12}+a_{31}\phi_{13} \\ \langle S(\phi_u),\phi_v\rangle = a_{11}\phi_{12}+a_{21}\phi_{22}+a_{31}\phi_{23} \\ \langle S(\phi_u),\phi_w\rangle = a_{11}\phi_{13}+a_{21}\phi_{23}+a_{31}\phi_{33}, \end{cases}$$

$$\begin{cases} \langle S(\phi_v),\phi_u\rangle = a_{12}\phi_{11}+a_{22}\phi_{12}+a_{32}\phi_{13} \\ \langle S(\phi_v),\phi_v\rangle = a_{12}\phi_{12}+a_{22}\phi_{22}+a_{32}\phi_{23} \\ \langle S(\phi_v),\phi_w\rangle = a_{12}\phi_{13}+a_{22}\phi_{23}+a_{32}\phi_{33}, \end{cases} \tag{7}$$

$$\begin{cases} \langle S(\phi_w),\phi_u\rangle = a_{13}\phi_{11}+a_{23}\phi_{12}+a_{33}\phi_{13} \\ \langle S(\phi_w),\phi_v\rangle = a_{13}\phi_{12}+a_{23}\phi_{22}+a_{33}\phi_{23} \\ \langle S(\phi_w),\phi_w\rangle = a_{13}\phi_{13}+a_{23}\phi_{23}+a_{33}\phi_{33}, \end{cases}$$

where

$$\begin{aligned} &\langle\phi_u,\phi_u\rangle=\phi_{11},\ \langle\phi_u,\phi_v\rangle=\phi_{12},\ \langle\phi_u,\phi_w\rangle=\phi_{13}, \\ &\langle\phi_v,\phi_v\rangle=\phi_{22},\ \langle\phi_v,\phi_w\rangle=\phi_{23},\ \langle\phi_w,\phi_w\rangle=\phi_{33}. \end{aligned} \tag{8}$$

Since the system $\{\phi_u,\phi_v,\phi_w\}$ is linearly independent, using the equations (2) and (8), we have

$$\|\phi_u\otimes\phi_v\otimes\phi_w\|^2=\begin{vmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{12} & \phi_{22} & \phi_{23} \\ \phi_{13} & \phi_{23} & \phi_{33} \end{vmatrix}\neq 0.$$

Also, 3-linear equation systems given by the equation 7 have the determinant

$$\begin{vmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{12} & \phi_{22} & \phi_{23} \\ \phi_{13} & \phi_{23} & \phi_{33} \end{vmatrix}=\Delta.$$

Because of the property $\|\phi_u\otimes\phi_v\otimes\phi_w\|^2=\Delta\neq 0$, these 3-linear equations systems can be solved by Cramer method. Then using the equations (6), (7) and (8) the matrix S of the Weingarten map in M^3 can be found. Although S is a symmetric linear operator, the matrix presentation (a_{ij}) of S with respect to $\{\phi_u,\phi_v,\phi_w\}$ is not necessary to be symmetric because the system $\{\phi_u,\phi_v,\phi_w\}$ is not orthonormal.

2.1 Special Case

If we take the orthogonal frame field $\{\phi_u,\phi_v,\phi_w\}$ of the hypersurface M^3, then we have $\phi_{12}=\phi_{13}=\phi_{23}=0$ from the equation (8). Then, the system $\left\{U=\frac{\phi_u}{\|\phi_u\|},\ V=\frac{\phi_v}{\|\phi_v\|},\ W=\frac{\phi_w}{\|\phi_w\|}\right\}$ is an orthonormal frame field. Furthermore, we can write the following equations

$$\begin{aligned} S(U)&=c_1U+c_2V+c_3W \\ S(V)&=c_2U+c_4V+c_5W \\ S(W)&=c_3U+c_5V+c_6W, \end{aligned} \tag{9}$$

then, the matrix of the Weingarten map can be calculated as follows:

$$S=\begin{pmatrix} c_1 & c_2 & c_3 \\ c_2 & c_4 & c_5 \\ c_3 & c_5 & c_6 \end{pmatrix}.$$

