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Semiprime and weakly compressible modules

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Abstract

An R-module M is called semiprime (resp. weakly compressible) if it is cogenerated by each of its essential submodules (resp. $\mathrm{Hom}_R(M, N)N$ is nonzero for every $0 \neq N \leq M_R$). We carry out a study of weakly compressible (semiprime) modules and show that there exist semiprime modules which are not weakly compressible. Weakly compressible modules with enough critical submodules are characterized in different ways. For certain rings R, including prime hereditary Noetherian rings, it is proved that M_R is weakly compressible (resp. semiprime) if and only if $M \in \mathrm{Cog}(\mathrm{Soc}(M) \oplus R)$ and $M/\mathrm{Soc}(M) \in \mathrm{Cog}(R)$ (resp. $M \in \mathrm{Cog}(\mathrm{Soc}(M) \oplus R)$). These considerations settle two questions, namely Qu 1, and Qu 2, in [6, p 92].

Keywords: Krull dimension, semiprime module, singular semi-Artinian ring, weakly compressible module.

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1. Introduction

Throughout this paper rings will have a nonzero identity, modules will be right and unitary. In [2], a module M_R is called *prime* if $\mathrm{Hom}_R(M, K)N \neq 0$ for all nonzero submodules $K, N \leq M_R$ and it is shown that M_R is prime if and only if it is cogenerated by each of its nonzero submodules. A semiprime notion for modules is then obtained in [4] by setting $K = N$ in the above definition of prime modules. These semiprime modules are precisely weakly compressible modules in the sense of [1]; see for example Theorem 2.5 below. Following [6], a module M_R is called *weakly compressible* if $\mathrm{Hom}_R(M, N)N \neq 0$ for all nonzero $N \leq M_R$. We also call M_R *semiprime* if every essential submodule of M_R cogenerates M_R. In this paper, prime module means the prime module in the sense of

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[2]; see [11, Sections 13, 14] for an excellent reference on the subject. Weakly compressible modules have applied in different situations. For example, in the study of weakly semisimple modules [12] and modules which have semiprime right Goldie endomorphism rings [3, Theorem 2.6]. They have also been appeared in the Cohen-Fishman's question about the semiprimeness of the smash product $A\#H$ when H is a semisimple Hopf algebra and A is a semiprime H-module algebra. In [6, Corollary 7.6] for certain semisimple Hopf algebra H, it is shown that $A\#H$ is a semiprime ring if and only if the $A\#H$-module A is weakly compressible.

In the present work, we carry out a study of weakly compressible (semiprime) modules and show that there are semiprime modules which are not weakly compressible (Examples and Remarks 2.8). Weakly compressible modules with enough critical submodules are characterized in different ways (Theorems 3.4 and 3.7). For certain rings R, including prime hereditary Noetherian rings, it is shown that M_R is weakly compressible (resp. semiprime) if and only if $M \in \mathrm{Cog}(\mathrm{Soc}(M) \oplus R)$ and $M/\mathrm{Soc}(M) \in \mathrm{Cog}(R)$ (resp. $M \in \mathrm{Cog}(\mathrm{Soc}(M) \oplus R)$). Furthermore, if R is a PID then M_R is weakly compressible if and only if $M/\mathrm{Soc}(M) \in \mathrm{Cog}(R)$ (Corollary 4.6). These considerations settle two questions, namely Qu 1, and qu 2, in [6, p 92] where it is asked whether there exists a weakly compressible module M which is not a subdirect product of prime modules. Such a module M cannot satisfy the conditions of Theorem 3.4 or 3.7 or 4.1 because of Remark 4.2. Any unexplained terminology and all the basic results on rings and modules that are used in the sequel can be found in [5] and [7].

2. General properties of weakly compressible modules

In this section, we investigate weakly compressible (semiprime) modules over any ring and show that semiprime modules are not necessarily weakly compressible. We give a characterization of weakly compressible modules and using this we state our main results in the next sections. Let M be an R-module and N be a submodule of M_R. We say that *M is N-weakly compressible* if for each nonzero submodule K of N, there exists an R-homomorphism $f : M \to K$ such that $f(K) \neq 0$. Thus M_R is weakly compressible if and only if M is M-weakly compressible if and only if M is N-weakly compressible for any $0 \neq N \leq M_R$. We use the notation $N \leq_{ess} M$ to denote N is an essential submodule of M. Also, if X and Y are R-modules, then $\cap\{\ker f \mid f : X_R \to Y_R\}$ is denoted by $\mathrm{Rej}(X, Y)$. The module X is cogenerated by Y (write $X \in \mathrm{Cog}(Y)$) if $\mathrm{Rej}(X, Y) = 0$. In the following, some properties of weakly compressible (semiprime) modules are collected.

2.1. Lemma. (a) *Let M be a semiprime R-module. If N is either an essential or fully invariant submodule of M_R, then N is a semiprime R-module.*
(b) *The class of weakly compressible modules is closed under co-products and taking submodules.*
(c) *The class of semiprime modules is closed under products and co-products.*
(d) *Let Λ be a non-empty set. Then M_R is semiprime if and only if $M_R^{(\Lambda)}$ is so.*
(e) *Every weakly compressible module is semiprime.*
(f) *Let M be a nonzero R-module and M_1, M_2 be submodules of M_R such that there is no nonzero R-module X which embeds in M_1 and M_2. Then M is $(M_1 \oplus M_2)$-weakly compressible if and only if M is M_i-weakly compressible for $i = 1, 2$.*
(g) *If M_R is semiprime then $\mathrm{ann}_R(M)$ is a semiprime ideal of R.*
(h) *M_R is weakly compressible (resp. semiprime) if and only if $M_{R/I}$ is weakly compressible (resp. semiprime) where $MI = 0$ and $I \rhd R$.*
(i) *If M_R is weakly compressible and N is a fully invariant closed submodule of M_R, then M/N is a weakly compressible R-module.*

