Asymptotic behavior of associated primes of certain ext modules

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Abstract

Let $R$ be a commutative Noetherian ring, $I$ an ideal of $R$ and $M$ a finitely generated $R$-module. It is shown that, whenever $I$ is principal, then for each integer $i$ the set of associated prime ideals $\text{Ass}_R \text{Ext}_i^R(R/I^n, M)$, $n = 1, 2, \ldots$, becomes independent of $n$, for large $n$.

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1. Introduction

Let $R$ denote a commutative Noetherian ring (with identity), $I$ an ideal of $R$, and $M$ a finitely generated $R$-module. In [7] L.J. Ratliff, Jr., conjectured about the asymptotic behavior of $\text{Ass}_R R/I^n$ when $R$ is a Noetherian domain. Subsequently, M. Brodmann [1] showed that $\text{Ass}_R M/I^n M$ is ultimately constant for large $n$. In [6], Melkersson and Schenzel asked whether the sets $\text{Ass}_R \text{Ext}_i^R(R/I^n, M)$ become stable for sufficiently large $n$. The aim of this paper is to show that, for all $i \geq 0$, the sets of prime ideals $\text{Ass}_R \text{Ext}_i^R(R/I^n, M)$, $n = 1, 2, \ldots$, becomes independent of $n$, for large $n$, whenever $I$ is principal, which is an affirmative answer to the above question in the case $I$ is principal.

Also, it is shown that, if $I$ is generated by an $R$-regular sequence and $\text{Ext}_i^R(R/I, M)$ is Artinian, then the set $\bigcup_{n=1}^{\infty} \text{Ass}_R \text{Ext}_{i+1}^R(R/I^n, M)$ is finite.

For any $R$-module $L$, the set $\{ p \in \text{Ass}_R L | \dim R/p = \dim L \}$ is denoted by $\text{Assh}_R L$.

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2. The Results

2.1. Lemma. Let $R$ be a Noetherian ring, $I$ an ideal of $R$ and $M$ a finitely generated $R$-module. Then the sequence $\text{Ass}_R \text{Ext}_1^0(R/\Gamma^n, M)$ becomes eventually constant, for large $n$.

Proof. See [4, Corollary 2.3].

2.2. Lemma. Let $x$ be an element of the Noetherian ring $R$. Let $M$ and $N$ be two finitely generated $R$-modules such that $\text{pd}(N) = t < \infty$. Then for each $i \geq t + 2$ and for all large $k$,

$\text{Ass}_R \text{Ext}_{i-k}^i(N/x^k N, M) = \text{Ass}_R \text{Ext}_{i-k}^i(N/\Gamma N, M)$,

and so the sets $\text{Ass}_R \text{Ext}_{i-k}^i(N/x^k N, M)$ are eventually constant.

Proof. Suppose that $i \geq t + 2$. As, $N$ is finitely generated, it follows that there is an integer $n$ such that

$\Gamma N := \bigcup_{i=0}^{\infty} (0 : M R^i) = (0 : N x^n) = (0 : N x^{n+1}) = \cdots$.

Now we claim that for any $k \geq n$, $\text{Ext}_{i-k}^i(N/x^k N, M) \cong \text{Ext}_{i-k}^{i-1}(N/\Gamma N, M)$.

To do this, as $(0 : N x^k) = \Gamma N$, it follows that $x^k N \cong N/\Gamma N$. Therefore for all $j \geq 0$ we have

$\text{Ext}_{i-k}^j(x^k N, M) \cong \text{Ext}_{i-k}^j(N/\Gamma N, M)$,

for all $k \geq n$. Now the exact sequence

$0 \rightarrow x^k N \rightarrow N \rightarrow N/x^k N \rightarrow 0$,

implies that

$\text{Ext}_{i-k}^i(N/x^k N, M) \cong \text{Ext}_{i-k}^{i-1}(x^k N, M) \cong \text{Ext}_{i-k}^{i-1}(N/\Gamma N, M)$,

(Note that $\text{pd}(N) = t$ and $i \geq t + 2$.) Hence we have

$\text{Ass}_R \text{Ext}_{i-k}^i(N/x^k N, M) = \text{Ass}_R \text{Ext}_{i-k}^{i-1}(N/\Gamma N, M)$,

for all $k \geq n$, as required.