By using the equations (4), (6) and (9), the coefficients $c_i \in \mathbb{R}$, $1 \leq i \leq 6$ can be calculated as follows:

$$\begin{aligned}
c_1 &= \langle S(U), U \rangle = \frac{1}{\|\phi_u\|^2} \left\langle \frac{\partial N}{\partial u}, \phi_u \right\rangle, \\
c_2 &= \langle S(U), V \rangle = \frac{1}{\|\phi_u\|} \frac{1}{\|\phi_v\|} \left\langle \frac{\partial N}{\partial u}, \phi_v \right\rangle, \\
c_3 &= \langle S(U), W \rangle = \frac{1}{\|\phi_u\|} \frac{1}{\|\phi_w\|} \left\langle \frac{\partial N}{\partial u}, \phi_w \right\rangle, \\
c_4 &= \langle S(V), V \rangle = \frac{1}{\|\phi_v\|^2} \left\langle \frac{\partial N}{\partial v}, \phi_v \right\rangle, \\
c_5 &= \langle S(V), W \rangle = \frac{1}{\|\phi_v\|} \frac{1}{\|\phi_w\|} \left\langle \frac{\partial N}{\partial v}, \phi_w \right\rangle, \\
c_6 &= \langle S(W), W \rangle = \frac{1}{\|\phi_w\|^2} \left\langle \frac{\partial N}{\partial w}, \phi_w \right\rangle.
\end{aligned} \tag{10}$$

By using the equation (5), we can also write six equations as below:

$$\begin{aligned}
&\left\langle \frac{\partial N}{\partial u}, \phi_u \right\rangle + \langle N, \phi_{uu} \rangle = 0, \\
&\left\langle \frac{\partial N}{\partial u}, \phi_v \right\rangle + \langle N, \phi_{uv} \rangle = 0, \\
&\left\langle \frac{\partial N}{\partial u}, \phi_w \right\rangle + \langle N, \phi_{uw} \rangle = 0, \\
&\left\langle \frac{\partial N}{\partial v}, \phi_v \right\rangle + \langle N, \phi_{vv} \rangle = 0, \\
&\left\langle \frac{\partial N}{\partial v}, \phi_w \right\rangle + \langle N, \phi_{vw} \rangle = 0, \\
&\left\langle \frac{\partial N}{\partial w}, \phi_w \right\rangle + \langle N, \phi_{ww} \rangle = 0.
\end{aligned} \tag{11}$$

Also, by using the equations (2) and (8), we find

$$\|\phi_u \otimes \phi_v \otimes \phi_w\|^2 = \begin{vmatrix} \phi_{22} & 0 & 0 \\ 0 & \phi_{11} & 0 \\ 0 & 0 & \phi_{33} \end{vmatrix} = \|\phi_u\|^2 \|\phi_v\|^2 \|\phi_w\|^2. \tag{12}$$

Hence we find the coefficients $c_1, c_2, c_3, c_4, c_5, c_6$ of the Weingarten matrix in the equation (9) as follows:

$$\begin{aligned}
c_1 &= -\frac{1}{\|\phi_u\|^3} \frac{1}{\|\phi_v\|} \frac{1}{\|\phi_w\|} \det(\phi_{uu}, \phi_u, \phi_v, \phi_w), \\
c_2 &= -\frac{1}{\|\phi_u\|^2} \frac{1}{\|\phi_v\|^2} \frac{1}{\|\phi_w\|} \det(\phi_{uv}, \phi_u, \phi_v, \phi_w), \\
c_3 &= -\frac{1}{\|\phi_u\|^2} \frac{1}{\|\phi_v\|} \frac{1}{\|\phi_w\|^2} \det(\phi_{uw}, \phi_u, \phi_v, \phi_w), \\
c_4 &= -\frac{1}{\|\phi_u\|} \frac{1}{\|\phi_v\|^3} \frac{1}{\|\phi_w\|} \det(\phi_{vv}, \phi_u, \phi_v, \phi_w), \\
c_5 &= -\frac{1}{\|\phi_u\|} \frac{1}{\|\phi_v\|^2} \frac{1}{\|\phi_w\|^2} \det(\phi_{vw}, \phi_u, \phi_v, \phi_w), \\
c_6 &= -\frac{1}{\|\phi_u\|} \frac{1}{\|\phi_v\|} \frac{1}{\|\phi_w\|^3} \det(\phi_{ww}, \phi_u, \phi_v, \phi_w).
\end{aligned} \tag{13}$$