Proof. (a) If $N \leq_{ess} M_R$, then it is easy to see that N_R is semiprime. Let N be a fully invariant of M_R and $K \leq_{ess} N$. There exists a submodule L of M_R such that $N \cap L = 0$ and $N \oplus L \leq_{ess} M$. Thus $K \oplus L \leq_{ess} M$. By our assumption $M \in \mathrm{Cog}(K \oplus L)$. Hence there exists an injective homomorphism $\theta : M \to K^I \oplus L^I$ for some set I. Since N is fully invariant of M_R, it is easy to see $\pi\theta(N) = 0$, where $\pi : K^I \oplus L^I \to L^I$ is the natural projection. It follows that $\theta(N)$, and hence N embeds in K^I, proving that N_R is semiprime.
(b) We only prove the co-product case. Let $\{M_i\}_{i\in I}$ be a family of weakly compressible R-modules and N be any nonzero submodule of $\oplus_{i\in I} M_i$. It is easy to verify that there exists subset J of I such that the canonical projection $\pi : \oplus_{i\in I} M_i \to \oplus_{j\in J} M_j =: W$ is one to one on N and $\pi(N) \cap M_j \neq 0$ for each $j \in J$; see also [9, Lemma 2.1]. Because M_j is weakly compressible for each $j \in J$, there are homomorphisms $f_j \in \mathrm{Hom}_R(M_j, \pi(N) \cap M_j)$ such that $f_j(\pi(N) \cap M_j) \neq 0$. Now let $f = \sum_{j\in J} f_j : W \to \pi(N)$ and $\theta = \pi^{-1} f \pi$. Then $\theta : M \to N$ such that $\theta|_N \neq 0$, as desired.
(c) Let N be an essential submodule of product $\prod_{i\in I} M_i$ where each M_i is a semiprime module (the co-product case has a similar proof). Note that for each $i \in I$ we have $(N \cap M_i) \leq_{ess} M_i$. Thus by our assumption, $M_i \in \mathrm{Cog}\ (N)$ for each $i \in I$. It follows that $\prod_{i\in I} M_i \in \mathrm{Cog}(N)$.
(d) The necessity follows by part (c). Conversely, let $M^{(\Lambda)}$ be semiprime and $N \leq_{ess} M$. Then $N^{(\Lambda)} \leq_{ess} M^{(\Lambda)}$. Thus $M^{(\Lambda)} \in \mathrm{Cog}(N^{(\Lambda)})$. This shows that $M \in \mathrm{Cog}(N)$, as desired.
(e) This is obtained by [6, Theorem 5.1(b)].
(f) Just note that if N is a nonzero submodule of $M_1 \oplus M_2$, then by our assumption, either $N \cap M_1 \neq 0$ or $N \cap M_2 \neq 0$.
(g) This follows by [6, Proposition 5.5(viii)].
(h) This has a routine argument.
(i) Let N be a fully invariant closed submodule of M_R. By [5, Proposition 6.32], there exists $K \leq M_R$ such that N is a complement to K in M. It follows that $K \oplus N/N$ is an essential submodule of M/N. Hence, it is enough to show that M/N is $(K \oplus N/N)$-weakly compressible. Now let $(x + N) \in (K \oplus N/N)$ for some nonzero element $x \in K$. Since M_R is weakly compressible, there exists a homomorphism $f : M \to xR$ such that $f(x) \neq 0$. We have $f(N) = 0$ because N is a fully invariant submodule of M. Thus f induces a homomorphism $\bar{f} : M/N \to xR \oplus N/N$ such that $\bar{f}(x + N) \neq 0$. The proof is complete. □

An R-module M is called *torsionless* if it is cogenerated by R. The following result may be already in the literature, but we cannot spot it, we give a proof for the sake of the reader.

2.2. Proposition. *Every torsionless module over a semiprime ring is weakly compressible.*

Proof. Let R be a semiprime ring and M be an R-submodule of R^I for some set I. Suppose that N is a nonzero submodule of M. Thus $\pi_i(N) \neq 0$ for some $i \in I$, where π_i is the canonical projection from R^I to R. Since R is a semiprime ring, $(\pi_i(N))^2 \neq 0$. Hence there exists $x \in N$ such that $x\pi_i(N) \neq 0$. Now let $f = \iota_x \pi$ where $\pi_i|_M = \pi$ and $\iota_x : R \to xR$ is left multiplication by x. Then $f : M \to N$ is a homomorphism such that $f(N) \neq 0$, proving that M_R is weakly compressible. □

2.3. Corollary. *Let R be a ring and $\{I_i\}_{i\in A}$ be a family of semiprime ideals in R. Then $\oplus_{i\in A}(R/I_i)^{\Lambda_i}$ is a weakly compressible R-module, where each Λ_i is a set.*

Proof. This follows by Proposition 2.2 and Lemma 2.1(b),(h). □

2.4. Lemma. *Every nonsingular R-module M contains an essential submodule isomorphic to $\oplus_i I_i$ where each I_i is a right ideal of R.*

Proof. Let x be any nonzero element of M_R. Then $\mathrm{ann}_R(x)$ is not an essential right ideal of R by our assumption on M_R. Thus there exists a nonzero right ideal I_x of R such that $\mathrm{ann}_R(x) \cap I_x = 0$. Note that $I_x \simeq xI_x$. Therefore every nonzero submodule of M contains a nonzero submodule that is isomorphic to a right ideal of R. Now suppose that $\Omega = \{N \leq M_R |$ there is $I \leq R_R$ such that $I \simeq N\}$. If $\{N_\lambda\}_{\lambda\in\Lambda}$ is a maximal independent family of submodules in Ω, then by what we have already proved, $\oplus_{\lambda\in\Lambda} N_\lambda$ is an essential submodule of M_R. $\square$

