Hacettepe Journal of Mathematics and Statistics
Volume 45 (2) (2016), 297 – 310

On M-term approximations of the Nikol'skii - Besov class

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Abstract

In this paper, we consider a Lebesgue space with a mixed norm of periodic functions of many variables. We obtain the exact estimation of the best M-term approximations of Nikol'skii's and Besov's classes in the Lebesgue space with the mixed norm.

Keywords: Lebesgue space, Nikol'skii - Besov class, approximation.

2000 AMS Classification: 41A10, 41A25.

Received : 03.07.2014 *Accepted* : 25.02.2015 *Doi* : 10.15672/HJMS.20164512492

1. Introduction

Let $\overline{x} = (x_1, ..., x_m) \in \mathbb{T}^m = [0, 2\pi)^m$ and $p_j \in [1, +\infty)$, $j = 1, ..., m$. $L_{\bar{p}}(\mathbb{T}^m)$ denotes the space of Lebesgue measureable functions $f(\bar{x})$ defined on $\mathbb{R}^m$, which have 2π period with respect to each variable such that

$$\|f\|_{\bar{p}} = \left[\int_0^{2\pi}\left[\cdots\left[\int_0^{2\pi}|f(\bar{x})|^{p_1}\,dx_1\right]^{\frac{p_2}{p_1}}\cdots\right]^{\frac{p_m}{p_{m-1}}}dx_m\right]^{\frac{1}{p_m}} < +\infty,$$

where $\overline{p} = (p_1, ..., p_m)$, $1 \leq p_j < +\infty$, $j = 1, ..., m$ (see [18], p. 128, [4], p. 54). In the case $p_1 = ... = p_m = p$, we write $L_p(\mathbb{T}^m)$.

Any function $f \in L_1(\mathbb{T}^m) = L(\mathbb{T}^m)$ can be expanded to the Fourier series

$$\sum_{\overline{n}\in\mathbb{Z}^m} a_{\overline{n}}(f)\, e^{i\langle\overline{n},\overline{x}\rangle},$$

where $\{a_{\overline{n}}(f)\}$ are Fourier coefficients of a function $f \in L_1(\mathbb{T}^m)$ with respect to a multiple trigonometric system $\{e^{i\langle\overline{n},\overline{x}\rangle}\}_{\bar{n}\in\mathbb{Z}^m}$ and $\mathbb{Z}^m$ is the space of points in $\mathbb{R}^m$ with integer coordinates.

For a function $f \in L(\mathbb{T}^m)$ and a number $s \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, let us introduce the notation

$$\delta_0(f, \bar{x}) = a_0(f)$$

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and

$$\delta_s(f,\overline{x}) = \sum_{\overline{n}\in\rho(s)} a_{\overline{n}}(f)e^{i\langle\overline{n},\overline{x}\rangle},$$

where $\langle\bar{y},\bar{x}\rangle = \sum\limits_{j=1}^{m} y_j x_j$ and

$$\rho(s) = \left\{\overline{k} = (k_1,...,k_m) \in \mathbb{Z}^m : \quad [2^{s-1}] \leq \max_{j=1,...,m} |k_j| < 2^s\right\},$$

where $[a]$ is the integer part of the number a.

Let us consider Nikol'skii's and Besov's classes ([4, 7, 18]). Let $1 < p_j < +\infty$, $j = 1,...,m$, $1 \leq \theta \leq \infty$, $r > 0$, and

$$H^r_{\overline{p}} = \left\{f \in L_{\overline{p}}(\mathbb{T}^m) : \sup_{s\in\mathbb{Z}_+} 2^{sr} \|\delta_s(f)\|_{\bar{p}} \leq 1\right\},$$

$$B^r_{\bar{p},\theta} = \left\{f \in L_{\overline{p}}(\mathbb{T}^m) : \Big(\sum_{s\in\mathbb{Z}_+} 2^{sr\theta} \|\delta_s(f)\|^{\theta}_{\bar{p}}\Big)^{\frac{1}{\theta}} \leq 1\right\}.$$

It is known that for $1 \leq \theta \leq \theta_1 \leq \infty$ the following holds

$$B^r_{\bar{p},1} \subset B^r_{\bar{p},\theta} \subset B^r_{\bar{p},\theta_1} \subset B^r_{\bar{p},\infty} = H^r_{\bar{p}}.$$

Let $f \in L_{\bar{p}}(\mathbb{T}^m)$ and $\left\{\bar{k}^{(j)}\right\}_{j=1}^{M}$ be a system of vectors $\bar{k}^{(j)} = (k_1^{(j)},...,k_m^{(j)})$ with integer coordinates. Consider the quantity

$$e_M\left(f\right)_{\bar{p}} = \inf_{\bar{k}^{(j)},b_j} \left\|f - \sum_{j=1}^{M} b_j e^{i\langle\bar{k}^{(j)},\bar{x}\rangle}\right\|_{\bar{p}},$$

where b_j is an arbitrary number. The quantity $e_M\left(f\right)_{\bar{p}}$ is called the best M-term approximation of a function $f \in L_{\bar{p}}(\mathbb{T}^m)$. For a given class $F \subset L_{\bar{p}}(\mathbb{T}^m)$ let

$$e_M\left(F\right)_{\bar{p}} = \sup_{f\in F} e_M\left(f\right)_{\bar{p}}.$$

The best M-term approximation was defined by S.B. Stechkin [22]. Estimations of M-term approximations of different classes were provided by R.S. Ismagilov [13], E.S. Belinsky [6], V.E. Maiorov [17], B.S. Kashin [14], R. DeVore [8], V.N. Temlyakov [23], A.S. Romanyuk [19], Dinh Dung [10], D.B. Bazarkhanov [5], L. Duan [11], M. Hansen and W. Sickel [12], S.A. Stasyuk [20, 21], and others (see bibliography in [1], [2], [8], [21], [23]).

For the case $p_1 = ... = p_m = p$ and $q_1 = ... = q_m = q$, R.A. De Vore and V.N. Temlyakov [9] proved the following theorem.

