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Simplicial homology groups of certain digital surfaces

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Abstract

In this paper we compute the simplicial homology groups of some digital surfaces.

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1. INTRODUCTION

Digital topology [19, 17] has been used in different image processing and computer graphics algorithms for several decades. It addresses the fundamental properties of binary object connectivity in two dimensional (2D) and three dimensional (3D) digital images. Concepts and results of Digital Topology are used to specify and justify some important low-level image processing algorithms including algorithms for thinning, boundary extraction, object counting, and contour filling. The properties of digital images with tools from Topology (including Algebraic Topology) are used by many researchers [$1-12, 16, 17, 19$].

Homology is a powerful topological invariant which characterizes an object by its p-dimensional holes. Intuitively the 0-dimensional holes can be seen as "tiny holes", 1-dimensional holes can be seen as tunnels, and 2-dimensional holes can be seen as cavities. The usage of homology groups is a new topic and is not widely spread. Simplicial homology groups of digital images have been studied by several researchers [1, 10, 16]. Boxer et al. [10] extend results of [1] about computing simplicial homology groups of digital images. In this work, we compute simplicial homology groups of certain minimal simple closed surfaces.

This paper is organized as follows. Section 2 provides some basic notions used in this paper. In section 3, we compute the simplicial homology groups of certain digital surfaces.

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2. PRELIMINARIES

Let $\mathbb{Z}^n$ be the set of lattice points in the n-dimensional Euclidean space where $\mathbb{Z}$ is the set of integers. For a positive integer l with $1 \le l \le n$ and two distinct points $p = (p_1, p_2, ..., p_n)$, $q = (q_1, q_2, ..., q_n) \in \mathbb{Z}^n$, p and q are c_l-*adjacent* [8] if

(1) there are at most l indices i such that $|p_i - q_i| = 1$; and

(2) for all other indices i such that $|p_i - q_i| \neq 1$, $p_i = q_i$.

Another commonly used notation for c_l-adjacency reflects the number of neighbors $q \in \mathbb{Z}^n$ that a given point $p \in \mathbb{Z}^n$ may have under the adjacency. For example, if $n = 1$ we have $c_1 = 2$-adjacency; if $n = 2$ we have $c_1 = 4$-adjacency and $c_2 = 8$-adjacency; if $n = 3$ we have $c_1 = 6$-adjacency, $c_2 = 18$-adjacency, and $c_3 = 26$-adjacency [8]. Given a natural number l in conditions (1) and (2) with $1 \le l \le n$, l determines each of the κ-adjacency relations of $\mathbb{Z}^n$ in terms of (1) and (2) [14] as follows.

$$\kappa \in \Big\{2n \ (n \ge 1), \ 3^n - 1 \ (n \ge 2), \ 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1 \ (2 \le r \le n-1, n \ge 3)\Big\} \tag{2.1}$$

The pair (X, κ) is considered in a digital picture $(\mathbb{Z}^n, \kappa, \overline{\kappa}, X)$ for $n \ge 1$ in [3, 4, 6, 13], which is called a *digital image* where $(\kappa, \overline{\kappa}) \in \{(\kappa, 2n), (2n, 3^n - 1)\}$. Each of κ and $\overline{\kappa}$ is one of the general κ-adjacency relations. We usually do not permit that κ and $\overline{\kappa}$ both equal $2n$ when $n > 1$, because of the digital connectivity paradox [18]. For instance, $(\kappa, \overline{\kappa}) \in \{(4, 8), (8, 4)\}$ and $\{(6, 18), (6, 26), (26, 6), (18, 6)\}$ are usually considered in $\mathbb{Z}^2$ and $\mathbb{Z}^3$, respectively [6, 13, 19, 20].

A *digital interval* is a set of the form $[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \le z \le b\}$ where $a, b \in \mathbb{Z}$ with $a < b$.

Let κ be an adjacency relation on $\mathbb{Z}^n$. A κ-neighbor of a lattice point p is κ-adjacent to p. A digital image $X \subset \mathbb{Z}^n$ is κ-*connected* [15] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, ..., x_r\}$ of points of a digital image X such that $x = x_0$, $y = x_r$ and x_i and x_{i+1} are κ-neighbors where $i = 0, 1, ..., r-1$. A κ-*component* of a digital image X is a maximal κ-connected subset of X.

Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0 and κ_1-adjacency respectively. Then the function $f : X \to Y$ is called (κ_0, κ_1)-*continuous* [6, 20] if for every κ_0-connected subset U of X, $f(U)$ is a κ_1-connected subset of Y. We say that such a function is digitally continuous. Similar notions are defined on discrete manifolds in [11]: Let D_1 and D_2 be two discrete manifolds and $f : D_1 \to D_2$ be a mapping. f is said to be an *immersion* from D_1 to D_2 or a *gradually varied operator* if x and y are adjacent in D_1 implies either $f(x) = f(y)$ or $f(x)$, $f(y)$ are adjacent in D_2.

Let X be a digital image with κ-adjacency. If $f : [0, m]_{\mathbb{Z}} \to X$ is a $(2, \kappa)$-continuous function such that $f(0) = x$ and $f(m) = y$, then f is called a *digital path* from x to y in X. If $f(0) = f(m)$ then the κ-path is said to be *closed*, and the function is called a κ-*loop*. Let $f : [0, m-1]_{\mathbb{Z}} \to X$ be a $(2, \kappa)$-continuous function such that $f(i)$ and $f(j)$ are κ-adjacent if and only if $j = i \pm 1 \ mod \ m$. Then the set $f([0, m-1]_{\mathbb{Z}})$ is called a *simple closed* κ-*curve*. A point $x \in X$ is called a κ-*corner*, if x is κ-adjacent to two and only two points $y, z \in X$ such that y and z are κ-adjacent to each other [4]. Moreover, the κ-corner x is called *simple* if y, z are not κ-corners and if x is the only point κ-adjacent to both y, z [3]. X is called a *generalized simple closed* κ-*curve* if what is obtained by removing all simple κ-corners of X is a simple closed κ-curve [4]. If (X, κ) is a κ-connected digital image in $\mathbb{Z}^3$, $|X|^x = N_3^*(x) \cap X$, where $N_3^*(x) = \{x' \in \mathbb{Z}^3 : x$ and x' are 26-adjacent$\}$ [3, 4]. Generally, if (X, κ) is a κ-connected digital image in $\mathbb{Z}^n$, $|X|^x = N_n^*(x) \cap X$, where $N_n^*(x) = \{x' \in \mathbb{Z}^n : x$ and x' are c_n-adjacent$\}$ [13].

Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0 and κ_1-adjacency respectively. A function $f : X \to Y$ is a (κ_0, κ_1)-*isomorphism* [9] (called (κ_0, κ_1)-homeomorphism in

[5]) if f is (κ_0,κ_1)-continuous, bijective and $f^{-1}: Y \to X$ is (κ_1,κ_0)-continuous, in which case we write $X \approx_{(\kappa_0,\kappa_1)} Y$.

2.1. Definition. [13] Let $c^* := \{x_0, x_1, ..., x_n\}$ be a closed κ-curve in $\mathbb{Z}^2$ where $\{\kappa, \overline{\kappa}\} = \{4, 8\}$. A point x of the complement $\overline{c^*}$ of a closed κ-curve c^* in $\mathbb{Z}^2$ is said to be in the *interior* of c^* if it belongs to the bounded $\overline{\kappa}$-connected component of $\overline{c^*}$. The set of all interior points of c^* is denoted by $Int(c^*)$.

2.2. Definition. [13] Let (X,κ) be a digital image in $\mathbb{Z}^n$, $n \geq 3$ and $\overline{X} = \mathbb{Z}^n - X$. Then X is called a *closed κ-surface* if it satisfies the following.

(1) In case that $(\kappa, \overline{\kappa}) \in \{(\kappa, 2n), (2n, 3^n - 1)\}$, where the κ-adjacency is taken from (2.1) with $\kappa \neq 3^n - 2^n - 1$ and $\overline{\kappa}$ is the adjacency on $\overline{X}$, then

(a) for each point $x \in X$, $|X|^x$ has exactly one κ-component κ-adjacent to x;

(b) $|\overline{X}|^x$ has exactly two $\overline{\kappa}$-components $\overline{\kappa}$-adjacent to x; we denote by C^{xx} and D^{xx} these two components; and

(c) for any point $y \in N_\kappa(x) \cap X$, $N_{\overline{\kappa}}(y) \cap C^{xx} \neq \emptyset$ and $N_{\overline{\kappa}}(y) \cap D^{xx} \neq \emptyset$, where $N_\kappa(x)$ means the κ-neighbors of x.

Further, if a closed κ-surface X does not have a simple κ-point, then X is called simple.

(2) In case that $(\kappa, \overline{\kappa}) = (3^n - 2^n - 1, 2n)$, then

(a) X is κ-connected,

(b) for each point $x \in X$, $|X|^x$ is a generalized simple closed κ-curve.

Further, if the image $|X|^x$ is a simple closed κ-curve, then the closed κ-surface X is called simple.

For a closed κ-surface S_κ, we denote by $\overline{S_\kappa}$ the complement of S_κ in $\mathbb{Z}^n$. Then a point x of $\overline{S_\kappa}$ is said to be *interior* of S_κ if it belongs to the bounded $\overline{\kappa}$-connected component of S_κ. The set of all interior points of S_κ is denoted by $int(S_\kappa)$.

The 3-dimensional digital images MSS_{18}^* and MSS_6^* which are obtained from the minimal simple closed curves MSC_8 and MSC_4 in $\mathbb{Z}^2$, respectively, are essentially used in establishing the notion of a connected sum [13].

Figure 1. Minimal simple closed curves MSC_4 and MSC_8.

- $MSS_6^* := MSS_6 \cup Int(MSS_6)$ where

$$MSS_6 \approx_{(6,6)} (MSC_4 \times [0,2]_{\mathbb{Z}}) \cup (Int(MSC_4) \times \{0,2\})$$

and MSC_4 is 4-isomorphic to the set

$$\{(1,0),(1,1),(0,1),(-1,1),(-1,0),(-1,-1),(0,-1),(1,-1)\}.$$

- $MSS_{18}^* := MSS_{18} \cup Int(MSS_{18})$ where

$$MSS_{18} \approx_{(18,18)} (MSC_8 \times \{1\}) \cup (Int(MSC_8) \times \{0,2\})$$

and MSC_8 is 8-isomorphic to the set

$$\{(0,0),(-1,1),(-2,0),(-2,-1),(-1,-2),(0,-1)\}.$$

2.3. Definition. [13] Let S_{κ_0} be a closed κ_0-surface in $\mathbb{Z}^{n_0}$ and S_{κ_1} be a closed κ_1-surface in $\mathbb{Z}^{n_1}$ for $n_0, n_1 \geq 3$. Consider $A'_{\kappa_0} \subset A_{\kappa_0} \subset S_{\kappa_0}$ such that

$$A'_{\kappa_0} \approx_{(\kappa_0,8)} Int(MSC_8^*),\ A'_{\kappa_0} \approx_{(\kappa_0,4)} Int(MSC_4^*) \text{ or } A'_{\kappa_0} \approx_{(\kappa_0,8)} Int(MSC_8'^*).$$

Let $f : A_{\kappa_0} \to f(A_{\kappa_0}) \subset S_{\kappa_1}$ be a (κ_0,κ_1)-isomorphism. Let $S'_{\kappa_i} = S_{\kappa_i} \setminus A'_{\kappa_i}$, $i \in \{0,1\}$. Then the connected sum, denoted by $S_{\kappa_0} \sharp S_{\kappa_1}$, is the quotient space $S'_{\kappa_0} \cup S'_{\kappa_1} / \sim$, where $i : A_{\kappa_0} \setminus A'_{\kappa_0} \to S'_{\kappa_0}$ is the inclusion map and $i(x) \sim f(x)$ for $x \in A_{\kappa_0} \setminus A'_{\kappa_0}$.

2.4. Definition. [21] Let S be a set of nonempty subsets of a digital image (X,κ). The members of S are called simplexes of (X,κ) if the following holds:

- ***(i)*** If p and q are distinct points of $s \in S$, then p and q are κ-adjacent.
- ***(ii)*** If $s \in S$ and $\emptyset \neq t \subset s$, then $t \in S$ (note this implies every point p that belongs to a simplex determines a simplex $\{p\}$).

An m-simplex is a simplex S such that $|S| = m+1$.

Let P be a digital m-simplex. If P' is a nonempty proper subset of P, then P' is called a face of P.

Since computing homology groups is easier than computing higher degree homotopy groups in algebraic topology, for the same reason computing homology groups of digital images is preferred to computing homotopy groups of digital images. The simplicial homology groups of n-dimensional digital images from algebraic topology have been introduced in [1].