In [6, Theorem 5.1], it is shown that an R-module M is weakly compressible if and only if $\mathrm{Hom}_R(M,N)^2 \neq 0$ for all nonzero $N \leq M$ if and only if $N \cap$ $\mathrm{Rej}(M,N) = 0$ for any nonzero $N \leq M_R$. In the following we give more equivalent conditions for a nonzero module M to be weakly compressible. We should note that in [1], a module M_R is called "weakly compressible" if for every $0 \neq N \leq M_R$ there exists $f \in \mathrm{Hom}_R(M,N)$ with $f^2 \neq 0$. Such a module M is clearly weakly compressible (in the sense of [6]), but we have been unable to find in the literature a proof to show the converse is true. A proof of this is given below for completeness. Recall that for any R-module M the set $\{m \in M \mid \mathrm{ann}_R(m) \leq_{ess} R_R\}$ is denoted by $\mathrm{Z}(M)$.

2.5. Theorem. *The following conditions are equivalent for a nonzero R-module M.*
(a) *M_R is weakly compressible.*
(b) *For every nonzero $N \leq M$, there exists $f \in Hom_R(M,N)$ such that $f^2 \neq 0$.*
(c) *$N \not\hookrightarrow Rej(M,N)$, for every nonzero $N \leq M_R$.*
(d) *$M_1 \not\hookrightarrow Rej(M,M_2)$ for all nonzero isomorphic R-modules M_1 and M_2.*
(e) *There exists an essential submodule N of M_R such that M is N-weakly compressible.*
(f) *There exists submodule N of M_R such that M is N-weakly compressible and M/N is weakly compressible.*
(g) *There exists a semiprime ideal I of R such that $MI = 0$ and M is $Rej(M,R/I)$-weakly compressible.*
(h) *M is $Z(M)$-weakly compressible and $M/Z_2(M) \in Cog(R/I)$ for some semiprime ideal $I \subseteq ann_R(M)$.*

Proof. (a) $\Rightarrow$ (b). Let N be a nonzero submodule of M_R and for every $f \in \mathrm{Hom}_R(M,N)$, $f^2 = 0$. It is easy to verify that $fg = -gf$ for all $f, g \in \mathrm{Hom}_R(M,N)$ (note that $(f+g)^2 = 0$). By (a), there exist $f \in \mathrm{Hom}_R(M,N)$ and $g \in \mathrm{Hom}_R(M,f(M))$ such that $f(N) \neq 0$ and $g(f(M)) \neq 0$. Since $gf = -fg$, we have $fg \neq 0$. If follows that $f^2(M) \neq 0$ because $g(M) \subseteq f(M)$. This contradicts our assumption.
(b) $\Rightarrow$ (c). Let N be any nonzero submodule of M_R. Suppose that there exists an injective homomorphism $\theta : N \to \mathrm{Rej}(M,N)$. Since $N \simeq \theta(N)$, $\mathrm{Rej}(M,\theta(N)) = \mathrm{Rej}(M,N)$. Hence, if $f \in \mathrm{Hom}_R(M,\theta(N))$ then $\mathrm{Im} f \subseteq \mathrm{Rej}(M,\theta(N))$. This shows that $f^2 = 0$ for every $f \in \mathrm{Hom}_R(M,\theta(N))$. This contradicts (b).
(c) $\Rightarrow$ (d). Just note that if $M_1 \hookrightarrow \mathrm{Rej}(M,M_2)$, then M_1 is isomorphic to a submodule N of M such that $N \hookrightarrow \mathrm{Rej}(M,N)$.
(d) $\Rightarrow$ (a), (a) $\Leftrightarrow$ (e) and (a) $\Rightarrow$ (f) are clear.
(a)$\Rightarrow$ (g). This is hold because $\mathrm{ann}_R(M)$ is a semiprime ideal of R by Lemma 2.1.
(g) $\Rightarrow$ (f). Let $N = \mathrm{Rej}(M,R/I)$. Then $M/N \in \mathrm{Cog}(R/I)$. Now apply Proposition 2.2 and Lemma 2.1(h).
(f) $\Rightarrow$ (a). Suppose (f) holds and K is a nonzero submodule of M_R. We shall show that there exists $g \in \mathrm{Hom}_R(M,K)$ with $g(K) \neq 0$. Now if $K \cap N \neq 0$, then we are done by our assumption on N. If $K \cap N = 0$, then consider the submodule $(N \oplus K)/N$ of M/N. Since M/N is weakly compressible, we can deduce such g exists.

(a) $\Rightarrow$ (h). First note that for any R-module M, we have $M/\mathrm{Z}_2(M)$ is a nonsingular R-module. Let $I = \mathrm{ann}_R(M)$. Thus $M/\mathrm{Z}_2(M) \in \mathrm{Cog}(R/I)$ by Lemmas 2.1(i) and 2.4, the proof is complete.
(h) $\Rightarrow$ (f). Since $\mathrm{Z}(M) \leq_{ess} \mathrm{Z}_2(M)$, it is clear that M_R is also $\mathrm{Z}_2(M)$-weakly compressible. The result is now obtained by Proposition 2.2. □

2.6. Corollary. (a) *If R is a right self injective ring, then M_R is weakly compressible if and only if M_R is $Z(M)$-weakly compressible.*
(b) *If R is a right V-ring (i.e., simple R-modules are injective) and $M/Soc(M)$ is a weakly compressible R-module, then M_R is weakly compressible.*

Proof. (a) Let R be a right self injective ring. For the sufficiency, let N be complement to $\mathrm{Z}(M)$ in M_R. By Theorem 2.5(e), we shall show that M is $\mathrm{Z}(M)\oplus N$-weakly compressible. Since R is right self injective, every nonsingular cyclic R-module is isomorphic to a direct summand of R_R and hence it is an injective R-module. It follows that M is N-weakly compressible. The proof is now completed by Lemma 2.1(f). The converse is clear.
(b) By Theorem 2.5(f). □