1.1. Theorem. *(see [9]). Let* $1 \leq p,q,\theta \leq \infty$, $r(p,q) = m\left(\frac{1}{p}-\frac{1}{q}\right)_+$ *if* $1 \leq p \leq q \leq 2$ *or* $1 \leq q \leq p < \infty$ *and* $r(p,q) = \max\left\{\frac{m}{p},\frac{m}{2}\right\}$ *in other cases. Then, for* $r > r(p,q)$, *the following relation holds*

$$e_M(B^r_{p,\theta})_q \asymp M^{-\frac{r}{m}+\left(\frac{1}{p}-\max\left\{\frac{1}{q},\frac{1}{2}\right\}\right)_+},$$

where $a_+ = \max\{a;0\}$.

Moreover, in the case of $m\left(\frac{1}{p}-\frac{1}{q}\right) < r < \frac{m}{p}$ and $1 < p \leq 2 < q < \infty$, S.A. Stasyuk [20, 21] proved that $e_M(B^r_{p,\theta})_q \asymp M^{-\frac{q}{2}\left(\frac{r}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)\right)}$.

The main goal of the present paper is to find the order of the quantity $e_M\left(F\right)_{\bar{q}}$ for the class $F = B^r_{\bar{p},\theta}$.

Let us denote by $C(p, q, r, y)$ positive quantities, which depend on the parameters in the parentheses, such that the parameters, in general, are distinct in distinct formulas. $A(y) \asymp B(y)$ means that there are positive numbers C_1 and C_2 such that $C_1 \cdot A(y) \leq B(y) \leq C_2 \cdot A(y)$.

To prove the main results, we need the following auxiliary results.

1.2. Theorem. *(see [24]). Let $\bar{n} = (n_1, ..., n_m)$, $n_j \in \mathbb{N}$, $j = 1, ..., m$, and*

$$T_{\bar{n}}(\bar{x}) = \sum_{|k_j| \leq n_j, j=1,...,m} c_{\bar{k}} e^{i\langle \bar{k}, \bar{x}\rangle}.$$

Then, for $1 \leq p_j < q_j \leq \infty$, $j = 1, ...m$, the following inequality holds

$$\|T_{\bar{n}}\|_{\bar{q}} \leq 2^m \prod_{j=1}^{m} n_j^{\frac{1}{p_j} - \frac{1}{q_j}} \|T_{\bar{n}}\|_{\bar{p}}.$$

1.3. Theorem. *(see [16]). Let $p \in (1, \infty)$. Then there exist positive constants $C_1(p)$ and $C_2(p)$ such that for each function $f \in L_p(\mathbb{T}^m)$ the following estimation is valid*

$$C_1(p)\|f\|_p \leq \left\| \left(\sum_{s=0}^{\infty} |\delta_s(f)|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_2(p)\|f\|_p.$$

Let Ω_M be a set containing no more than M vectors $\bar{k} = (k_1, ..., k_m)$ with integer coordinates and $P(\Omega_M, \bar{x})$ be any trigonometric polynomial, which consists of harmonics with “indices“ in Ω_M.

1.4. Lemma. *(see [2]). Let $2 < q_j < +\infty$ and $j = 1, ..., m$. Then, for any trigonometric polynomial $P(\Omega_N)$ and for any natural number $M < N$, there exists a trigonometric polynomial $P(\Omega_M)$ such that the following estimation holds*

$$\|P(\Omega_N) - P(\Omega_M)\|_{\bar{q}} \leq C_1(NM^{-1})^{\frac{1}{2}} \|P(\Omega_N)\|_2,$$

and, moreover, $\Omega_M \subset \Omega_N$.

2. Main results

Let us prove the main results.

2.1. Theorem. *Let $\bar{p} = (p_1, ..., p_m)$, $\bar{q} = (q_1, ..., q_m)$, $1 < p_j \leq 2 < q_j < \infty$, and $1 \leq \theta \leq \infty$.*

1. If $\sum_{j=1}^{m} \left(\frac{1}{p_j} - \frac{1}{q_j} \right) < r < \sum_{j=1}^{m} \frac{1}{p_j}$, then

$$e_M \left(B_{\bar{p},\theta}^r \right)_{\bar{q}} \asymp M^{-\left(2 \sum_{j=1}^{m} \frac{1}{q_j}\right)^{-1} \left(r - \sum_{j=1}^{m} \left(\frac{1}{p_j} - \frac{1}{q_j} \right) \right)}.$$

2. If $r = \sum_{j=1}^{m} \frac{1}{p_j}$, then

$$e_M \left(B_{\bar{p},\theta}^r \right)_{\bar{q}} \asymp M^{-\frac{1}{2}} (\log(1 + M))^{1 - \frac{1}{\theta}}.$$

3. If $r > \sum_{j=1}^{m} \frac{1}{p_j}$, then

$$e_M \left(B_{\bar{p},\theta}^r \right)_{\bar{q}} \asymp M^{-\frac{1}{m} \left(r + \sum_{j=1}^{m} \left(\frac{1}{2} - \frac{1}{p_j} \right) \right)}.$$

Proof. Firstly, we are going to consider the upper bound in the first item. Taking into account the inclusion $B^r_{\bar{p},\theta} \subset H^r_{\bar{p}}$, $1 \le \theta < +\infty$, it suffices to prove it for the class $H^r_{\bar{p}}$.

Let $1 \le p_j < q_j < \infty$ and $\mathbb{N}$ be the set of natural numbers. For a number $M \in \mathbb{N}$ choose a natural number n such that $2^{nm} < M \le 2^{(n+1)m}$. For a function $f \in H^r_{\bar{p}}$, it is known that

$$f(\bar{x}) = \sum_{s=0}^{\infty} \delta_s(f, \bar{x})$$

and

$$\|\delta_s(f)\|_{\bar{p}} \le 2^{-sr}, \quad 1 < p_j < \infty, \quad j = 1, ..., m.$$

We will seek an approximation polynomial $P(\Omega_M, \bar{x})$ in the form

$$P(\Omega_M, \bar{x}) = \sum_{s=0}^{n-1} \delta_s(f, \bar{x}) + \sum_{n \le s < \alpha n} P(\Omega_{N_s}, \bar{x}), \tag{1}$$

where the polynomials $P(\Omega_{N_s}, \bar{x})$ will be constructed for each $\delta_s(f, \bar{x})$ in accordance with Lemma 1.4 and the number $\alpha > 1$ will be chosen during the construction.