So, by taking into account the equations (4), (12) and (13) we have the symmetric Weingarten matrix

$$S=\begin{pmatrix} \frac{\varphi_{11}}{\phi_{11}} & \frac{\varphi_{12}}{\sqrt{\phi_{11}\phi_{22}}} & \frac{\varphi_{13}}{\sqrt{\phi_{11}\phi_{33}}} \\ \frac{\varphi_{12}}{\sqrt{\phi_{11}\phi_{22}}} & \frac{\varphi_{22}}{\phi_{22}} & \frac{\varphi_{23}}{\sqrt{\phi_{22}\phi_{33}}} \\ \frac{\varphi_{13}}{\sqrt{\phi_{11}\phi_{33}}} & \frac{\varphi_{23}}{\sqrt{\phi_{22}\phi_{33}}} & \frac{\varphi_{33}}{\phi_{33}} \end{pmatrix} \tag{14}$$

where

$$\begin{array}{l} \varphi_{11}=-\langle \phi_{uu},N\rangle,\ \varphi_{12}=-\langle \phi_{uv},N\rangle,\ \varphi_{13}=-\langle \phi_{uw},N\rangle, \\ \varphi_{22}=-\langle \phi_{vv},N\rangle,\ \varphi_{23}=-\langle \phi_{vw},N\rangle,\ \varphi_{33}=-\langle \phi_{ww},N\rangle. \end{array}$$

Finally the following theorem can be given for hypersurface M^3 in $\mathbb{E}^4$:

Theorem 1. *Let M^3 be an oriented hypersurface in $\mathbb{E}^4$. Then the Gaussian curvature and the mean curvature of M^3 can be given by:*

$$K=\frac{\varphi_{11}\varphi_{22}\varphi_{33}+2\varphi_{12}\varphi_{13}\varphi_{23}-\varphi_{12}^2\varphi_{33}-\varphi_{13}^2\varphi_{22}-\varphi_{23}^2\varphi_{11}}{\phi_{11}\phi_{22}\phi_{33}}$$

and

$$H=\frac{1}{3}\left(\frac{\varphi_{11}}{\phi_{11}}+\frac{\varphi_{22}}{\phi_{22}}+\frac{\varphi_{33}}{\phi_{33}}\right),$$

respectively.

Proof. By using the equation (14) and the definitions of the Gaussian curvature K and the mean curvature H, the theorem can be easily proved. ■

Example 2. *Let M^3 be an oriented hypersurface with the implicit equation $xy=1$ in $\mathbb{E}^4$. The parametric equation of M^3 can be given by*

$$\phi(u,v,w)=\left(u,\frac{1}{u},v,w\right).$$

Then, we obtain $\phi_u\otimes\phi_v\otimes\phi_w=\left(-\frac{1}{u^2},-1,0,0\right)$ and the unit normal field $N=\frac{1}{\sqrt{1+u^4}}\left(-1,-u^2,0,0\right)$. By using the orthonormal basis $\left\{\frac{\phi_u}{\|\phi_u\|},\frac{\phi_v}{\|\phi_v\|},\frac{\phi_w}{\|\phi_w\|}\right\}$, we have

$$\begin{array}{l} S\left(\frac{\phi_u}{\|\phi_u\|}\right)=\frac{2u^3}{\left(1+u^4\right)^{3/2}}\frac{\phi_u}{\|\phi_u\|}, \\ S\left(\frac{\phi_v}{\|\phi_v\|}\right)=0, \\ S\left(\frac{\phi_w}{\|\phi_w\|}\right)=0. \end{array}$$

So, we find the Weingarten matrix S as:

$$S=\begin{pmatrix} \frac{2u^3}{\left(1+u^4\right)^{3/2}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Example 3. *Let S^3 be a hypersphere with the implicit equation $x^2+y^2+z^2+t^2=1$ in $\mathbb{E}^4$. The parametric equation of S^3 can be given by*

$$\phi(u,v,w)=(\sin u\cos v\sin w,\sin u\sin v\sin w,\cos u\sin w,\cos w).$$

Then, $\{\phi_u, \phi_v, \phi_w\}$ is an orthogonal system. Also we have the orthonormal basis $\{U, V, W\}$ of S^3 such that

$$U = \frac{\phi_u}{\|\phi_u\|} = (\cos u \cos v, \cos u \sin v, -\sin u, 0),$$

$$V = \frac{\phi_v}{\|\phi_v\|} = (-\sin v, \cos v, 0, 0),$$

$$W = \frac{\phi_w}{\|\phi_w\|} = (\sin u \cos v \cos w, \sin u \sin v \cos w, \cos u \cos w, -\sin w).$$

Furthermore, the unit normal vector field N can be found:

$$N = (-\sin u \cos v \sin w, -\sin u \sin v \sin w, -\cos u \sin w, -\cos w).$$

Then using the equation (15), we obtain $S = I_3$.