2.7. Proposition. *The following statements hold for an extending module M_R.*
(a) *M_R is weakly compressible if and only if $Z_2(M)$ and $M/Z_2(M)$ are weakly compressible R-modules.*
(b) *If $Soc(R_R) \leq_{ess} R_R$, then M_R is semiprime if and only if $Z_2(M)$ and $M/Z_2(M)$ are semiprime R-modules.*

Proof. Let $N = \mathrm{Z}_2(M)$. Since M is extending, it is known that $M \simeq N \oplus M/N$.
(a) Apply Theorem 2.5(f) and note that N is weakly compressible if and only if M is N-weakly compressible.
(b) Since $\mathrm{Soc}(R_R) \leq_{ess} R_R$, it is easy to verify that $\mathrm{Z}(V^\Lambda) = (\mathrm{Z}(\mathrm{V}))^\Lambda$ for any R-module V and any set Λ. Now let M_R be semiprime. By Lemma 2.1(a), N_R is semiprime. Suppose that $K/N \leq_{ess} M/N$. Then $K \leq_{ess} M$ and so there exists an injective homomorphism $\theta : M \to K^\Lambda$. Define $\alpha : M/N \to K^\Lambda/N^\Lambda$ by $\alpha(m+N) = \theta(m) + N^\Lambda$. Clearly α is a homomorphism. If $\alpha(m+N) = 0$ then $\theta(m) = \{n_\lambda\}_{\lambda\in\Lambda} \in N^\Lambda$. For each λ, we have $n_\lambda J_\lambda \subseteq \mathrm{Z}(M)$ where $J_\lambda \leq_{ess} R_R$. Thus $\theta(mJ) \subseteq (\mathrm{Z}(M))^\Lambda$ where $\cap_\lambda J_\lambda = J$. By our assumption on R, $J \leq_{ess} R_R$ and $\theta(mJ) \subseteq \mathrm{Z}(K^\Lambda)$. It follows that $mJ \subseteq \mathrm{Z}(M)$ because θ is one to one. Hence $m \in N$, proving that α is injective and so M/N is weakly compressible. □

For every module M_R the intersection of all maximal submodule of M is denoted by $\mathrm{Rad}(M)$. If M does not have maximal submodules, we put $\mathrm{Rad}(M) = M$.

2.8. Examples and Remarks. (a) There are modules N such that $\mathrm{Rad}(N) = 0$ but N is not semiprime. Let P be the set of all prime integer numbers and $p \in P$. Consider the $\mathbb{Z}$-module $N = \{m/p^n \mid m, n \in \mathbb{Z}, n \geq 1\}$. Then for each $q \in P \setminus \{p\}$, qN is a maximal submodule of $N_\mathbb{Z}$. To see this, note that $qN \neq N$ and suppose that K is any submodule of $N_\mathbb{Z}$ such that $qN \subsetneq K$ and $m/p^t \in K \setminus qN$. Hence $(m, q) = 1$. Also, if $a/p^r \in K$ for some $r \geq 1$ and $(a, q) = 1$, then $1/p^r \in K$. It follows that $1/p^n \in K$ for all $n \geq 1$ (take $n \geq t$ or $n \leq t$). Therefore $K = N$ and so qN is a maximal submodule. Clearly $\bigcap_{q\neq p} qN = 0$ and hence Rad $(N) = 0$. Now if $N_\mathbb{Z}$ is semiprime, then $\mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z}) \neq 0$ and since N is uniform, we must have $N \hookrightarrow \mathbb{Z}$, contradiction.

(b) A direct summand of a semiprime module is not necessarily a semiprime module. Assume that P and N are as stated in (a). Let $W = \oplus_{p\in P}\mathbb{Z}_p$ and $L = W \oplus N$. We show that $L_\mathbb{Z}$ is semiprime. Since $\bigcap_{q\neq p} qN = 0$, $N \in \mathrm{Cog}(W)$. Thus from $\mathrm{Soc}(L) = W$, we have $L \in Cog(\mathrm{Soc}(L))$. It follows that L is semiprime as a $\mathbb{Z}$-module because every essential submodule of L contains $\mathrm{Soc}(L)$.

(c) Lemma 2.1(b) and part (b) show that the $\mathbb{Z}$-module L in (b) is semiprime which is not weakly compressible. Furthermore, let R be a commutative regular ring which is not semi-Artinian (for example $R = \prod \mathbb{Z}_2$). Since R is a regular ring, $\mathrm{Rad}(M) = 0$ for all R-modules. Hence every R-module embeds in a semiprime R-module by Lemma 2.1(c). On the other hand, since R is not semi-Artinian, there exists an R-module M which is not weakly compressible by [10, Corollary 3.5]. Now if M embeds in a semiprime R-module L, then L is not weakly compressible by Lemma 2.1(b).

(d) The condition (h) in Theorem 2.5 shows that the study of weakly compressible modules can be reduced to the study of such modules when they are either singular or nonsingular; see Proposition 2.7. However we shall note that, in general, the condition M is $\mathrm{Z}(M)$-weakly compressible is stronger than $\mathrm{Z}(M)$ is a weakly compressible R-module. For example, if $R = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{bmatrix}$, then $Z(R_R) = \begin{bmatrix} 0 & 0 \\ \mathbb{Z}_2 & 0 \end{bmatrix} =: I$. Thus I_R is weakly compressible, but R is not I-weakly compressible because $\mathrm{Hom}_R(R, I)(I) = 0$.