Let $\sum\limits_{j=1}^{m} (\frac{1}{p_j} - \frac{1}{q_j}) < r < \sum\limits_{j=1}^{m} \frac{1}{p_j}$. Suppose

$$N_s = \left[2^{nm} 2^{s\left(\sum\limits_{j=1}^{m} \frac{1}{p_j} - r\right)} 2^{-n\alpha\left(\sum\limits_{j=1}^{m} \frac{1}{p_j} - r\right)} \right] + 1,$$

where $[y]$ is the integer part of the number y.

Now we are going to show that the polynomials (1) have no more than M harmonics (in terms of order). By the definition of the number N_s, we have

$$\sum_{s=0}^{n-1} \sharp\{\bar{k} = (k_1, ..., k_m) : [2^{s-1}] \le \max_{j=1,...,m} |k_j| < 2^s\} + \sum_{n \le s < \alpha n} N_s \le C 2^{nm} +$$

$$+ \sum_{n \le s < \alpha n} \left(2^{nm} 2^{s\left(\sum\limits_{j=1}^{m} \frac{1}{p_j} - r\right)} 2^{-n\alpha\left(\sum\limits_{j=1}^{m} \frac{1}{p_j} - r\right)} + 1 \right) \le C 2^{nm} + (\alpha - 1)\, n \le C 2^{nm} \asymp M,$$

where $\sharp A$ denotes the number of elements in the set A.

Next, by the property of the norm, we have

$$\|f - P(\Omega_M)\|_{\bar{q}} \le C \left\| \sum_{n \le s < \alpha n} (\delta_s(f) - P(\Omega_{N_s})) \right\|_{\bar{q}} +$$

$$+ \left\| \sum_{\alpha n \le s < +\infty} \delta_s(f) \right\|_{\bar{q}} = J_1(n) + J_2(n). \tag{2}$$

Let us estimate $J_2(n)$. Applying the inequality of different metrics for trigonometric polynomials (Theorem 1.2), we can obtain

$$J_2(n) \le \sum_{\alpha n \le s < +\infty} \|\delta_s(f)\|_{\bar{q}} \le C \sum_{\alpha n \le s < +\infty} 2^{s \sum\limits_{j=1}^{m} \left(\frac{1}{p_j} - \frac{1}{q_j}\right)} \|\delta_s(f)\|_{\bar{p}}.$$

Therefore, taking into account $f \in H^r_{\bar{p}}$ and $\sum\limits_{j=1}^{m}(\frac{1}{p_j}-\frac{1}{q_j})<r$, we get

$$J_2(n) \le C \sum_{\alpha n \le s < +\infty} 2^{-s\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)} \le C2^{-n\alpha\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}. \quad (3)$$

Let us estimate $J_1(n)$. Using the property of the norm, Lemma 1.4 and the inequality of different metrics (Theorem 1.2), we get

$$J_1(n) \le \sum_{n \le s < \alpha n} \|\delta_s(f) - P(\Omega_{N_s})\|_{\bar{q}} \le C \sum_{n \le s < \alpha n} \left(N_s^{-1} 2^{sm}\right)^{\frac{1}{2}} \|\delta_s(f)\|_2 \le$$

$$\le C \sum_{n \le s < \alpha n} \left(N_s^{-1} 2^{sm}\right)^{\frac{1}{2}} 2^{s\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{2}\right)} \|\delta_s(f)\|_{\bar{p}} \le$$

$$\le C \sum_{n \le s < \alpha n} N_s^{-\frac{1}{2}} 2^{s\sum\limits_{j=1}^{m}\frac{1}{p_j}} 2^{-sr} \le$$

$$\le C2^{-\frac{nm}{2}} 2^{\frac{n\alpha}{2}\left(\sum\limits_{j=1}^{m}\frac{1}{p_j}-r\right)} \sum_{n \le s < \alpha n} 2^{s\left(\sum\limits_{j=1}^{m}\frac{1}{p_j}-r\right)\frac{1}{2}} \le C2^{-\frac{nm}{2}} 2^{\frac{n\alpha}{2}\left(\sum\limits_{j=1}^{m}\frac{1}{p_j}-r\right)}. \quad (4)$$

Suppose $\alpha = m\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}$. Then, from the inequality (4), we get

$$J_1(n) \le C2^{-nm\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)} \asymp M^{-\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}. \quad (5)$$

For $\alpha = m\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}$, using the inequality (3) and taking into account $2^{nm} \asymp M$, we obtain

$$J_2(n) \le CM^{-\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}. \quad (6)$$

By (5) and (6), we get from the inequality (2) the following

$$\|f - P(\Omega_M)\|_{\bar{q}} \le CM^{-\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)},$$

for any function $f \in H^r_{\bar{p}}$ in the case of $\sum\limits_{j=1}^{m}(\frac{1}{p_j}-\frac{1}{q_j}) < r < \sum\limits_{j=1}^{m}\frac{1}{p_j}$.

From the inclusion $B^r_{\bar{p},\theta} \subset H^r_{\bar{p}}$ and the definition of the M-term approximation, it follows that

$$e_M\left(B^r_{\bar{p},\theta}\right)_{\bar{q}} \le CM^{-\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}$$

in the case of $\sum\limits_{j=1}^{m}(\frac{1}{p_j}-\frac{1}{q_j}) < r < \sum\limits_{j=1}^{m}\frac{1}{p_j}$.

Let us consider the lower bound. We will use the well-known formula (see [19], p. 79)

$$e_M(f)_{\bar{q}} = \inf_{\Omega_M} \sup_{P \in L_M^{\perp}, \|P\|_{\bar{q}'} \le 1} \left| \int_{\mathbb{T}^m} f(\bar{x})\bar{P}(\bar{x})d\bar{x} \right|, \quad (7)$$

where $\bar{q}' = (q_1{}', ..., q_m{}')$, $\frac{1}{q_j} + \frac{1}{q_j{}'} = 1$, $j = 1, ..., m$, and $L_M^{\perp}$ is the set of functions that are orthogonal to the subspace of trigonometric polynomials with harmonics in the set Ω_M.