2.3. Theorem. Let $R$ be a Noetherian ring and let $x$ be an element of $R$. Let $M$ be a finitely generated $R$-module and $i$ a non-negative integer. Then the sequence $\text{Ass}_R \text{Ext}_{i-k}^i(R/Rx^k, M)$, of associated primes is ultimately constant for large $k$, and if $i \geq 2$, then

$\text{Ass}_R \text{Ext}_{i-k}^i(R/Rx^k, M) = \text{Ass}_R \text{Ext}_{i-k}^{i-1}(R/\Gamma N, M)$,

for all large $k$.

Proof. The result follows from Lemmas 2.1 and 2.2.

2.4. Proposition. Let $R$ be a Noetherian ring and let $M, N$ be tow finitely generated $R$-modules. Let $x$ be an $N$-regular element of $R$. Then, for any given integer $j \geq 0$, the set

$\bigcup_{n=1}^{\infty} \text{Ass}_R \text{Ext}_{j-k}^j(N/x^n N, M)$,

of associated prime ideals, is finite.
Proof. If \( j = 0 \), then
\[
\text{Ass}_R \text{Hom}_R(N/x^n N, M) = \text{Ass}_R \text{Hom}_R(N, \text{Hom}_R(R/Rx, M)),
\]
and so
\[
\bigcup_{n=1}^{\infty} \text{Ass}_R \text{Ext}_R^0(N/x^n N, M)
\]
is a finite set. Suppose then that \( j \geq 1 \), and we use the exact sequence
\[
0 \longrightarrow N \overset{x^n} \longrightarrow N/x^n N \longrightarrow 0,
\]
to obtain the exact sequence
\[
\cdots \rightarrow \text{Ext}_R^{j-1}(N, M) \overset{x^n} \longrightarrow \text{Ext}_R^{j-1}(N, M) \longrightarrow \text{Ext}_R^j(N/x^n N, M)
\]
\[
\longrightarrow \text{Ext}_R^j(N, M) \overset{x^n} \longrightarrow \text{Ext}_R^j(N, M) \longrightarrow \cdots
\]
Hence we have the following exact sequence,
\[
0 \rightarrow \text{Ext}_R^{j-1}(N, M)/x^n \text{Ext}_R^{j-1}(N, M) \rightarrow \text{Ext}_R^j(N/x^n N, M) \rightarrow (0 : \text{Ext}_R^j(N, M) x^n) \rightarrow 0.
\]
Consequently, it follows from Brodmann’s result (see [1]) that the set
\[
\bigcup_{n=1}^{\infty} \text{Ass}_R \text{Ext}_R^j(N/x^n N, M)
\]
is finite. \( \square \)

2.5. Lemma. Let \( R \) be a Noetherian ring and let \( M \) be an \( R \)-module. Let \( N \) be an Artinian submodule of \( M \). Then
\[
\text{Ass}_R M/N/\text{Supp} N = \text{Ass}_R M/\text{Supp} N.
\]
Proof. As \( N \) is an Artinian \( R \)-module, it follows that the set \( \text{Supp} N \subseteq \text{Max} R \) is finite. Let \( \text{Supp} N = \{m_1, \ldots, m_n\} \) and \( J := m_1 \cdots m_n \). Then we have
\[
\text{Ass}_R M/\text{Supp} N = \text{Ass}_R M/\Gamma J(M) = \text{Ass}_R M/N/\text{Supp} N,
\]
as required. \( \square \)

Following we let \( H^j_I(M) \) denote the \( j^{\text{th}} \) local cohomology module of \( M \) with respect to an ideal \( I \) of a Noetherian ring \( R \) (cf. [2] and [3]).

2.6. Theorem. Let \( R \) be a Noetherian ring and let \( I \) be an ideal of \( R \) which is generated by an \( R \)-regular sequence. Let \( M \) be a finitely generated \( R \)-module and let \( i \) be a non-negative integer such that the \( R \)-module \( \text{Ext}_R^i(R/I, M) \) is Artinian. Then the set
\[
\bigcup_{n=1}^{\infty} \text{Ass}_R \text{Ext}_R^{i+1}(R/I^n, M),
\]
is finite. In particular, the set \( \text{Ass}_R H^{i+1}_I(M) \) is finite.