2.5. Definition. [1] Let (X,κ) be a finite collection of digital m-simplices, $0 \leq m \leq d$ for some nonnegative integer d. If the following statements hold, then (X,κ) is called a finite digital simplicial complex:

- **(1)** If P belongs to X, then every face of P also belongs to X.
- **(2)** If $P, Q \in X$, then $P \cap Q$ is either empty or a common face of P and Q.

The dimension of a digital simplicial complex X is the biggest integer m such that X has an m-simplex.

$C_q^\kappa(X)$ is a free abelian group with basis all digital (κ, q)-simplices in X [1].

2.6. Corollary. [10] Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. Then for all $q > m$, $C_q^\kappa(X)$ is a trivial group.

Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. The homomorphism $\partial_q : C_q^\kappa(X) \to C_{q-1}^\kappa(X)$ defined by

$$\partial_q(< p_0, p_1, ..., p_q >) = \begin{cases} \sum_{i=0}^{q} (-1)^i < p_0, p_1, ..., \widehat{p_i}, ..., p_q >, & q \leq m; \\ 0, & \text{q>m} \end{cases}$$

is called a boundary homomorphism where $\widehat{p_i}$ means deleting the point p_i. Then for all $1 \leq q \leq m$, we have $\partial_{q-1} \circ \partial_q = 0$ [1].

2.7. Theorem. [1] Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. Then

$$C_*^\kappa(X) : 0 \xrightarrow{\partial_{m+1}} C_m^\kappa(X) \xrightarrow{\partial_m} C_{m-1}^\kappa(X) \xrightarrow{\partial_{m-1}} ... \xrightarrow{\partial_1} C_0^\kappa(X) \xrightarrow{\partial_0} 0$$

is a chain complex.

Let (X,κ) be a digital simplicial complex. The group of digital simplicial q-cycles is $Z_q^\kappa(X) = Ker\ \partial_q = \{\sigma \in C_q^\kappa(X) | \partial_q(\sigma) = 0\}$ and the group of digital simplicial q-boundaries is $B_q^\kappa(X) = Im\ \partial_{q+1} = \{\tau \in C_q^\kappa(X) | \partial_{q+1}(\sigma) = \tau \text{ for } \sigma \in C_{q+1}^\kappa(X)\}$. The qth digital simplicial homology group is $H_q^\kappa(X) = Z_q^\kappa(X)/B_q^\kappa(X)$ [1].

2.8. Theorem. [1] If $f : X \to Y$ is a digital (κ_0, κ_1)-isomorphism, then for all q

$$H_q^{\kappa_0}(X) \cong H_q^{\kappa_1}(Y).$$

2.9. Theorem. [10] Let (X,κ) be a directed digital simplicial complex of dimension m.

- ***(1)*** $H_q^\kappa(X)$ is a finitely generated abelian group for every $q \geq 0$.
- ***(2)*** $H_q^\kappa(X)$ is a trivial group for all $q > m$.
- ***(3)*** $H_q^\kappa(X)$ is a free abelian group, possibly zero.

2.10. Definition. [10] Let (X, κ) be a digital image of dimension m, and for each $q \geq 0$, let α_q be the number of digital (κ, q)-simplexes in X. The Euler characteristic of X, denoted by $\chi(X, \kappa)$, is defined by

$$\chi(X, \kappa) = \sum_{q=0}^{m} (-1)^q \alpha_q.$$

2.11. Theorem. [10] If (X, κ) is a digital image of dimension m, then

$$\chi(X, \kappa) = \sum_{q=0}^{m} (-1)^q rank\ H_q^{\kappa}(X).$$

2.12. Example. [10] By the definition of Euler characteristic, we have

$$\chi(MSS_6, 6) = \alpha_0 - \alpha_1 = 26 - 48 = -22$$
$$\chi(MSS_6 \sharp MSS_6, 6) = \alpha_0 - \alpha_1 = 42 - 80 = -38$$
$$\chi(MSS_{18}, 18) = \alpha_0 - \alpha_1 + \alpha_2 = 10 - 20 + 8 = -2$$
$$\chi(MSS_{18} \sharp MSS_{18}, 18) = \alpha_0 - \alpha_1 + \alpha_2 = 14 - 28 + 8 = -6$$

3. MAIN RESULTS

Simplicial homology groups of several digital surfaces have been computed in [10]. By using an argument similar to that of [10], we have the following theorems.

3.1. Theorem. The digital simplicial homology groups of $MSS_{18} \sharp MSS_{18}$ are

$$H_q^{18}(MSS_{18} \sharp MSS_{18}) = \begin{cases} \mathbb{Z}, & q = 0; \\ \mathbb{Z}^7, & q = 1; \\ 0, & q \geq 2. \end{cases}$$

Figure 2. $MSS_{18} \sharp MSS_{18}$

Proof. Let

$$\begin{aligned} MSS_{18} \sharp MSS_{18} = \{ & c_0 = (1,0,1), c_1 = (1,1,1), c_2 = (1,2,1), \\ & c_3 = (0,3,1), c_4 = (-1,2,1), c_5 = (-1,1,1), \\ & c_6 = (-1,0,1), c_7 = (0,-1,1), c_8 = (0,2,2), \\ & c_9 = (0,1,2), c_{10} = (0,0,2), c_{11} = (0,2,0), \\ & c_{12} = (0,1,0), c_{13} = (0,0,0)\}. \end{aligned}$$

Then we can direct $MSS_{18} \sharp MSS_{18}$ by the ordering $c_6 < c_5 < c_4 < c_7 < c_{13} < c_{10} < c_{12} < c_9 < c_{11} < c_8 < c_3 < c_0 < c_1 < c_2$. We have the following simplicial chain complexes:
$C_0^{18}(MSS_{18} \sharp MSS_{18})$ has for a basis $\{\langle c_0 \rangle, \langle c_1 \rangle, ..., \langle c_{13} \rangle\}$,
$C_1^{18}(MSS_{18} \sharp MSS_{18})$ has for a basis

$$\begin{aligned} \{ & \langle c_7c_0 \rangle, \langle c_{10}c_0 \rangle, \langle c_{13}c_0 \rangle, \langle c_0c_1 \rangle, \langle c_9c_1 \rangle, \langle c_{12}c_1 \rangle, \langle c_1c_2 \rangle, \langle c_8c_2 \rangle, \langle c_{11}c_2 \rangle, \langle c_3c_2 \rangle, \langle c_4c_3 \rangle, \\ & \langle c_8c_3 \rangle, \langle c_{11}c_3 \rangle, \langle c_5c_4 \rangle, \langle c_4c_8 \rangle, \langle c_4c_{11} \rangle, \langle c_6c_5 \rangle, \langle c_5c_9 \rangle, \langle c_5c_{12} \rangle, \langle c_6c_7 \rangle, \langle c_6c_{10} \rangle, \langle c_6c_{13} \rangle, \\ & \langle c_7c_{10} \rangle, \langle c_7c_{13} \rangle, \langle c_9c_8 \rangle, \langle c_{10}c_9 \rangle, \langle c_{12}c_{11} \rangle, \langle c_{13}c_{12} \rangle\}, \end{aligned}$$