Example 4. *Let M^3 be an oriented hypersurface with implicit equation $t = x^2 + y^2 + z^2$ in $\mathbb{E}^4$. The parametric equation of M^3 can be given by*

$$\phi(u, v, w) = \left(u \cos v \cos w, u \cos v \sin w, u \sin v, u^2\right).$$

By calculating

$$\phi_u \otimes \phi_v \otimes \phi_w = \left(-2u^3 \cos^2 v \cos w, -2u^3 \cos^2 v \sin w, -2u^3 \sin v \cos v, u^2 \cos v\right)$$

we obtain the unit normal vector field

$$N = \frac{1}{\sqrt{1+4u^2}} (-2u \cos v \cos w, -2u \cos v \sin w - 2u \sin v, 1).$$

By using the orthonormal basis $\{\phi_u, \phi_v, \phi_w\}$, we have the following matrix form of the shape operator:

$$S = \begin{pmatrix} -\frac{2}{\left(1+4u^2\right)^{3/2}} & 0 & 0 \\ 0 & -\frac{2}{\left(1+4u^2\right)^{1/2}} & 0 \\ 0 & 0 & -\frac{2}{\left(1+4u^2\right)^{1/2}} \end{pmatrix}.$$

Theorem 5. *Let M^3 be an oriented hypersurface in $\mathbb{E}^4$ and let $\{X_P, Y_P, Z_P\}$ be a linearly independent vector system of the tangent space $T_{M^3}(P)$. Then, we have*

i. $S(X_P) \otimes S(Y_P) \otimes S(Z_P) = K(P)(X_P \otimes Y_P \otimes Z_P)$
ii. $(S(X_P) \otimes Y_P \otimes Z_P) + (X_P \otimes S(Y_P) \otimes Z_P) + (X_P \otimes Y_P \otimes S(Z_P)) = 3H(P)(X_P \otimes Y_P \otimes Z_P)$,

where K and H are the Gaussian curvature and the mean curvature of M^3, respectively.

Proof. By using (i), (ii) parts of the equation (2) and considering the definitions of the Gaussian curvature K and the mean curvature H the theorem can be easily proved. ■

In [4], it is proved that these equations are also provided for closed hypersurfaces.

Theorem 6. *Let M^3 be an oriented hypersurface in $\mathbb{E}^4$ and let I^q, K, H be the q-th fundamental forms, the Gaussian curvature and the mean curvature, respectively. Then we have*

$$I^4 - 3H I^3 + \frac{3K}{h} I^2 - K I = 0 \tag{15}$$

where h is the harmonic mean of the non-zero principal curvatures of M^3.

Proof. Let k_1, k_2, k_3 be the characteristic values of the Weingarten map S (or the principal curvatures of M^3). Then we obtain the characteristic polynomial $P_S(\lambda)$ of the Weingarten map S of M^3 as

$$P_S(\lambda) = \det(\lambda I_3 - S) = \lambda^3 - (k_1 + k_2 + k_3)\lambda^2 + (k_1 k_2 + k_1 k_3 + k_2 k_3)\lambda - (k_1 k_2 k_3).$$

By using the Cayley-Hamilton theorem, we obtain

$$S^3-(k_1+k_2+k_3)S^2+(k_1k_2+k_1k_3+k_2k_3)S-(k_1k_2k_3)I_3=0.$$

By using the definitions of the $q-th$ fundamental forms, the Gaussian curvature, the mean curvature and the harmonic mean

$$h=\frac{3}{\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}}$$

of the principal curvature k_1,k_2,k_3, we obtain the equation (15). ■

3. Dupin indicatrix of the hypersurface in $\mathbb{E}^4$

Let X,Y,Z be three principal vectors according to the principal curvatures k_1,k_2,k_3 of M^3. If we consider the orthonormal basis $\{X,Y,Z\}$ of M^3 then for any tangent vector $W_P\in T_{M^3}(P)$, we can write $W_P=xX_P+yY_P+zZ_P$, where $x,y,z\in\mathbb{R}$, and

$$\begin{aligned}S(W_P)&=xS(X_P)+yS(Y_P)+zS(Z_P)\\&=xk_1X_P+yk_2Y_P+zk_3Z_P\end{aligned}$$