Consider the function

$$F_{\bar{q},n}(\bar{x}) = \sum_{\max\limits_{j=1,\dots,m}|k_j|\le 2^{\left[nm\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}\right]}} e^{i\langle\bar{k},\bar{x}\rangle}.$$

Let Ω_M be a set of M vectors with integer coordinates. Suppose

$$g(\bar{x}) = F_{\bar{q},n}(\bar{x}) - \sum_{\bar{k}\in\Omega_M}^{*} e^{i\langle\bar{k},\bar{x}\rangle},$$

where the sum $\sum\limits_{\bar{k}\in\Omega_M}^{*}$ contains those terms in the function $F_{\bar{q},n}(\bar{x})$ with indices only in Ω_M. By the inequality (see [18], p. 88)

$$\Big\| \sum_{\max\limits_{j=1,\dots,m}|k_j|\le 2^l} e^{i\langle\bar{k},\bar{x}\rangle}\Big\|_{\bar{p}} \le C2^{l\sum\limits_{j=1}^{m}(1-\frac{1}{p_j})} \tag{8}$$

and Perseval's equality for $1 < {q_j}' < 2$, $j = 1, ..., m$, we obtain

$$\|g\|_{\bar{q}'} \le \|F_{\bar{q},n}\|_{\bar{q}'} + (2\pi)^{\sum\limits_{j=1}^{m}(\frac{1}{q_j}-\frac{1}{2})}\Big\|\sum_{\bar{k}\in\Omega_M}^{*} e^{i\langle\bar{k},\bar{x}\rangle}\Big\|_2 \le C(2^{\frac{nm}{2}} + M^{\frac{1}{2}}) \le C2^{\frac{nm}{2}}. \tag{9}$$

Now we consider the function

$$P_1(\bar{x}) = C_2 2^{\left(-\frac{nm}{2}\right)} g(\bar{x}). \tag{10}$$

Then (9) implies that the function P_1 satisfies the assumptions of the formula (7) for some constant $C_2 > 0$.

Consider the function

$$f_1(\bar{x}) = C_3 2^{-nm\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-1\right)\right)} F_{\bar{q},n}(\bar{x}). \tag{11}$$

By the inequality (8), we get

$$\sum_{s=0}^{\infty} 2^{sr}\left\|\delta_s(f_1)\right\|_{\bar{p}} \le$$

$$\le C2^{-nm\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-1\right)\right)} \sum_{s=0}^{\left[nm\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}\right]} 2^{sr}2^{s\sum\limits_{j=1}^{m}\left(1-\frac{1}{p_j}\right)} \le C_3.$$

Hence $C_3^{-1} f_1 \in B^r_{\bar{p},1}$.

For the functions (10) and (11), we have, by the formula (7), the following

$$e_M(f_1)_{\bar{q}} \ge \inf_{\Omega_M}\left|\int_{\mathbb{T}^m} f_1(\bar{x})\,\bar{P}_1(\bar{x})\,d\bar{x}\right| \ge$$

$$\ge C2^{-nm\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-1\right)\right)} 2^{-\frac{nm}{2}}\left(\|F_{\bar{q},n}\|_2^2 - M\right) \ge$$

$$\ge C2^{-nm\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}. \tag{12}$$

Hence, it follows from (12) by the inclusion $B^r_{\bar{p},1} \subset B^r_{\bar{p},\theta}$ that

$$e_M(f_1)_{\bar{q}} \ge C2^{-nm\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}$$

in the case of $\sum\limits_{j=1}^{m}(\frac{1}{p_j}-\frac{1}{q_j})<r<\sum\limits_{j=1}^{m}\frac{1}{p_j}$. So we have proved the first item.

Now we consider the case $r=\sum\limits_{j=1}^{m}\frac{1}{p_j}$. Let $f\in B^r_{\bar{p},\theta}$. Suppose $\alpha=m\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}$ and

$$N_s=\left[2^{nm}n^{\frac{1}{\theta}-1}\left\|\delta_s(f)\right\|_{\bar{p}}2^{sr}\right]+1.$$

Then, by definition of the numbers N_s and Holder's inequality, we obtain

$$\sum_{s=0}^{n-1}\sharp\{\bar{k}=(k_1,...,k_m):[2^{s-1}]\le\max_{j=1,...,m}|k_j|<2^s\}+\sum_{n\le s<\alpha n}N_s\le$$

$$\le C2^{nm}+(\alpha-1)n+2^{nm}n^{\frac{1}{\theta}-1}((\alpha-1)n)^{\frac{1}{\theta'}}\left(\sum_{s=0}^{\infty}\left\|\delta_s(f)\right\|_{\bar{p}}^{\theta}2^{sr\theta}\right)^{\frac{1}{\theta}}\le C2^{nm}\asymp M.$$

Suppose $\beta=\max\{q_1,...,q_m\}$. Then

$$J_1(n)=\left\|\sum_{n\le s<\alpha n}(\delta_s(f)-P(\Omega_{N_s}))\right\|_{\bar{q}}\le C\left\|\sum_{n\le s<\alpha n}(\delta_s(f)-P(\Omega_{N_s}))\right\|_{\beta}.$$

Next, by Theorem 1.3, we have

$$J_1(n)\le C\left\|\left(\sum_{n\le s<\alpha n}|\delta_s(f)-P(\Omega_{N_s})|^2\right)^{\frac{1}{2}}\right\|_{\beta}.$$

Since $\beta>2$, then by applying the property of the norm, Lemma 1.4 and the inequality of different metrics for trigonometric polynomials (see Theorem 1.2), we obtain

$$J_1(n)\le\left(\sum_{n\le s<\alpha n}\left\|\delta_s(f)-P(\Omega_{N_s})\right\|_{\beta}^2\right)^{\frac{1}{2}}\le C\left(\sum_{n\le s<\alpha n}N_s^{-1}2^{sm}\left\|\delta_s(f)\right\|_2^2\right)^{\frac{1}{2}}\le$$

$$\le C\left(\sum_{n\le s<\alpha n}N_s^{-1}2^{sm}2^{2s\sum\limits_{j=1}^{m}(\frac{1}{p_j}-\frac{1}{2})}\left\|\delta_s(f)\right\|_{\bar{p}}^2\right)^{\frac{1}{2}}. \tag{13}$$