and $C_2^{18}(MSS_{18} \sharp MSS_{18})$ has for a basis

$$\{\langle c_7c_{13}c_0 \rangle, \langle c_7c_{10}c_0 \rangle, \langle c_8c_3c_2 \rangle, \langle c_{11}c_3c_2 \rangle, \langle c_4c_8c_3 \rangle, \langle c_4c_{11}c_3 \rangle, \langle c_6c_7c_{10} \rangle, \langle c_6c_7c_{13} \rangle\}.$$

Thus, we obtain the following short sequence:

$$0 \xrightarrow{\partial_3} C_2^{18}(MSS_{18}\sharp MSS_{18}) \xrightarrow{\partial_2} C_1^{18}(MSS_{18}\sharp MSS_{18}) \xrightarrow{\partial_1} C_0^{18}(MSS_{18}\sharp MSS_{18}) \xrightarrow{\partial_0} 0.$$

By Theorem 2.9, $H_q^{18}(MSS_{18}\sharp MSS_{18})$ is a trivial group for all $q > 2$.

We determine the kernel of ∂_2. If

$$\begin{aligned}\partial_2(a_1\langle c_7c_{13}c_0\rangle + a_2\langle c_7c_{10}c_0\rangle + a_3\langle c_8c_3c_2\rangle + a_4\langle c_{11}c_3c_2\rangle + a_5\langle c_4c_8c_3\rangle + a_6\langle c_4c_{11}c_3\rangle \\ + a_7\langle c_6c_7c_{10}\rangle + a_8\langle c_6c_7c_{13}\rangle) = a_1\langle c_{13}c_0\rangle + (-a_1 - a_2)\langle c_7c_0\rangle + (a_1 + a_8)\langle c_7c_{13}\rangle \\ + a_2\langle c_{10}c_0\rangle + (a_2 + a_7)\langle c_7c_{10}\rangle + (a_3 + a_4)\langle c_3c_2\rangle - a_3\langle c_8c_2\rangle + (a_3 + a_5)\langle c_8c_3\rangle \\ - a_4\langle c_{11}c_2\rangle + (a_4 + a_6)\langle c_{11}c_3\rangle + (-a_5 - a_6)\langle c_4c_3\rangle + a_5\langle c_4c_8\rangle \\ + a_6\langle c_4c_{11}\rangle - a_7\langle c_6c_{10}\rangle + (a_7 + a_8)\langle c_6c_7\rangle - a_8\langle c_6c_{13}\rangle = 0,\end{aligned}$$

then one easily sees that $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = 0$. Therefore, $Z_2^{18}(MSS_{18}\sharp MSS_{18}) = \{0\}$ and hence $H_2^{18}(MSS_{18}\sharp MSS_{18}) = \{0\}$.

Since $Ker\ \partial_2 = Z_2^{18}(MSS_{18}\sharp MSS_{18}) = \{0\}$, $Im\ \partial_2 \cong C_2^{8}(MSS_{18}\sharp MSS_{18})$, and so $B_1^{18}(MSS_{18}\sharp MSS_{18}) \cong \mathbb{Z}^8$.

We can use standard methods to determine that $Z_1^{18}(MSS_{18}\sharp MSS_{18}) \cong \mathbb{Z}^{15}$, from which it follows easily that $B_0^{18}(MSS_{18}\sharp MSS_{18}) \cong \mathbb{Z}^{13}$. However, the direct calculation of $Z_1^{18}(MSS_{18}\sharp MSS_{18})$ is very long. Since our goal is to calculate $H_1^{18}(MSS_{18}\sharp MSS_{18})$, we will do so below without showing a direct calculation of $Z_1^{18}(MSS_{18}\sharp MSS_{18})$.

By using the short sequence again, we have

$$Z_0^{18}(MSS_{18}\sharp MSS_{18}) = \Big\{ \sum_{i=0}^{13} a_i\langle c_i\rangle \mid a_i \in \mathbb{Z}, i = 0, 1, ..., 13\Big\} \cong \mathbb{Z}^{14}$$

Any 0-cycle $w_0 = \sum_{i=0}^{13} a_i\langle c_i\rangle$ can be written as

$$\begin{aligned}w_0 = \partial_1((-a_7)\langle c_7c_0\rangle + (a_1 + a_2 + a_3)\langle c_0c_1\rangle + (a_2 + a_3)\langle c_1c_2\rangle \\ + (-a_3)\langle c_3c_2\rangle + a_{11}\langle c_4c_{11}\rangle + (a_4 + a_{11})\langle c_5c_4\rangle + a_{12}\langle c_5c_{12}\rangle \\ + (a_4 + a_5 + a_{11} + a_{12})\langle c_6c_5\rangle + a_{13}\langle c_6c_{13}\rangle \\ + (-a_4 - a_5 - a_6 - a_{11} - a_{12} - a_{13})\langle c_6c_{10}\rangle + a_8\langle c_9c_8\rangle \\ + (a_8 + a_9)\langle c_{10}c_9\rangle + (a_0 + a_1 + a_2 + a_3 + a_7)\langle c_{10}c_0\rangle) + \sum_{i=0}^{13} a_i\langle c_{10}\rangle.\end{aligned}$$

So w_0 is homologous to 0-chain $\sum_{i=0}^{13} a_i\langle c_{10}\rangle$. Hence the 0-chain is homologous to an integral multiple of $\langle c_{10}\rangle$. Thus we deduce $H_0^{18}(MSS_{18}\sharp MSS_{18}) \cong \mathbb{Z}$.