Here, the Dupin indicatrix $\mathbb{D}$ of M^3 can be defined by

$$\mathbb{D}=\left\{\ W_P=(x,y,z)\in T_{M^3}(P)|\,\langle S(W_P),W_P\rangle=k_1x^2+k_2y^2+k_3z^2=\pm1\ \right\}.$$

In another words, the Dupin indicatrix corresponds to a hypercylinder which has the equation

$$k_1x^2+k_2y^2+k_3z^2=\pm1.$$

Now, we will examine the Dupin indicatrix according to the Gaussian curvature K :

1) Let $K(P)>0$.

- If $k_1,\ k_2,\ k_3>0$ then for equation of the Dupin indicatrix, we can write $k_1x^2+k_2y^2+k_3z^2=\pm1$. Hence, the Dupin indicatrix is the ellipsoidal class and this equation is called ellipsoidal cylinder in E^4. In this condition, $P\in M^3$ is called an ellipsoidal point.
- If $k_1>0,\ k_2,\ k_3<0$ or $k_2>0,\ k_1,\ k_3<0$ or $k_3>0\ k_1,\ k_2<0$ then for equation of the Dupin indicatrix, we can write $k_1x^2-k_2y^2-k_3z^2=\pm1$. Hence, the Dupin indicatrix is the hyperboloidical class and this equation is called hyperboloidical cylinder one or two sheets in $\mathbb{E}^4$. In this condition, $P\in M^3$ is called a hyperboloidical point.

2) Let $K(P)<0$.

- If only one of k_i's, $i=1,2,3$ is negative, then for the equation of the Dupin indicatrix, we can write

  $$\begin{cases}k_1x^2+k_2y^2-k_3z^2=\pm1,\\k_1x^2-k_2y^2+k_3z^2=\pm1,\\-k_1x^2+k_2y^2+k_3z^2=\pm1.\end{cases}$$

  The above equations are called one or two sheeted hyperboloidical cylinder in $\mathbb{E}^4$. Then $P\in M^3$ is called a hyperboloidical point.
- If $k_1,k_2,k_3<0$ then the Dupin indicatrix is the ellipsoidal class and this equation is called ellipsoidal cylinder in $\mathbb{E}^4$. So $P\in M^3$ is called a ellipsoidal point.

3) Let $K(P)=0$.

- If $k_1=0$ or $k_2=0$ or $k_3=0$, then for the equation of the Dupin indicatrix for each case, we get

  i If $k_1=0,\ k_2,\ k_3$ are the same or different signs then $k_2y^2+k_3z^2=\pm1$.
  ii If $k_2=0,\ k_1,k_3$ are the same or different signs then $k_1x^2+k_3z^2=\pm1$.
  iii If $k_3=0,\ k_1,\ k_2$ are the same or different signs then $k_1x^2+k_2y^2=\pm1$.

  These equations are called elliptic cylinder or hyperbolic cylinder in $\mathbb{E}^4$. In this condition, $P\in M^3$ is called an elliptic cylinder or hyperbolic cylinder point.
- If $k_1=k_2=k_3=0$ then the point $P\in M^3$ is a flat point.
- If any two of k_i's, $i=1,2,3$ are zero and other positive or negative then $k_3z^2=\pm1$ or $k_2y^2=\pm1$ or $k_1x^2=\pm1$.

4. Conclusions

In this present study, the basic notations of hypersurface theory in $\mathbb{E}^4$ are examined firstly. After that, analog to calculation of the matrix presentation of Weingarten Map for 2-surfaces in $\mathbb{E}^3$ and $\mathbb{E}^4$, a practical and an efficient method is established for the 3-surfaces (hypersurfaces). The method and the results are given in this study lead light to practical calculations of other algebraic invariants of the Weingarten Map and the investigations of the special hypersurfaces in 4-dimensional space.

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