Next, since $r=\sum\limits_{j=1}^{m}\frac{1}{p_j}$, we have, by the definition of the numbers N_s and using Holder's inequality, the following

$$J_1(n)\le C(2^{-nm}n^{1-\frac{1}{\theta}})^{\frac{1}{2}}\left(\sum_{n\le s<\alpha n}2^{sr}\left\|\delta_s(f)\right\|_{\bar{p}}\right)^{\frac{1}{2}}\le$$

$$\le C(2^{-nm}n^{1-\frac{1}{\theta}})^{\frac{1}{2}}\left(\sum_{n\le s<\alpha n}2^{sr\theta}\left\|\delta_s(f)\right\|_{\bar{p}}^{\theta}\right)^{\frac{1}{2\theta}}\left(\sum_{n\le s<\alpha n}1\right)^{\frac{1}{2}(1-\frac{1}{\theta})}$$

$$\le C2^{-\frac{nm}{2}}n^{1-\frac{1}{\theta}}\asymp M^{-\frac{1}{2}}(\log(1+M))^{1-\frac{1}{\theta}}.$$

Thus,

$$J_1(n)\le CM^{-\frac{1}{2}}(\log(1+M))^{1-\frac{1}{\theta}} \tag{14}$$

in the case of $r=\sum\limits_{j=1}^{m}\frac{1}{p_j}$.

To estimate $J_2(n)$, we apply Holder's inequality, and taking into account $r = \sum_{j=1}^{m} \frac{1}{p_j}$ and $\alpha = m\left(2\sum_{j=1}^{m}\frac{1}{q_j}\right)^{-1}$, we obtain

$$J_2(n) \le C \sum_{n\alpha \le s < +\infty} 2^{s\sum_{j=1}^{m}(\frac{1}{p_j}-\frac{1}{q_j})} \|\delta_s(f)\|_{\bar{p}} \le \tag{15}$$

$$\le C\left(\sum_{s=0}^{\infty} 2^{sr\theta}\|\delta_s(f)\|_{\bar{p}}^{\theta}\right)^{\frac{1}{\theta}}\left(\sum_{n\alpha\le s<+\infty} 2^{-s\theta'\sum_{j=1}^{m}\frac{1}{q_j}}\right)^{\frac{1}{\theta'}} \le C2^{-n\alpha\sum_{j=1}^{m}\frac{1}{q_j}} = C2^{-\frac{nm}{2}} \asymp M^{-\frac{1}{2}}.$$

By (14) and (15), the inequality (2) implies that

$$\|f - P(\Omega_M)\|_{\bar{q}} \le CM^{-\frac{1}{2}}(\log(1+M))^{1-\frac{1}{\theta}}$$

in the case of $r = \sum_{j=1}^{m}\frac{1}{p_j}$. It proves the upper bound estimation in the second item.

Let $r > \sum_{j=1}^{m}\frac{1}{p_j}$. Suppose

$$N_s = \left[2^{n\left(r-\sum_{j=1}^{m}\left(\frac{1}{p_j}-1\right)\right)}2^{-s\left(r-\sum_{j=1}^{m}\frac{1}{p_j}\right)}\right]+1.$$

Then

$$\sum_{s=0}^{n-1}\sharp\{\bar{k}=(k_1,...,k_m):[2^{s-1}] \le \max_{j=1,\dots,m}|k_j| < 2^s\} + \sum_{n\le s<\alpha n} N_s \le$$
$$\le C2^{nm} + (\alpha-1)n \le C2^{nm} \le CM.$$

If $f \in H_{\bar{p}}^r$, then, by using the definition of the numbers N_s and $r > \sum_{j=1}^{m}\frac{1}{p_j}$, we obtain from (13) the following

$$J_1(n) \le \left(\sum_{n\le s<\alpha n} N_s^{-1}2^{sm}2^{2s\sum_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{2}\right)}\|\delta_s(f)\|_{\bar{p}}^2\right)^{\frac{1}{2}} \le$$

$$\le C2^{-\frac{n}{2}\left(r-\sum_{j=1}^{m}\left(\frac{1}{p_j}-1\right)\right)}\left(\sum_{n\le s<\alpha n}2^{-s\left(r-\sum_{j=1}^{m}\frac{1}{p_j}\right)}\right)^{\frac{1}{2}} \le C2^{-n\left(r+\sum_{j=1}^{m}\left(\frac{1}{2}-\frac{1}{p_j}\right)\right)}.$$

Thus,

$$J_1(n) \le CM^{-\frac{1}{m}\left(r+\sum_{j=1}^{m}(\frac{1}{2}-\frac{1}{p_j})\right)} \tag{16}$$

in the case of $r > \sum_{j=1}^{m}\frac{1}{p_j}$.

To estimate $J_2(n)$, we suppose $\alpha = \left(r+\sum_{j=1}^{m}\left(\frac{1}{2}-\frac{1}{p_j}\right)\right)\left(r+\sum_{j=1}^{m}\left(\frac{1}{q_j}-\frac{1}{p_j}\right)\right)^{-1}$ and get

$$J_2(n) \le C\sum_{n\alpha\le s<\infty}2^{-s\left(r+\sum_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)} \le C2^{-n\left(r+\sum_{j=1}^{m}\left(\frac{1}{2}-\frac{1}{p_j}\right)\right)} \le$$

$$\le CM^{-\frac{1}{m}\left(r+\sum_{j=1}^{m}\left(\frac{1}{2}-\frac{1}{p_j}\right)\right)} \tag{17}$$

for a function $f \in H^r_{\bar{p}}$. By (16) and (17), it follows from (2) that

$$\|f - P\left(\Omega_M\right)\|_{\bar{q}} \le CM^{-\frac{1}{m}\left(r + \sum\limits_{j=1}^{m}\left(\frac{1}{2} - \frac{1}{p_j}\right)\right)}$$

for any function $f \in H^r_{\bar{p}}$ in the case of $r > \sum\limits_{j=1}^{m} \frac{1}{p_j}$.

It follows from $B^r_{\bar{p},\theta} \subset H^r_{\bar{p}}$ that

$$e_M\left(B^r_{\bar{p},\theta}\right)_{\bar{q}} \le e_M\left(H^r_{\bar{p}}\right)_{\bar{q}} \le CM^{-\frac{1}{m}\left(r + \sum\limits_{j=1}^{m}\left(\frac{1}{2} - \frac{1}{p_j}\right)\right)}$$

in the case of $r > \sum\limits_{j=1}^{m} \frac{1}{p_j}$. It proves the upper bound estimation in the item 3.