To compute the $H_1^{18}(MSS_{18}\sharp MSS_{18})$, we can use the results in [10]. By Example 2.12, we know that $\chi(MSS_{18}\sharp MSS_{18}, 18) = -6$. From Theorem 2.11,

$$\begin{aligned}\chi(MSS_{18}\sharp MSS_{18}, 18) = \sum_{q=0}^{2}(-1)^q rank\ H_q^{18}(MSS_{18}\sharp MSS_{18}) \\ -6 = 1 - rank\ H_1^{18}(MSS_{18}\sharp MSS_{18}) + 0\end{aligned}$$

Thus we get $rank\ H_1^{18}(MSS_{18}\sharp MSS_{18}) = 7$ which in turn gives us

$$H_1^{18}(MSS_{18}\sharp MSS_{18}) \cong \mathbb{Z}^7.$$

□

3.2. Theorem. The digital simplicial homology groups of MSS_6 are

$$H_q^6(MSS_6) = \begin{cases} \mathbb{Z}, & q = 0; \\ \mathbb{Z}^{23}, & q = 1; \\ 0, & q \neq 0, 1. \end{cases}$$

Figure 3. MSS_6

Proof. If we take

$$\begin{aligned} MSS_6 = \{ & c_0 = (-1,-1,0), c_1 = (0,-1,0), c_2 = (1,-1,0), c_3 = (1,0,0), \\ & c_4 = (0,0,0), c_5 = (-1,0,0), c_6 = (-1,1,0), c_7 = (0,1,0), \\ & c_8 = (1,1,0), c_9 = (1,1,1), c_{10} = (0,1,1), c_{11} = (-1,1,1), \\ & c_{12} = (-1,0,1), c_{13} = (1,0,1), c_{14} = (1,-1,1), c_{15} = (0,-1,1), \\ & c_{16} = (-1,-1,1), c_{17} = (-1,-1,2), c_{18} = (0,-1,2), c_{19} = (1,-1,2), \\ & c_{20} = (1,0,2), c_{21} = (0,0,2), c_{22} = (-1,0,2), c_{23} = (-1,1,2), \\ & c_{24} = (0,1,2), c_{25} = (1,1,2)\}, \end{aligned}$$

then we can direct MSS_6 by the ordering $c_0 < c_{16} < c_{17} < c_5 < c_{12} < c_{22} < c_6 < c_{11} < c_{23} < c_1 < c_{15} < c_{18} < c_4 < c_{21} < c_7 < c_{10} < c_{24} < c_2 < c_{14} < c_{19} < c_3 < c_{13} < c_{20} < c_8 < c_9 < c_{25}$.

We have the following simplicial chain complexes:
$C_0^6(MSS_6)$ has for a basis $\{\langle c_0\rangle, \langle c_1\rangle, ..., \langle c_{25}\rangle\}$, and $C_1^6(MSS_6)$ has for a basis

$$\begin{aligned} \{ & \langle c_0c_1\rangle, \langle c_0c_5\rangle, \langle c_0c_{16}\rangle, \langle c_1c_2\rangle, \langle c_1c_4\rangle, \langle c_1c_{15}\rangle, \langle c_2c_{14}\rangle, \langle c_2c_3\rangle, \langle c_4c_3\rangle, \langle c_3c_8\rangle, \langle c_3c_{13}\rangle, \\ & \langle c_5c_4\rangle, \langle c_4c_7\rangle, \langle c_5c_6\rangle, \langle c_5c_{12}\rangle, \langle c_6c_7\rangle, \langle c_6c_{11}\rangle, \langle c_7c_8\rangle, \langle c_7c_{10}\rangle, \langle c_8c_9\rangle, \langle c_{10}c_9\rangle, \langle c_{13}c_9\rangle, \\ & \langle c_9c_{25}\rangle, \langle c_{11}c_{10}\rangle, \langle c_{10}c_{24}\rangle, \langle c_{12}c_{11}\rangle, \langle c_{11}c_{23}\rangle, \langle c_{16}c_{12}\rangle, \langle c_{12}c_{22}\rangle, \langle c_{14}c_{13}\rangle, \langle c_{13}c_{20}\rangle, \\ & \langle c_{15}c_{14}\rangle, \langle c_{14}c_{19}\rangle, \langle c_{16}c_{15}\rangle, \langle c_{15}c_{18}\rangle, \langle c_{16}c_{17}\rangle, \langle c_{17}c_{18}\rangle, \langle c_{17}c_{22}\rangle, \langle c_{18}c_{19}\rangle, \langle c_{18}c_{21}\rangle, \\ & \langle c_{19}c_{20}\rangle, \langle c_{21}c_{20}\rangle, \langle c_{20}c_{25}\rangle, \langle c_{22}c_{21}\rangle, \langle c_{21}c_{24}\rangle, \langle c_{22}c_{23}\rangle, \langle c_{23}c_{24}\rangle, \langle c_{24}c_{25}\rangle\}. \end{aligned}$$

Thus we get the following short sequence:

$$0 \xrightarrow{\partial_2} C_1^6(MSS_6) \xrightarrow{\partial_1} C_0^6(MSS_6) \xrightarrow{\partial_0} 0.$$

By Theorem 2.9, we have $H_q^6(MSS_6) = \{0\}$ for every $q > 1$.

Direct calculation yields that $Z_1^6(MSS_6) \cong \mathbb{Z}^{23}$, from which it follows easily that $B_0^6(MSS_6) \cong \mathbb{Z}^{25}$. However, direct calculation of $Z_1^6(MSS_6)$ is very long. Since our goal is to calculate $H_1^6(MSS_6)$, we do so below without showing a direct calculation of $Z_1^6(MSS_6)$.

By using the short sequence, we have

$$Z_0^6(MSS_6) = \Big\{ \sum_{i=0}^{25} a_i\langle c_i\rangle \mid a_i \in \mathbb{Z},\ i = 0, 1, ..., 25 \Big\} \cong \mathbb{Z}^{26}.$$