Proof. For \( n \geq 0 \), the exact sequence
\[
0 \longrightarrow I^n/I^{n+1} \longrightarrow R/I^{n+1} \longrightarrow R/I^n \longrightarrow 0
\]
induces the exact sequence
\[
\text{Ext}_R^i(I^n/I^{n+1}, M) \rightarrow \text{Ext}_R^{i+1}(R/I^n, M) \rightarrow \text{Ext}_R^{i+1}(R/I^{n+1}, M) \rightarrow \text{Ext}_R^{i+1}(I^n/I^{n+1}, M).
\]
Since $I$ is generated by an $R$-regular sequence, by [5, page 125] $I^n/I^{n+1}$ is a finitely generated free $R/I$-module, and so the sets
\[
\text{Ass}_R \text{Ext}_R^{i+1}(I^n/I^{n+1}, M) = \text{Ass}_R \text{Ext}_R^{i+1}(R/I, M),
\]
and
\[
\text{SuppExt}_R^{i}(I^n/I^{n+1}, M) = \text{SuppExt}_R^{i}(R/I, M)
\]
as finite, (note that the $R$-module $\text{Ext}_R^{i}(R/I, M)$ is Artinian). Therefore in view of the above exact sequence and Lemma 2.5, the set
\[
\text{Ass}_R \text{Ext}_R^{i+1}(R/I^n, M) \setminus \text{SuppExt}_R^{i}(R/I, M)
\]
is a subset of
\[
(\text{Ass}_R \text{Ext}_R^{i+1}(R/I^n, M) \setminus \text{SuppExt}_R^{i}(R/I, M)) \cup \text{Ass}_R \text{Ext}_R^{i+1}(R/I, M),
\]
and so the set $\bigcup_{n=1}^{\infty} \text{Ass}_R \text{Ext}_R^{i+1}(R/I^n, M)$ is finite, as required. The second assertion follows from the fact that
\[
\text{Ass}_R H_R^{i+1}(M) \subseteq \bigcup_{n=1}^{\infty} \text{Ass}_R \text{Ext}_R^{i+1}(R/I^n, M).
\]

2.7. Corollary. Let $R$ be a Noetherian ring and let $I$ be an ideal of $R$ which is generated by an $R$-regular sequence. Let $M$ be a finitely generated $R$-module and let $i$ be a non-negative integer such that $\text{Ext}_R^i(R/I, M) = 0$. Then the sequence
\[
\text{Ass}_R \text{Ext}_R^{i+1}(R/I^k, M),
\]
of associated primes is increasing and ultimately constant for all large $k$.

**Proof.** Since $I^k/I^{k+1}$ is a free $R/I$-module, it follows that $\text{Ext}_R^i(I^k/I^{k+1}, M) = 0$, for all $k \geq 1$. Hence the exact sequence
\[
0 \rightarrow \text{Ext}_R^{i+1}(R/I^k, M) \rightarrow \text{Ext}_R^{i+1}(R/I^{k+1}, M) \rightarrow \text{Ext}_R^{i+1}(I^k/I^{k+1}, M),
\]
implies that
\[
\text{Ass}_R \text{Ext}_R^{i+1}(R/I^k, M) \subseteq \text{Ass}_R \text{Ext}_R^{i+1}(R/I^{k+1}, M).
\]
Now the result follows from Theorem 2.6. 

2.8. Lemma. Let $(R, m)$ be a Noetherian local ring of depth $d$. Let $M$ be a finitely generated $R$-module and $N$ an Artinian submodule of $M$. Then for all $i \leq d-1$,
\[
\text{Ext}_R^i(M, R) \cong \text{Ext}_R^i(M/N, R).
\]

**Proof.** The exact sequence
\[
0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0
\]
induces the exact sequence
\[
\text{Ext}_R^{-1}(N, R) \rightarrow \text{Ext}_R^i(M/N, R) \rightarrow \text{Ext}_R^i(M, R) \rightarrow \text{Ext}_R^i(N, R).
\]
As $N$ has finite length and depth $R = d$, it follows that
\[
\text{Ext}_R^i(N, R) = 0 = \text{Ext}_R^{-1}(N, R).
\]
Hence the result follows. 

2.9. Lemma. Let $(R, m)$ be a local Cohen-Macaulay ring of dimension $d$ and $I$ an ideal of $R$. Then for any $p \in \text{Ass}_R \text{Ext}_R^{i+i}(R/I, R)$, 
\[
\text{height } p = \text{grade } I.
\]
Proof. Let \( \text{grade } I = t \). The assertion is clear when \( t = 0 \). Now suppose that, \( t \geq 1 \).