Let us consider the lower bound estimation in the case of $r = \sum\limits_{j=1}^{m} \frac{1}{p_j}$. Consider the function

$$g_1(\bar{x}) = \sum_{s=1}^{n} \sum_{\bar{k} \in \rho(s)} \prod_{j=1}^{m} k_j^{-1} \cos k_j x_j. \tag{18}$$

Then

$$\delta_s(g_1, \bar{x}) = \sum_{\bar{k} \in \rho(s)} \prod_{j=1}^{m} k_j^{-1} \cos k_j x_j.$$

It is known that for a function $d_s(\bar{x}) = \sum\limits_{\bar{k} \in \rho(s)} \prod\limits_{j=1}^{m} \cos k_j x_j$ the following relation holds

$$\|d_s\|_{\bar{p}} \asymp 2^{s\sum\limits_{j=1}^{m}(1 - \frac{1}{p_j})}, \ \ 1 < p_j < +\infty, \ \ j = 1, ..., m.$$

Therefore, by the Marcinkiewicz theorem on multipliers (see [18]), we have

$$\|\delta_s(g_1)\|_{\bar{p}} \le C2^{-sm}\|d_s\|_{\bar{p}} \le C2^{-s\sum\limits_{j=1}^{m}\frac{1}{p_j}}.$$

Hence, since $r = \sum\limits_{j=1}^{m} \frac{1}{p_j}$, we obtain

$$\left(\sum_{s=0}^{\infty} 2^{sr\theta} \|\delta_s(g_1)\|^{\theta}_{\bar{p}}\right)^{\frac{1}{\theta}} \le C_1 n^{\frac{1}{\theta}}.$$

Therefore, the function $f_2(\bar{x}) = C_1^{-1} n^{-\frac{1}{\theta}} g_1(\bar{x})$ belongs to the class $B^r_{\bar{p},\theta}$, $1 < p_j < +\infty$, $j = 1, ..., m$.

Now, we are going to construct a function P_1, which satisfies the conditions of the formula (7). Let

$$v_1(\bar{x}) = \sum_{s=1}^{n} \sum_{\bar{k} \in \rho(s)} \prod_{j=1}^{m} \cos k_j x_j$$

and Ω_M be an arbitrary set of M vectors $\bar{k} = (k_1, ..., k_m)$ with integer coordinates. Consider the function

$$u_1(\bar{x}) = \sum_{\bar{k} \in \Omega_M}^{*} \prod_{j=1}^{m} \cos k_j x_j$$

which contains only those terms in (18) with indices in Ω_M. Suppose $w_1(\bar{x}) = v_1(\bar{x}) - u_1(\bar{x})$. Then, since $1 < {q_j}' < 2$, $j = 1, ..., m$, we obtain, by Perseval's equality, the following

$$\|w_1\|_{\bar{q}'} \le \|v_1\|_{\bar{q}'} + \|u_1\|_2 \le \|v_1\|_{\bar{q}'} + CM^{\frac{1}{2}}.$$

By the property of the norm and the estimation of the norm of the Dirichlet kernel in the Lebesgue space, we have

$$\|v_1\|_{\bar{q}'} \leq \sum_{s=1}^{n} \|\delta_s(v_1)\|_{\bar{q}'} \leq$$

$$\leq C \sum_{s=1}^{n} 2^{s \sum\limits_{j=1}^{m}\left(1-\frac{1}{q_j'}\right)} \leq C 2^{n \sum\limits_{j=1}^{m} \frac{1}{q_j}}.$$

Therefore, taking into account $\frac{1}{q_j} < \frac{1}{2}$, $j = 1, ..., m$, we get

$$\|w_1\|_{\bar{q}'} \leq C(2^{\frac{nm}{2}} + M^{\frac{1}{2}}) \leq C_2 2^{\frac{nm}{2}}.$$

Hence, the function

$$P_1(\bar{x}) = C_2^{-1} 2^{-\frac{nm}{2}} w_1(\bar{x})$$

satisfies the conditions of the formula (7). Then, by substituting the functions f_2 and P_1 into (7) and by orthogonality of the trigonometric system, we obtain

$$e_M\left(f_2\right)_{\bar{q}} \geq C \sum_{n_1 \leq s < n} \sum_{\bar{k} \in \rho(s)} \prod_{j=1}^{m} k_j^{-1} 2^{-\frac{nm}{2}} n^{-\frac{1}{\theta}} \geq$$

$$\geq C(\ln 2)^m \sum_{n_1 \leq s < n} 2^{-\frac{nm}{2}} n^{-\frac{1}{\theta}} = C(\ln 2)^m 2^{-\frac{nm}{2}} n^{-\frac{1}{\theta}}(n - n_1) \geq$$

$$\geq C(\ln 2)^m 2^{-\frac{nm}{2}} n^{1-\frac{1}{\theta}} \asymp M^{-\frac{1}{2}} (\log(1+M))^{1-\frac{1}{\theta}},$$

where n_1 is a natural number such that $n_1 \leq \frac{n}{2}$.

So, for the function $f_2 \in B^r_{\bar{p},\theta}$, it has been proved that

$$e_M\left(f_2\right)_{\bar{q}} \geq C M^{-\frac{1}{2}} (\log(1+M))^{1-\frac{1}{\theta}}$$

in the case of $r = \sum\limits_{j=1}^{m} \frac{1}{p_j}$. Hence

$$e_M\left(B^r_{\bar{p},\theta}\right)_{\bar{q}} \geq C M^{-\frac{1}{2}} (\log(1+M))^{1-\frac{1}{\theta}}$$

in the case of $r = \sum\limits_{j=1}^{m} \frac{1}{p_j}$. It proves the lower bound estimation in the second item.

Let us prove the lower bound estimation for the case $r > \sum\limits_{j=1}^{m} \frac{1}{p_j}$. Since in this case an upper bound estimation of the quantity $e_M\left(B^r_{\bar{p},\theta}\right)_{\bar{q}}$ does not depend on θ and $B^r_{\bar{p},1} \subset B^r_{\bar{p},\theta}$, $1 < \theta \leq +\infty$, it suffices to prove the lower bound estimation for $B^r_{\bar{p},1}$.

For a number $M \in \mathbb{N}$, we choose a natural number n such that $2^{nm} < M \leq 2^{(n+1)m}$ and $2M \leq \sharp\rho(n)$, where $\sharp\rho(n)$ denotes the number of elements in the set $\rho(n)$.