Any 0-cycle $w_0 = \sum_{i=0}^{25} a_i\langle c_i\rangle$ can be written as

$$\begin{aligned}
w_0 = \partial_1((-a_6)\langle c_6c_{11}\rangle &+ (-a_6 - a_{11})\langle c_{11}c_{23}\rangle + (a_6 + a_{11} + a_{23})\langle c_{22}c_{23}\rangle \\
&+ (a_6 + a_{11} + a_{22} + a_{23})\langle c_{12}c_{22}\rangle + (a_6 + a_{11} + a_{12} + a_{22} + a_{23})\langle c_5c_{12}\rangle \\
&+ (a_5 + a_6 + a_{11} + a_{12} + a_{22} + a_{23})\langle c_0c_5\rangle \\
&+ (-a_0 - a_5 - a_6 - a_{11} - a_{12} - a_{22} - a_{23})\langle c_0c_{16}\rangle \\
&+ (-a_0 - a_5 - a_6 - a_{11} - a_{12} - a_{16} - a_{22} - a_{23})\langle c_{16}c_{17}\rangle \\
&+ (-a_0 - a_5 - a_6 - a_{11} - a_{12} - a_{16} - a_{17} - a_{22} - a_{23})\langle c_{17}c_{18}\rangle \\
&+ a_{15}\langle c_1c_{15}\rangle + (-a_1 - a_{15})\langle c_1c_4\rangle \\
&+ (-a_1 - a_4 - a_{15})\langle c_4c_7\rangle + (-a_1 - a_4 - a_7 - a_{15})\langle c_7c_{10}\rangle \\
&+ (-a_1 - a_4 - a_7 - a_{10} - a_{15})\langle c_{10}c_{24}\rangle \\
&+ (a_1 + a_4 + a_7 + a_{10} + a_{15} + a_{24})\langle c_{21}c_{24}\rangle \\
&+ (a_1 + a_4 + a_7 + a_{10} + a_{15} + a_{21} + a_{24})\langle c_{18}c_{21}\rangle + (-a_8)\langle c_8c_9\rangle \\
&+ (-a_8 - a_9)\langle c_9c_{25}\rangle + (a_8 + a_9 + a_{25})\langle c_{20}c_{25}\rangle \\
&+ (a_8 + a_9 + a_{20} + a_{25})\langle c_{13}c_{20}\rangle + (a_8 + a_9 + a_{13} + a_{20} + a_{25})\langle c_3c_{13}\rangle \\
&+ (a_3 + a_8 + a_9 + a_{13} + a_{20} + a_{25})\langle c_2c_3\rangle \\
&+ (-a_2 - a_3 - a_8 - a_9 - a_{13} - a_{20} - a_{25})\langle c_2c_{14}\rangle \\
&+ (-a_2 - a_3 - a_8 - a_9 - a_{13} - a_{14} - a_{20} - a_{25})\langle c_{14}c_{19}\rangle \\
&+ (a_2 + a_3 + a_8 + a_9 + a_{13} + a_{14} + a_{19} + a_{20} + a_{25})\langle c_{18}c_{19}\rangle) + \sum_{i=0}^{25} a_i\langle c_{18}\rangle.
\end{aligned}$$

So w_0 is homologous to 0-chain $\sum_{i=0}^{25} a_i\langle c_{18}\rangle$. Hence the 0-chain is homologous to an integral multiple of $\langle c_{18}\rangle$. Thus we get

$$H_0^6(MSS_6) \cong \mathbb{Z}.$$

We use the results in [10] to compute the $H_1^6(MSS_6)$. From Example 2.12, we have $\chi(MSS_6, 6) = -22$. From Theorem 2.11,

$$\begin{aligned}
\chi(MSS_6, 6) &= \sum_{q=0}^{1}(-1)^q rank\ H_q^6(MSS_6) \\
-22 &= 1 - rank\ H_1^6(MSS_6)
\end{aligned}$$

Thus we get $rank\ H_1^6(MSS_6) = 23$ which gives us

$$H_1^6(MSS_6) \cong \mathbb{Z}^{23}.$$

□

3.3. Theorem. The digital simplicial homology groups of $MSS_6\sharp MSS_6$ are

$$H_q^6(MSS_6\sharp MSS_6) = \begin{cases} \mathbb{Z}, & q = 0; \\ \mathbb{Z}^{39}, & q = 1; \\ 0, & q \neq 0, 1. \end{cases}$$

Figure 4. $MSS_6\sharp MSS_6$

Proof. Let

$$\begin{aligned} MSS_6\sharp MSS_6 = \{&c_0=(0,0,0), c_1=(1,0,0), c_2=(2,0,0), c_3=(2,1,0),\\ &c_4=(1,1,0), c_5=(0,1,0), c_6=(0,2,0), c_7=(1,2,0),\\ &c_8=(2,2,0), c_9=(2,3,0), c_{10}=(1,3,0), c_{11}=(0,3,0),\\ &c_{12}=(0,4,0), c_{13}=(1,4,0), c_{14}=(2,4,0), c_{15}=(2,4,1),\\ &c_{16}=(1,4,1), c_{17}=(0,4,1), c_{18}=(0,3,1), c_{19}=(2,3,1),\\ &c_{20}=(2,2,1), c_{21}=(0,2,1), c_{22}=(0,1,1), c_{23}=(2,1,1),\\ &c_{24}=(2,0,1), c_{25}=(1,0,1), c_{26}=(0,0,1), c_{27}=(0,0,2),\\ &c_{28}=(1,0,2), c_{29}=(2,0,2), c_{30}=(2,1,2), c_{31}=(1,1,2),\\ &c_{32}=(0,1,2), c_{33}=(0,2,2), c_{34}=(1,2,2), c_{35}=(2,2,2),\\ &c_{36}=(2,3,2), c_{37}=(1,3,2), c_{38}=(0,3,2), c_{39}=(0,4,2),\\ &c_{40}=(1,4,2), c_{41}=(2,4,2)\}. \end{aligned}$$

We can direct $MSS_6\sharp MSS_6$ by the ordering $c_0 < c_{26} < c_{27} < c_5 < c_{22} < c_{32} < c_6 < c_{21} < c_{33} < c_{11} < c_{18} < c_{38} < c_{12} < c_{17} < c_{39} < c_1 < c_{25} < c_{28} < c_4 < c_{31} < c_7 < c_{34} < c_{10} < c_{37} < c_{13} < c_{16} < c_{40} < c_2 < c_{24} < c_{29} < c_3 < c_{23} < c_{30} < c_8 < c_{20} < c_{35} < c_9 < c_{19} < c_{36} < c_{14} < c_{15} < c_{41}$.