(e) In view of the condition (c) in Theorem 2.5, we note that the condition $N \hookrightarrow Rej(M, N)$ is weaker than $N \subseteq Rej(M, N)$. For if we consider I as left ideal in R, then $I \simeq \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{Z}_2 \end{bmatrix} =: J$ as left R-modules and $\mathrm{Rej}(R, J) = \mathrm{l.ann}_R(J) = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_2 & 0 \end{bmatrix}$. Hence $J \hookrightarrow \mathrm{Rej}(R, J)$, but $J \nsubseteq \mathrm{Rej}(R, J)$.

In the following nonsingular weakly compressible modules are characterized and some corollaries are given. For certain module M_R, the condition (c) of Theorem 2.5 is reduced to the ideals of R; see below.

2.9. Proposition. *Let M be a module over a semiprime ring R and $Z(Rej(M, R)) = 0$. Then the following statements are equivalent.*
(a) *M_R is weakly compressible.*
(b) *For all nonzero right ideal I of R, $I \not\hookrightarrow Rej(M, I)$.*
(c) *$M \in Cog(R)$.*

Proof. (a) $\Rightarrow$ (b). By Theorem 2.5(c).
(b) $\Rightarrow$ (c). If $\mathrm{Rej}(M, R)$ is nonzero then by Lemma 2.4, $I \hookrightarrow \mathrm{Rej}(M, R)$ for some nonzero right ideal I of R. It follows that $I \hookrightarrow \mathrm{Rej}(M, I)$, a contradiction. Therefore $\mathrm{Rej}(M, R) = 0$ and so (c) holds.
(c) $\Rightarrow$ (a). By Proposition 2.2. □

2.10. Corollary. *Let M be a nonsingular R-module. Then M_R is weakly compressible if and only if there exists a semiprime ideal $I \subseteq ann_R(M)$ such that $M \in Cog(R/I)$.*

Proof. Note that $\mathrm{Z}(M_{R/I}) \subseteq \mathrm{Z}(M_R)$, for any ideal I of R. The result is now obtained by Proposition 2.9. □

A ring R is called *right (left) duo* ring if every right(left) ideal of R is two sided.

2.11. Corollary. *Let M be a faithful module over a right(left) duo ring R. Then M_R is weakly compressible if and only if M_R is $Z(M)$-weakly compressible, $M/Z(M) \in Cog(R)$ and R is a semiprime ring.*

Proof. It is easy to verify that every semiprime right(left) duo ring must be reduced and hence it is a nonsingular ring [5, Lemma 7.8]. Thus $\mathrm{Z}(M) = \mathrm{Z}_2(M)$. Suppose now M is weakly compressible, then R must be a semiprime ring because M_R is faithful. Also $M/\mathrm{Z}(M)$ is weakly compressible by Lemma 2.1(i), and so $M/\mathrm{Z}(M) \in \mathrm{Cog}(R)$ by Proposition 2.9. The converse is obtained by Theorem 2.5(h). $\square$

3. Weakly compressible modules with enough critical submodules

We are now going to investigate semiprime and weakly compressible modules over rings with Krull dimensions. Let M be an R-module. Following [7, Chapter 6], the *Krull dimension* of M_R, will be denoted by $\mathrm{K.dim}(M)$. Modules with Krull dimensions are known to have *finite uniform dimensions* [7, Lemma 6.2.6]. Let $\alpha \geq 0$ be an ordinal number. A module M_R is called *α-critical* if $\mathrm{K.dim}(M) = \alpha$ and $\mathrm{K.dim}(M/N) < \alpha$ for every nonzero submodule N of M_R. A module is then called *critical* if it is β-critical for some ordinal number β. The submodule $\bigcap\{\ K \leq M_R \mid M/K \text{ is } \alpha\text{-critical }\ \}$ is denoted by $\mathrm{J}_\alpha(M)$.

3.1. Lemma. *Let M be a semiprime R-module, T be any nonzero submodule of M. If there exist submodules W and N of M such that $N \in Cog(T)$, $T \notin Cog(W)$ and $(N \oplus W) \leq_{ess} M$, then $T \nsubseteq Rej(M,T)$.*

Proof. Since M_R is semiprime, there exists an injective homomorphism $f : M \to N^A \oplus W^A$ for some set A. Since $T \notin \mathrm{Cog}(W)$, $\pi f(T) \neq 0$, where $\pi : N^A \oplus W^A \to N^A$ is natural projection. By our assumption, $N^A \in \mathrm{Cog}(T)$. Hence there exists a homomorphism $\varphi : N^A \to T$ such that $\varphi\pi f(T) \neq 0$, proving that $T \nsubseteq \mathrm{Rej}(M,T)$. $\square$

The following lemma is needed. That is just obtained by the definition of critical submodules.

3.2. Lemma. *Let U and V be critical R-modules and $f : U \to V$ be a nonzero homomorphism. Then either $Kerf = 0$ or $K.dim(V) < K.dim(U)$.*

We say that a module M_R *has enough critical submodules* if every nonzero submodule has a nonzero submodule with Krull dimension (note, modules with Krull dimension have critical submodules).

3.3. Lemma. *Suppose that M_R has enough critical submodules and $\alpha = Min\{K.dim(N) \mid 0 \neq N \leq M_R\}$. If M_R is semiprime, then $N \nsubseteq Rej(M,N)$ for every submodule $N \leq M_R$ with $K.dim(N) = \alpha$.*

Proof. Let $N \leq M_R$ and $\mathrm{K.dim}(N) = \alpha$. By [7, Lemma 6.2.10], there exists a critical submodule $T \leq N$. By choosing of α, T is α-critical. Let $\Lambda = \{T^{'} \leq M_R \mid T^{'} \in Cog(T)\}$, $\{T_i^{'}\}_{i\in I}$ be a maximal independent family of elements in Λ and $N' = \oplus_{i\in I} T_i^{'}$. Since M_R has enough critical submodules, $N' \oplus W \leq_{ess} M_R$ where W is a direct sum of critical submodules. Therefore by Lemma 3.2, $T \notin Cog(W)$ and so $T \nsubseteq \mathrm{Rej}(M,T)$ by Lemma 3.1. The proof is complete. $\square$

A module M_R is called *compressible* if it embeds in every submodule of M. By Lemma 3.2 critical weakly compressible modules are compressible.