Consider the following function

$$f_3(\bar{x}) = 2^{-n\left(r + \sum\limits_{j=1}^{m}\left(1-\frac{1}{p_j}\right)\right)} \sum_{\bar{k} \in \rho(n)} e^{i\langle \bar{k}, \bar{x}\rangle}.$$

Then $\|\delta_s(f_3)\|_{\bar{p}} = 0$ provided $s \neq n$ and

$$\|\delta_n(f_3)\|_{\bar{p}} = 2^{-n\left(r + \sum\limits_{j=1}^{m}\left(1-\frac{1}{p_j}\right)\right)} \prod_{j=1}^{m} \Big\| \sum_{k_j=2^{n-1}}^{2^n-1} e^{ik_jx_j} \Big\|_{p_j}.$$

By the estimation of the norm of the Dirichlet kernel (see [18], p. 181), we have

$$\Big\|\sum_{k_j=2^{n-1}}^{2^n-1} e^{ik_jx_j}\Big\|_{p_j} \le C2^{n(1-\frac{1}{p_j})},$$

for $p_j \in (1,\infty)$, $j=1,...,m$. Therefore

$$\|\delta_n(f_3)\|_{\bar p} \le C2^{-nr}.$$

Hence

$$\sum_{s=0}^{\infty} 2^{sr}\,\|\delta_s(f_3)\|_{\bar p} \le C_3,$$

i.e. the function $C_3^{-1}f_3 \in B^r_{\bar p,1}$. Next, we consider the functions

$$v_2(\bar x) = \sum_{\bar k\in\rho(n)} e^{i\langle \bar k,\bar x\rangle}$$

and

$$u_2(\bar x) = \sum_{\bar k\in\rho(n)\cap\Omega_M} e^{i\langle \bar k,\bar x\rangle}.$$

Suppose $w_2(\bar x)=v_2(\bar x)-u_2(\bar x)$. By Perseval's equality,

$$\|u_2\|_2 \le M^{\frac12}, \quad \|v_2\|_2 \le C2^{\frac{nm}{2}}.$$

From these relations, we obtain, by the properties of the norm, the following

$$\|w_2\|_2 \le \|v_2\|_2 + \|u_2\|_2 \le C_4 2^{\frac{nm}{2}}.$$

Therefore, the function $P_2(\bar x)=C_4^{-1}2^{-\frac{nm}{2}}w_2(\bar x)$ satisfies the conditions of the formula (7). Since $2<q_j<\infty$, $j=1,...,m$, we have $e_M\left(f_3\right)_2 \le Ce_M\left(f_3\right)_{\bar q}$. Now, by the formula (7), we get

$$e_M\left(f_3\right)_{\bar q} \ge Ce_M\left(f_3\right)_2 \ge$$

$$\ge C\inf_{\Omega_M}\int_{\mathbb{T}^m} f_3(\bar x)\bar P_2(\bar x)d\bar x =$$

$$= C_2^{-1}2^{-\frac{nm}{2}}2^{-n\left(r+\sum\limits_{j=1}^{m}\left(1-\frac{1}{p_j}\right)\right)}\inf_{\Omega_M}\left[\sharp\rho(n)-\sharp\left(\rho(n)\cap\Omega_M\right)\right] \ge$$

$$\ge C2^{-\frac{nm}{2}}2^{-n\left(r+\sum\limits_{j=1}^{m}\left(1-\frac{1}{p_j}\right)\right)}\left[\sharp\rho(n)-M\right] \ge$$

$$\ge C2^{-\frac{nm}{2}}2^{-n\left(r+\sum\limits_{j=1}^{m}\left(1-\frac{1}{p_j}\right)\right)}\left[\sharp\rho(n)-\frac{\sharp\rho(n)}{2}\right] \ge$$

$$\ge C2^{-\frac{nm}{2}}2^{-n\left(r-\sum\limits_{j=1}^{m}\frac{1}{p_j}\right)}.$$

It follows from the relation $2^{nm}\asymp M$ that

$$e_M\left(f_3\right)_{\bar q} \ge CM^{-\frac{1}{m}\left(r+\sum\limits_{j=1}^{m}\left(\frac12-\frac{1}{p_j}\right)\right)}$$

in the case of $r>\sum\limits_{j=1}^{m}\frac{1}{p_j}$ for the function $C_3^{-1}f_3\in B^r_{\bar p,1}$. Hence

$$e_M\left(B^r_{\bar p,1}\right)_{\bar q} \ge CM^{-\frac{1}{m}\left(r+\sum\limits_{j=1}^{m}\left(\frac12-\frac{1}{p_j}\right)\right)}.$$

Therefore,

$$e_M\left(B^r_{\bar{p},\theta}\right)_{\bar{q}} \geq CM^{-\frac{1}{m}\left(r+\sum\limits_{j=1}^{m}\left(\frac{1}{2}-\frac{1}{p_j}\right)\right)}$$

in the case of $r > \sum\limits_{j=1}^{m}\frac{1}{p_j}$. So Theorem 2.1 has been proved.

2.2. Theorem. *Let $\bar{p}=(p_1,...,p_m)$, $\bar{q}=(q_1,...,q_m)$, $1 < p_j < q_j \leq 2$, and $1 \leq \theta \leq +\infty$. If $r > \sum\limits_{j=1}^{m}(\frac{1}{p_j}-\frac{1}{q_j})$, then*

$$e_M\left(B^r_{\bar{p},\theta}\right)_{\bar{q}} \asymp M^{-\frac{1}{m}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}.$$

Proof. For a number $M \in \mathbb{N}$, we choose a natural number n such that $M \asymp 2^{nm}$. By the inequality of distinct metrics (see Theorem 1.2) and by Holder's inequality, we have

$$\|f-\sum_{s=0}^{n}\delta_s(f)\|_{\bar{q}} \leq \sum_{s=n}^{\infty}\|\delta_s(f)\|_{\bar{q}} \leq$$

$$\leq \left[\sum_{s=0}^{\infty}2^{sr\theta}\|\delta_s(f)\|^{\theta}_{\bar{q}}\right]^{\frac{1}{\theta}}\left[\sum_{s=n}^{\infty}2^{s\theta'\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}\right]^{\frac{1}{\theta'}} \leq$$

$$\leq C2^{n\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)} \leq CM^{-\frac{1}{m}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}$$

for $f \in B^r_{\bar{p},\theta}$, $\frac{1}{\theta}+\frac{1}{\theta'}=1$. Therefore

$$e_M(f)_{\bar{q}} \leq \|f-\sum_{s=0}^{n}\delta_s(f)\|_{\bar{q}} \leq CM^{-\frac{1}{m}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}.$$

Hence

$$e_M\left(B^r_{\bar{p},\theta}\right)_{\bar{q}} \leq CM^{-\frac{1}{m}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}.$$

It proves the upper bound estimation.