We have the following simplicial chain complexes:
$C_0^6(MSS_6\sharp MSS_6)$ has for a basis $\{\langle c_0\rangle, \langle c_1\rangle, ..., \langle c_{41}\rangle\}$, and
$C_1^6(MSS_6\sharp MSS_6)$ has for a basis

$$\begin{aligned} &\{\langle c_0c_1\rangle, \langle c_0c_5\rangle, \langle c_0c_{26}\rangle, \langle c_1c_4\rangle, \langle c_1c_2\rangle, \langle c_1c_{25}\rangle, \langle c_2c_3\rangle, \langle c_2c_{24}\rangle, \langle c_4c_3\rangle, \langle c_3c_8\rangle, \langle c_3c_{23}\rangle,\\ &\langle c_4c_7\rangle, \langle c_5c_4\rangle, \langle c_5c_6\rangle, \langle c_5c_{22}\rangle, \langle c_6c_{11}\rangle, \langle c_6c_{21}\rangle, \langle c_6c_7\rangle, \langle c_7c_{10}\rangle, \langle c_7c_8\rangle, \langle c_8c_9\rangle, \langle c_8c_{20}\rangle,\\ &\langle c_9c_{14}\rangle, \langle c_{10}c_9\rangle, \langle c_9c_{19}\rangle, \langle c_{10}c_{13}\rangle, \langle c_{11}c_{10}\rangle, \langle c_{11}c_{12}\rangle, \langle c_{11}c_{18}\rangle, \langle c_{12}c_{13}\rangle, \langle c_{12}c_{17}\rangle,\\ &\langle c_{13}c_{16}\rangle, \langle c_{13}c_{14}\rangle, \langle c_{14}c_{15}\rangle, \langle c_{16}c_{15}\rangle, \langle c_{19}c_{15}\rangle, \langle c_{15}c_{41}\rangle, \langle c_{17}c_{16}\rangle, \langle c_{16}c_{40}\rangle, \langle c_{18}c_{17}\rangle,\\ &\langle c_{17}c_{39}\rangle, \langle c_{21}c_{18}\rangle, \langle c_{18}c_{38}\rangle, \langle c_{20}c_{19}\rangle, \langle c_{19}c_{36}\rangle, \langle c_{23}c_{20}\rangle, \langle c_{20}c_{35}\rangle, \langle c_{22}c_{21}\rangle, \langle c_{21}c_{33}\rangle,\\ &\langle c_{26}c_{22}\rangle, \langle c_{22}c_{32}\rangle, \langle c_{24}c_{23}\rangle, \langle c_{23}c_{30}\rangle, \langle c_{25}c_{24}\rangle, \langle c_{24}c_{29}\rangle, \langle c_{26}c_{25}\rangle, \langle c_{25}c_{28}\rangle, \langle c_{26}c_{27}\rangle,\\ &\langle c_{27}c_{28}\rangle, \langle c_{27}c_{32}\rangle, \langle c_{28}c_{29}\rangle, \langle c_{28}c_{31}\rangle, \langle c_{29}c_{30}\rangle, \langle c_{30}c_{35}\rangle, \langle c_{31}c_{30}\rangle, \langle c_{32}c_{31}\rangle, \langle c_{31}c_{34}\rangle,\\ &\langle c_{32}c_{33}\rangle, \langle c_{33}c_{34}\rangle, \langle c_{33}c_{38}\rangle, \langle c_{34}c_{35}\rangle, \langle c_{34}c_{37}\rangle, \langle c_{35}c_{36}\rangle, \langle c_{31}c_{34}\rangle, \langle c_{32}c_{33}\rangle, \langle c_{33}c_{34}\rangle,\\ &\langle c_{33}c_{38}\rangle, \langle c_{34}c_{35}\rangle, \langle c_{34}c_{37}\rangle, \langle c_{35}c_{36}\rangle, \langle c_{31}c_{34}\rangle, \langle c_{32}c_{33}\rangle, \langle c_{33}c_{34}\rangle, \langle c_{33}c_{38}\rangle, \langle c_{34}c_{35}\rangle,\\ &\langle c_{34}c_{37}\rangle, \langle c_{35}c_{36}\rangle, \langle c_{27}c_{32}\rangle, \langle c_{28}c_{29}\rangle, \langle c_{28}c_{31}\rangle, \langle c_{29}c_{30}\rangle, \langle c_{30}c_{35}\rangle, \langle c_{31}c_{30}\rangle, \langle c_{32}c_{31}\rangle,\\ &\langle c_{31}c_{34}\rangle, \langle c_{32}c_{33}\rangle, \langle c_{33}c_{34}\rangle, \langle c_{33}c_{38}\rangle, \langle c_{34}c_{35}\rangle, \langle c_{34}c_{37}\rangle, \langle c_{35}c_{36}\rangle, \langle c_{37}c_{36}\rangle, \langle c_{36}c_{41}\rangle,\\ &\langle c_{38}c_{37}\rangle, \langle c_{37}c_{40}\rangle, \langle c_{38}c_{39}\rangle, \langle c_{39}c_{40}\rangle, \langle c_{40}c_{41}\rangle\}. \end{aligned}$$

Thus we obtain the following short sequence:

$$0 \xrightarrow{\partial_2} C_1^6(MSS_6\sharp MSS_6) \xrightarrow{\partial_1} C_0^6(MSS_6\sharp MSS_6) \xrightarrow{\partial_0} 0.$$

By Theorem 2.9, $H_q^6(MSS_6\sharp MSS_6)$ is a trivial group for $q > 1$.

Direct calculation yields that $Z_1^6(MSS_6\sharp MSS_6) \cong \mathbb{Z}^{39}$, from which it follows easily that $B_0^6(MSS_6\sharp MSS_6) \cong \mathbb{Z}^{41}$. However, direct calculation of the group $Z_1^6(MSS_6\sharp MSS_6)$

of digital simplicial 1-cylces is very long. Since our goal is to calculate $H_1^6(MSS_6\sharp MSS_6)$, we do so below without showing a direct calculation of $Z_1^6(MSS_6\sharp MSS_6)$.

By using the short sequence again, we have

$$Z_0^6(MSS_6\sharp MSS_6) = \Big\{ \sum_{i=0}^{41} a_i\langle c_i\rangle \mid a_i \in \mathbb{Z},\ i = 0, 1, ..., 41 \Big\} \cong \mathbb{Z}^{42}.$$

Any 0-cycle $w_0 = \sum_{i=0}^{41} a_i\langle c_i\rangle$ can be written as

$$\begin{aligned} w_0 = \partial_1(&- a_{12}\langle c_{12}c_{17}\rangle + (-a_{12} - a_{17})\langle c_{17}c_{39}\rangle + (a_{12} + a_{17} + a_{39})\langle c_{38}c_{39}\rangle \\ &+ (a_{12} + a_{17} + a_{38} + a_{39})\langle c_{18}c_{38}\rangle \\ &+ (a_{12} + a_{17} + a_{18} + a_{38} + a_{39})\langle c_{11}c_{18}\rangle \\ &+ (a_{11} + a_{12} + a_{17} + a_{18} + a_{38} + a_{39})\langle c_6c_{11}\rangle \\ &+ (-a_6 - a_{11} - a_{12} - a_{17} - a_{18} - a_{38} - a_{39})\langle c_6c_{21}\rangle \\ &+ (-a_6 - a_{11} - a_{12} - a_{17} - a_{18} - a_{21} - a_{38} - a_{39})\langle c_{21}c_{33}\rangle \\ &+ (a_6 + a_{11} + a_{12} + a_{17} + a_{18} + a_{21} + a_{33} + a_{38} + a_{39})\langle c_{32}c_{33}\rangle \\ &+ (a_6 + a_{11} + a_{12} + a_{17} + a_{18} + a_{21} + a_{32} + a_{33} + a_{38} + a_{39})\langle c_{22}c_{32}\rangle \end{aligned}$$