3.4. Theorem. *Suppose that M_R has enough critical submodules and $\beta = Sup\{K.dim(N) \mid N$ is a critical submodule of $M_R\}$. Then the following statements are equivalent.*
(a) *M_R is semiprime module and $J_\beta(M) = 0$.*
(b) *M_R embeds in a product of β-critical compressible submodules of M_R.*
(c) *M_R embeds in a product of β-critical compressible R-modules.*
Furthermore, each of the above conditions implies that M_R is weakly compressible.

Proof. (a) $\Rightarrow$ (b). We first show that every critical submodule of M_R is β-critical. Let C be any critical submodule of M_R. By our assumption, $C \nsubseteq \mathrm{J}_\beta(M)$. It follows that there exists a homomorphism f from M_R to a β-critical module T_R such that $f(C) \neq 0$. By Lemma 3.2, f is one to one on C. Thus C_R is β-critical, as desired. Now since M_R has enough critical submodules, $\beta = \mathrm{Min}\{\mathrm{K.dim}(N) \mid 0 \neq N \leq M_R\}$. Therefore M_R is weakly compressible by Lemma 3.3. Hence, every critical submodule of M_R is also weakly compressible as well as compressible. The proof is now complete because M contains an essential submodule that is a direct sum of β-critical compressible submodules.
(b) $\Rightarrow$ (c). This is clear.
(c) $\Rightarrow$ (a). It is easy to see that $\mathrm{J}_\beta(M) = 0$. As we see in the proof of (a) $\Rightarrow$ (b), for every critical submodule C of M_R there exist a β-critical compressible R-module T and homomorphism $\alpha : M \to T$ such that α is one to one on C. Since now T_R is compressible, there exists an injective homomorphism $f : T \to C$. Thus $f\alpha(C) \neq 0$, proving that $C \nsubseteq \mathrm{Rej}(M, C)$. It follows that M_R is weakly compressible, hence semiprime. □

3.5. Remark. Let $R = \mathbb{Z}$, $M = \mathbb{Z}_2 \oplus \mathbb{Z}$ and β be as stated in Theorem 3.4. Then M_R is weakly compressible and $\beta = 1$, but $\mathrm{J}_\beta(M) \neq 0$ because $M \notin \mathrm{Cog}(R)$.

3.6. Lemma. *Suppose that M is an R-module, $\{V_i\}_{i\in I}$ is a family of nonzero submodules of M_R, $\{W_j\}_{j\in J}$ is a family of R-modules and the following conditions (a), (b) hold,*
(a) *For every nonzero submodule N of M_R, there exists $V_i \subseteq N$ for some $i \in I$.*
(b) *For every $i \in I$, there exist $j \in J$ and homomorphism $f : M \to W_j$ such that $Kerf \cap V_i = 0$.*
If M_R has finite uniform dimension, then there exists a finite subset A of J such that M_R embeds in $\oplus_{j\in A} W_j$.

Proof. Let $\Lambda = \{\mathrm{u.dim}(\mathrm{Ker}f) \mid f \in \mathrm{Hom}_R(M, \oplus_{j\in A} W_j)$ and A is a finite set $\}$. By hypothesis Λ is a nonempty set. Let n be the smallest element in Λ, and $f : M \to \oplus_{j\in A} W_j$ such that $\mathrm{u.dim}(\mathrm{Ker}f) = n$. Let $K = \mathrm{Ker}f$. If $K \neq 0$, then by (a), there exists $i \in I$ such that $V_i \subseteq K$ and by (b) there exists a homomorphism $g : M \to W_t$ such that $\mathrm{Ker}g \cap V_i = 0$ for some $t \in J$. Now, define $h : M \to \oplus_{j\in A} W_j \oplus W_t$ by $h(m) = (f(m), g(m))$ for all $m \in M$. It is clear that $\mathrm{Ker}h = \mathrm{K} \cap \mathrm{Ker}g$. Since $\mathrm{Ker}h \cap V_i = 0$, $\mathrm{Ker}h$ is not essential submodule of K. Hence $\mathrm{u.dim}(\mathrm{Ker}h) < \mathrm{u.dim}(\mathrm{Ker}f)$. This contradicts the choice of f. Therefore $K = 0$ and so M embeds in $\oplus_{j\in A} W_j$, as desired. □

3.7. Theorem. *The following statements are equivalent for a nonzero module M_R.*
(a) *M_R is weakly compressible with finite uniform dimension and $Z(M)$ has Krull dimension.*
(b) *M_R is weakly compressible with finite uniform dimension and $Z(M)$ has enough critical submodules.*
(c) *M_R embeds in a finite direct sum $\oplus_i W_i$ of cyclic compressible submodules of M_R such that each W_i is either uniform nonsingular or critical singular.*
(d) *M_R embeds in $W \oplus V$ such that W_R and V_R are weakly compressible, W is nonsingular with finite uniform dimension and V is singular with Krull dimension.*

Proof. (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) are clear. (d) $\Rightarrow$ (a) is obtained by Lemma 2.1(b) and the fact that modules with Krull dimensions have finite uniform dimensions. We shall show that (b) $\Rightarrow$ (c).
Apply Lemma 3.6 for $\{V_i\}_{ß\in I} = \{W_j\}_{æ\in J=I} = \{C \leq M_R \mid C$ is either uniform nonsingular or critical singular$\}$. By our hypothesis, the condition (a) of Lemma 3.6 holds. Note that every endomorphism of the above submodules C is either injective or zero (Lemma 3.2). Hence, by the weakly compressible condition on M, we have the submodules C

are compressible and the condition (b) of Lemma 3.6 holds. The proof is now complete because any compressible module embeds in each of its cyclic submodule. □

3.8. Corollary. *The following statements are equivalent for a nonzero module M_R.*
(a) *M_R is weakly compressible with Krull dimension.*
(b) *M_R is weakly compressible with finite uniform dimension and it has enough critical submodules.*
(c) *M_R embeds in a finite direct sum of critical compressible submodules of M_R.*
(d) *M_R embeds in a finite direct sum of critical compressible R-modules.*

Proof. This follows by Theorem 3.7. □

The following result is a consequence of Theorem 3.7 which should be compared with Corollary 2.10.