For the lower bound estimation, let us consider the function

$$f_4(\bar{x}) = n^{-r+\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-1\right)}V_n(\bar{x}),$$

where $V_n(\bar{x})$ is a multiple of the Valle-Poisson sum.

Next, following the proof in [9] (pp. 46-47) and applying Theorem 1.2, we obtain the lower bound estimation of the quantity $e_M\left(B^r_{\bar{p},\theta}\right)_{\bar{q}}$.

2.3. Theorem. *Let $\bar{p}=(p_1,...,p_m)$, $\bar{q}=(q_1,...,q_m)$, $2 \leq p_j < q_j < \infty$, $j=1,...,m$, and $1 \leq \theta \leq +\infty$. If $r > \frac{m}{2}$, then*

$$e_M\left(B^r_{\bar{p},\theta}\right)_{\bar{q}} \asymp M^{-\frac{r}{m}}.$$

Proof. By the inclusion $B^r_{\bar{p},\theta} \subset B^r_{2,\theta} \subset H^r_2$, we have

$$e_M\left(B^r_{\bar{p},\theta}\right)_{\bar{q}} \leq e_M\left(B^r_{2,\theta}\right)_{\bar{q}} \leq e_M\left(H^r_2\right)_{\bar{q}}.$$

By Theorem 2.1,

$$e_M\left(H^r_2\right)_{\bar{q}} \leq CM^{-\frac{r}{m}},$$

for $p_j=2$, $j=1,...,m$. Hence

$$e_M\left(B^r_{\bar{p},\theta}\right)_{\bar{q}} \leq CM^{-\frac{r}{m}}.$$

It proves the upper bound estimation.

Let us consider the lower bound estimation. Consider Rudin-Shapiro's polynomial (see [15], p. 155) of the type

$$R_s(x) = \sum_{s=2^{s-1}}^{2^s} \varepsilon_k e^{ikx}, \;\; x \in [0, 2\pi], \;\; \varepsilon_k = \pm 1.$$

It is known that $\|R_s\|_\infty = \max\limits_{x\in[0,2\pi]} |R_s(x)| \le C 2^{\frac{s}{2}}$ (see [15], p. 155). For a given number M choose a number n such that $M \asymp 2^{nm}$. Now we consider the function

$$f_5(\bar{x}) = 2^{-n(\frac{m}{2}+r)} \sum_{s=1}^{n} \prod_{1}^{m} R_s(x_j).$$

Then, by the continuity, we have $f_5 \in L_{\bar{p}}(\mathbb{T}^m)$ and

$$\sum_{s=0}^{\infty} 2^{s\theta r} \|\delta_s(f_5)\|_{\bar{p}}^{\theta} = 2^{-n(\frac{m}{2}+r)} \sum_{s=1}^{n} 2^{s\theta r} \| \prod_{1}^{m} R_s(x_j)\|_{\bar{p}}^{\theta} \le$$

$$\le 2^{-n(\frac{m}{2}+r)} \sum_{s=1}^{n} 2^{s(\frac{m}{2}+r)\theta} \le C_5.$$

Hence, the function $C_5^{-1} f_5 \in B^r_{\bar{p},\theta}$. Now, we construct a function $P(\bar{x})$, which satisfies the conditions in the formula (7). Suppose

$$v_3(\bar{x}) = \sum_{s=1}^{n} \prod_{1}^{m} R_s(x_j), \;\; u_3(\bar{x}) = \sum_{s}^{*} \prod_{1}^{m} R_s(x_j),$$

where the sign $*$ means that the polynomial $u_3(\bar{x})$ contains only those harmonics of v_3, which have indices in Ω_M. Suppose $w_3(\bar{x}) = v_3(\bar{x}) - u_3(\bar{x})$. Then, since $1 < {q_j}' = \frac{q_j}{q_j - 1} < 2$, $j = 1, ..., m$, we have the following (by Perseval's equality)

$$\|w_3\|_{\bar{q}'} \le \|w_3\|_2 \le C_1 2^{\frac{nm}{2}}.$$

Therefore, for the function $P_3(\bar{x}) = C_1^{-1} 2^{-\frac{nm}{2}} w_3(\bar{x})$ the inequality $\|P_3\|_{\bar{q}'} \le 1$ holds. Now, using the formula (7), we obtain

$$e_M\left(B^r_{\bar{p},\theta}\right)_{\bar{q}} \ge e_M(f_3)_{\bar{q}} \ge 2^{-n(\frac{m}{2}+r)} 2^{-\frac{nm}{2}} (2^{nm} - M) \ge C 2^{-n(m+r)} 2^{nm} \ge C M^{-\frac{r}{m}}.$$

So

$$e_M\left(B^r_{\bar{p},\theta}\right)_{\bar{q}} \ge C M^{-\frac{r}{m}}.$$

It proves Theorem 2.3.

Remark. In the case $p_j = p$, $q_j = q$, $j = 1, ..., m$, and $r > m(\frac{1}{p} - \frac{1}{q})$, the results of R.A. DeVore and V.N. Temlyakov [9] follow from Theorem 2.1 - 2.3. If $1 < p \le 2 < q < \infty$ and $m(\frac{1}{p} - \frac{1}{q}) < r \le \frac{m}{p}$, the results of S.A. Stasyuk [20, 21] follow from the first and second items of Theorem 2.1. Theorem 2.1 - 2.3 were announced in [3].

Acknowledgements. This work was supported by the Ministry of Education and Science of Republic Kazakhstan (Grant no. 5129GF4) and by the Competitiveness Enhancement Program of the Ural Federal University (Enactment of the Government of the Russian Federation of March 16, 2013 no. 211, agreement no. 02.A03. 21.0006 of August 27, 2013).

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