There exists an \( R \)-regular sequence \( x_1, \ldots, x_t \in I \). As

\[
\text{Ext}_R^{\text{grade } I}(R/I, R) \cong \text{Hom}_{R/(x_1, \ldots, x_t)}(R/I, R/(x_1, \ldots, x_t)),
\]

and \( R/(x_1, \ldots, x_t) \) is a Cohen-Macaulay ring it follows that

\[
\text{Ass}_R \text{Ext}_R^{\text{grade } I}(R/I, R) \subseteq \text{Ass}_R R/(x_1, \ldots, x_t),
\]

that implies for any \( p \in \text{Ass}_R \text{Ext}_R^{\text{grade } I}(R/I, R) \),

\[
\text{height } p = \text{grade } I,
\]
as required.

\[\square\]

2.10. Theorem. Let \((R, m)\) be a local Cohen-Macaulay ring of dimension \( d \geq 3 \). Let \( I \) be an ideal of \( R \) such that \( 1 \leq \text{grade } I \leq d - 2 \). Then

\[
\text{depth } \text{Ext}_R^{\text{grade } I}(R/I, R) \geq 2,
\]

and if \( \text{grade } I \leq d - 3 \) then the equality holds if and only if \( m \in \text{Ass}_R \text{Ext}_R^{1 + \text{grade } I}(R/I, R) \).

Proof. Set \( t := \text{grade } I \). Let \( \Gamma_m(R/I) := J/I \) for some ideal \( J \) of \( R \) with \( I \subseteq J \). Then it is easy to see that \( m \notin \text{Ass}_R R/J \) and \( \dim R/J = \dim R/J \). Hence as \( R \) is a Cohen-Macaulay ring, it follows that \( \text{grade } I = \text{grade } J \). Moreover, since \( J/I \) has finite length, it follows from Lemma 2.8 that

\[
\text{Ext}_R^t(R/I, R) \cong \text{Ext}_R^t(R/J, R) \text{ and } \text{Ext}_R^{t+1}(R/I, R) \cong \text{Ext}_R^{t+1}(R/J, R).
\]

Therefore, we may and do replace \( I \) with \( J \) in the following. Since \( m \notin \text{Ass}_R R/J \), it follows that there exists an element \( x \in R \) such that \( x \) is \( R/J \)-regular sequence. Then, as \( \dim R/(J + Rx) = \dim R/J - 1 \) and \( R \) is a Cohen-Macaulay ring, it follows that

\[
\text{grade } (J + Rx) = \text{grade } J + 1.
\]

Now the exact sequence

\[
0 \rightarrow R/J \rightarrow R/J + Rx \rightarrow 0
\]

induces the exact sequence

\[
0 \rightarrow \text{Ext}_R^t(R/J, R) \rightarrow \text{Ext}_R^t(R/J, R) \rightarrow \text{Ext}_R^{t+1}(R/J, R).
\]

Hence

\[
\text{Ass}_R \text{Ext}_R^t(R/J, R) \subseteq \text{Ass}_R \text{Ext}_R^{t+1}(R/J, R),
\]

and since \( 1 + \text{grade } J \leq d - 1 \), it follows from Lemma 2.9 that

\[
\text{depth } \text{Ext}_R^t(R/J, R) \geq 2.
\]

Now, let \( \text{grade } J \leq d - 3 \). Then we have the following exact sequence,

\[
0 \rightarrow \text{Ext}_R^t(R/J, R) \rightarrow \text{Ext}_R^{t+1}(R/J + Rx, R) \rightarrow 0.
\]

Since \( \text{grade } (J + Rx) = t + 1 \) and \( t + 1 \leq d - 2 \), it follows from the first part that

\[
\text{depth } \text{Ext}_R^{t+1}(R/J + Rx, R) \geq 2.
\]

Therefore it follows from the exact sequence

\[
0 \rightarrow \text{Hom}_R(R/m, \text{Ext}_R^{t+1}(R/J, R)) \rightarrow \text{Ext}_R^t(R/m, \text{Ext}_R^t(R/J, R)) \rightarrow 0.
\]

that \( \text{depth } \text{Ext}_R^t(R/J, R) = 2 \) if and only if \( \text{Hom}_R(R/m, \text{Ext}_R^{t+1}(R/J, R)) \neq 0 \). Consequently \( \text{depth } \text{Ext}_R^t(R/J, R) = 2 \) if and only if \( m \in \text{Ass}_R \text{Ext}_R^{t+1}(R/J, R) \), as required. \[\square\]
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References