3.9. Corollary. *If M_R is a nonsingular weakly compressible module with finite uniform dimension, then M_R embeds in a finitely generated free R-module.*

Proof. Note that every nonsingular compressible R-module embeds in R (Lemma 2.4). Thus the result is obtained by Theorem 3.7(c). □

4. Weakly compressible modules over singular semi-Artinian rings

In [8, Main Theorem], it is shown that a $\mathbb{Z}$-module M is weakly compressible if and only if $\mathrm{Z}(M)$ is semisimple and $M/\mathrm{Z}(M)$ is torsionless. We conclude the paper with a characterization of weakly compressible (semiprime) modules over certain rings including prime hereditary Noetherian rings. If R is a hereditary Noetherian ring, then by [7, Proposition 5.4.5], *every nonzero singular R-module has a nonzero socle.* We call such rings R *right singular semi-Artinian.*

4.1. Theorem. *Suppose that R is a right singular semi-Artinian ring, M_R is nonzero and $MI = 0$ for some ideal I of R. If M_R is semiprime then $M \in Cog(Soc(M) \oplus R/I)$. The converse holds if I is a prime ideal of R.*

Proof. Since R/I is also a right singular semi-Artinian ring, we can suppose that $I = 0$. Let M_R be semiprime and $\mathrm{Soc}(\mathrm{Z}(M)) \oplus K \leq_{ess} M_R$ where $K \leq M_R$. By our assumption on R, we have $\mathrm{Z}(K) = 0$. Thus $M \in \mathrm{Cog}(\mathrm{Soc}(M) \oplus R)$ by Lemma 2.4.
Conversely, assume that $M \in \mathrm{Cog}(\mathrm{Soc}(M) \oplus R)$ and R is a prime ring. Let N be any essential submodule of M_R. We have $\mathrm{Soc}(\mathrm{Z}(N)) \oplus L \leq_{ess} N$ such that $L \simeq \oplus_{i \in I} I_i$ where each I_i is a right ideal of R. Since $\mathrm{Soc}(M)$ lies in any essential submodule of M_R, we deduce from the hypothesis that $M \in \mathrm{Cog}(\mathrm{Soc}(N) \oplus L \oplus R)$. Now $\mathrm{Rej}(R, L) = \mathrm{ann}_R(L) = 0$ because R is prime ring. It follows that $R \in \mathrm{Cog}(L)$ and hence $M \in \mathrm{Cog}(N)$, proving that M_R is semiprime. □

4.2. Remark. Let R be any ring and M be a nonzero R-module. Then M_R is a subdirect product of prime modules if and only if M is cogenerated by prime modules. Now let M_R be a weakly compressible R-module and $A = \mathrm{ann}_R(M)$. Note that R/A is subdirect product of prime R-modules. Therefore if M_R satisfies the conditions of Theorem 3.4 or 3.7 or 4.1, then M is cogenerated by prime modules and hence it is a subdirect product of prime R-modules. This gives a partially answer to the open problem 1 of [6].

4.3. Proposition. *Let M_R be semiprime and $L \leq M_R$. Then the following statements hold.*
(a) *If $Soc(L)$ is finitely generated then $Soc(L)$ is a direct summand of M. In particular, if M has acc on direct summands, then $Soc(M)$ is a direct summand of M.*
(b) *If every cyclic submodules of L has a finitely generated socle then $Soc(L)$ is a closed*

submodule of M.
(c) *M is Soc(M)-weakly compressible and Soc(M)$\cap$ Rad(M) $= 0$.*

Proof. (a) Suppose that the length of $\mathrm{Soc}(L) = n$. Let T_1 be a simple submodule of L_R, $N = \sum\{T' \leq M_R \mid T' \simeq T_1\}$ and W be a complement to N in M. Then $T_1 \notin \mathrm{Cog}(W)$ and so by Lemma 3.1, $T_1 \nsubseteq Rej(M, T_1)$. Hence there exists a nonzero homomorphism $f : M \to T_1$ such that $f(T_1) \neq 0$. Clearly $\mathrm{Ker}(f)$ is a maximal submodule of M_R. It follows that $M = T_1 \oplus A_1$ where $A_1 \ker f$, and hence $L = T_1 \oplus (L \cap A_1)$. If $\mathrm{Soc}(L \cap A_1) = 0$, then $\mathrm{Soc}(L) = T_1$ and we are done. If not, consider the simple submodule T_2 of $L \cap A_1$. Again we deduce that T_2 is a direct summand of M and hence of A_1. Thus $M = T_1 \oplus T_2 \oplus A_2$ for some $A_2 \leq M$ and $L = T_1 \oplus T_2 \oplus (L \cap A_2)$. Continue to obtain $T_1 \oplus T_2 \oplus ... \oplus T_n$ is a direct summand of M_R, as desired. The last statement is now clear.
(b) If $\mathrm{Soc}(L) \leq_{ess} C$ and $x \in C \leq L$, then $\mathrm{Soc}(xR) \leq_{ess} xR$. Hence by our assumption and (a), we must have $xR = \mathrm{Soc}(xR) \subseteq \mathrm{Soc}(L)$. It follows that $\mathrm{Soc}(L) = C$.
(c) This is obtained by (a) and the fact that every nonzero cyclic submodule in $\mathrm{Rad}(M)$ is a small submodule of M and so cannot be a direct summand. □