$$
\begin{aligned}
&+ (a_6 + a_{11} + a_{12} + a_{17} + a_{18} + a_{21} + a_{22} + a_{32} + a_{33} + a_{38} + a_{39})\langle c_5 c_{22}\rangle \\
&+ (a_5 + a_6 + a_{11} + a_{12} + a_{17} + a_{18} + a_{21} + a_{22} + a_{32} + a_{33} \\
&+ a_{38} + a_{39})\langle c_0 c_5\rangle + (-a_0 - a_5 - a_6 - a_{11} - a_{12} - a_{17} - a_{18} - a_{21} \\
&- a_{22} - a_{32} - a_{33} - a_{38} - a_{39})\langle c_0 c_{26}\rangle + (-a_0 - a_5 - a_6 - a_{11} - \\
&- a_{12} - a_{17} - a_{18} - a_{21} - a_{22} - a_{26} - a_{32} - a_{33} - a_{38} - a_{39})\langle c_{26} c_{27}\rangle \\
&+ (-a_0 - a_5 - a_6 - a_{11} - a_{12} - a_{17} - a_{18} - a_{21} - a_{22} - a_{26} - a_{27} - a_{32} \\
&- a_{33} - a_{38} - a_{39})\langle c_{27} c_{28}\rangle + a_{25}\langle c_1 c_{25}\rangle + (-a_1 - a_{25})\langle c_1 c_4\rangle \\
&+ (-a_1 - a_4 - a_{25})\langle c_4 c_7\rangle + (-a_1 - a_4 - a_7 - a_{25})\langle c_7 c_{10}\rangle \\
&+ (-a_1 - a_4 - a_7 - a_{10} - a_{25})\langle c_{10} c_{13}\rangle \\
&+ (-a_1 - a_4 - a_7 - a_{10} - a_{13} - a_{25})\langle c_{13} c_{16}\rangle \\
&+ (-a_1 - a_4 - a_7 - a_{10} - a_{13} - a_{16} - a_{25})\langle c_{16} c_{40}\rangle \\
&+ (a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} + a_{25} + a_{40})\langle c_{37} c_{40}\rangle \\
&+ (a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} + a_{25} + a_{37} + a_{40})\langle c_{34} c_{37}\rangle \\
&+ (a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} + a_{25} + a_{34} + a_{37} + a_{40})\langle c_{31} c_{34}\rangle \\
&+ (a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} + a_{25} + a_{31} + a_{34} + a_{37} + a_{40})\langle c_{28} c_{31}\rangle \\
&+ (-a_{14})\langle c_{14} c_{15}\rangle + (-a_{14} - a_{15})\langle c_{15} c_{41}\rangle + (a_{14} + a_{15} + a_{41})\langle c_{36} c_{41}\rangle \\
&+ (a_{14} + a_{15} + a_{36} + a_{41})\langle c_{19} c_{36}\rangle + (a_{14} + a_{15} + a_{19} + a_{36} + a_{41})\langle c_9 c_{19}\rangle \\
&+ (a_9 + a_{14} + a_{15} + a_{19} + a_{36} + a_{41})\langle c_8 c_9\rangle \\
&+ (-a_8 - a_9 - a_{14} - a_{15} - a_{19} - a_{36} - a_{41})\langle c_8 c_{20}\rangle \\
&+ (-a_8 - a_9 - a_{14} - a_{15} - a_{19} - a_{20} - a_{36} - a_{41})\langle c_{20} c_{35}\rangle \\
&+ (a_8 + a_9 + a_{14} + a_{15} + a_{19} + a_{20} + a_{35} + a_{36} + a_{41})\langle c_{30} c_{35}\rangle \\
&+ (a_8 + a_9 + a_{14} + a_{15} + a_{19} + a_{20} + a_{30} + a_{35} + a_{36} + a_{41})\langle c_{23} c_{30}\rangle \\
&+ (a_8 + a_9 + a_{14} + a_{15} + a_{19} + a_{20} + a_{23} + a_{30} + a_{35} + a_{36} + a_{41})\langle c_3 c_{23}\rangle \\
&+ (a_3 + a_8 + a_9 + a_{14} + a_{15} + a_{19} + a_{20} + a_{23} + a_{30} + a_{35} + a_{36} + a_{41})\langle c_2 c_3\rangle \\
&+ (-a_2 - a_3 - a_8 - a_9 - a_{14} - a_{15} - a_{19} - a_{20} - a_{23} - a_{30} - a_{35} - a_{36} \\
&- a_{41})\langle c_2 c_{24}\rangle + (-a_2 - a_3 - a_8 - a_9 - a_{14} - a_{15} - a_{19} - a_{20} - a_{23} - a_{24} - a_{30} \\
&- a_{35} - a_{36} - a_{41})\langle c_{24} c_{29}\rangle + (a_2 + a_3 + a_8 + a_9 + a_{14} + a_{15} + a_{19} \\
&\quad + a_{20} + a_{23} + a_{24} + a_{29} + a_{30} + a_{35} + a_{36} + a_{41})\langle c_{28} c_{29}\rangle) + \sum_{i=0}^{41} a_i \langle c_{28}\rangle.
\end{aligned}
$$

So w_0 is homologous to 0-chain $\sum_{i=0}^{41} a_i\langle c_{28}\rangle$. Hence the 0-cycle is homologous to an integral multiple of $\langle c_{28}\rangle$. Thus we get $H_0^6(MSS_6\sharp MSS_6) \cong \mathbb{Z}$.

From Example 2.12, Theorem 2.11, and the above, we have

$$
\begin{aligned}
-38 = \chi(MSS_6\sharp MSS_6) =& rank\ H_0^6(MSS_6\sharp MSS_6) - rank\ H_1^6(MSS_6\sharp MSS_6) \\
=& 1 - rank\ H_1^6(MSS_6\sharp MSS_6).
\end{aligned}
$$

Therefore, $rank\ H_1^6(MSS_6\sharp MSS_6) = 39$. It follows from Theorem 2.9 that $H_1^6(MSS_6\sharp MSS_6) \cong \mathbb{Z}^{39}$. □

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