4.4. Lemma. *Suppose S and R are two rings, $T = R \oplus S$ and M be a T-module. Then $M = K \oplus L$ where K and L are modules over R and S respectively. In this case, $Z(M_T)$ $= Z(K_R)\oplus$ $Z(L_S)$ and $Soc(M_T)$ $= Soc(K_R)\oplus$ $Soc(L_S)$.*

Proof. Just note that if M is a T-module then $M = Me_1 \oplus Me_2$ where $e_1 = 1_R$ and $e_2 = 1_S$ are central orthogonal idempotents in T such that $e_1 S = e_2 R = 0$. Clearly Me_1 and Me_2 are naturally R-module and S-module respectively. □

4.5. Theorem. *Suppose that M is a nonzero R-module and $MI = 0$ for some semiprime ideal I of R. If $M \in$ Cog(Soc(M)$\oplus R/I$) and $M/$Soc(M) $\in$ Cog(R/I), then M_R is weakly compressible. The converse holds if R is a right singular semi-Artinian ring such that every cyclic R-module has a finitely generated socle or acc on direct summands.*

Proof. We may suppose that $I = 0$. Let $N = \mathrm{Soc}(M)$. By Proposition 2.2, M/N is a weakly compressible R-module. Hence by Theorem 2.5(f), we need to show that M is N-weakly compressible. Assume that S is a simple submodule of M, and by hypothesis let $\theta : M \hookrightarrow (N \oplus R)^{\Lambda} =: L$ for some set Λ. Then $\pi_\lambda \theta(S)$ is nonzero for some canonical projection π_λ $(\lambda \in \Lambda)$ on L. Let $U = \pi_\lambda \theta(S)$ and note that $U \simeq S$. Since now any minimal right ideal in a semiprime ring R is a direct summand of R_R, we deduce that there exists R-homomorphism $f : M \to U$ such that $f(S) \neq 0$. It follows $S \nsubseteq \mathrm{Rej}(M, S)$, as desired.
Conversely, suppose that R satisfies the above hypothesis and M_R is weakly compressible. By Theorem 4.1, $M \in \mathrm{Cog}(\mathrm{Soc}(M) \oplus R)$. It remains to show that $M/\mathrm{Soc}(M) \in \mathrm{Cog}(R)$. Since R is assumed to be a semiprime ring, $\mathrm{Soc}(R_R)$ is a direct summand of R by Proposition 4.3(a). It follows that $R \simeq A \oplus B$ where A is a semisimple ring and B is a ring with zero socle. By Lemma 4.4, $M = K \oplus L$ and $\mathrm{Soc}(M) = K \oplus$ $\mathrm{Soc}(L)$. Thus it is enough to show that $L/\ \mathrm{Soc}(L) \in \mathrm{Cog}(B)$. Now L is a weakly compressible B-module. Since B is a right singular semi-Artinian ring, $\mathrm{Z}(L)$ has an essential socle, and since $\mathrm{Soc}(B_B) = 0$, every simple B-module is singular. Thus $\mathrm{Soc}(L) \leq_{ess} \mathrm{Z}(L)$. On the other hand, if C is a cyclic submodule L_B, then an application of Proposition 4.3(a) for C_B shows that $\mathrm{Soc}(C)$ is a direct summand of C. Hence $\mathrm{Soc}(C)$ is cyclic. It follows that $\mathrm{Soc}(L)$ is a closed submodule of L by Proposition 4.3(b). Therefore $\mathrm{Soc}(L) = \mathrm{Z}(L) = \mathrm{Z}_2(L)$. The proof is now completed by Lemma 2.1(i). □

4.6. Corollary. *Let R be a prime right singular semi-Artinian ring such that cyclic R-modules have finite uniform dimensions. Then the following statements hold for M_R.*
(a) *$M \in Cog(Soc(M) \oplus R)$ and $M/Soc(M) \in Cog(R)$ if and only if M_R is weakly compressible.*
(b) *$M \in Cog(Soc(M) \oplus R)$ if and only if M_R is semiprime.*
(c) *If M_R is semiprime, then either M_R is semisimple or $Z(M) = Soc(M)$.*
(d) *Furthermore, if R is a PID then $M/Soc(M) \in Cog(R)$ if and only if M_R is weakly compressible.*

Proof. (a) and (b). These follow from Theorems 4.1 and 4.5.
(c). By Proposition 4.3(a), $\mathrm{Soc}(R_R)$ is a direct summand of R. Since now R is a prime ring, R is semisimple or $\mathrm{Soc}(R_R) = 0$. If R is a semisimple ring then M_R is semisimple. In case $\mathrm{Soc}(R_R) = 0$, as we see in the proof of Theorem 4.5, $\mathrm{Z}(M) = \mathrm{Soc}(M)$.
(d) The sufficiency holds by part (a). Conversely, let $N = \mathrm{Soc}(M)$ and $M/N \in \mathrm{Cog}(R)$. It follows that $\mathrm{Z}(M) \subseteq N$ and M/N is weakly compressible. Thus we need to show that M is N-weakly compressible. Let S be a simple submodule of M_R and $P = \mathrm{ann}_R(S)$. Let $P = pR$ for some prime element $p \in R$. If $0 \neq x \in S \subseteq MP$, then $x = mp$ for some $m \in M$ and so $p^2R \subseteq \mathrm{ann}_R(m) \subsetneq pR$. Hence $p^2R = \mathrm{ann}_R(m)$. This implies that $mR \subseteq N$, a contradiction. Therefore, $S \cap MP = 0$. Since now $M/MP \simeq S^{(\Lambda)}$ for some set Λ, we can deduce that $S \nsubseteq \mathrm{Rej}(M, S)$. The proof is complete. □

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