

Clean property in amalgamated algebras along an ideal

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Abstract

Let $f : A \rightarrow B$ be a ring homomorphism and let J be an ideal of B . In this paper, we give a characterization for the amalgamation of A with B along J with respect to f (denoted by $R \bowtie^f J$) to be (uniquely) clean.

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1. Introduction

Throughout this paper, all rings are commutative with identity. We denote respectively by $\text{Nilp}(A)$, $\text{Rad}(A)$, and $\text{Idem}(A)$ the ideal of all nilpotent elements of the ring A , Jacobson radical of A , and the set of all idempotent of A .

Let A and B be two rings with unity, let J be an ideal of B and let $f : A \rightarrow B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

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called *the amalgamation of A with B along J with respect to f* (introduced and studied by D'Anna, Finocchiaro, and Fontana in [10, 11]). This construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in [12, 13, 14]). Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation ([10, Examples 2.5 and 2.6]) and other classical constructions, such as the Nagata's idealization (cf. [17, page 2]), and the CPI extensions (in the sense of Boisen and Sheldon [6]) are strictly related to it ([10, Example 2.7 and Remark 2.8]).

On the other hand, the amalgamation $A \bowtie^f J$ is related to a construction proposed by Anderson in [1] and motivated by a classical construction due to Dorroh [9], concerning the embedding of a ring without identity in a ring with identity. An ample introduction on the genesis of the notion of amalgamation is given in [10, Section 2]. Also, the authors consider the iteration of the amalgamation process, giving some geometrical applications of it.

One of the key tools for studying $A \bowtie^f J$ is based on the fact that the amalgamation can be studied in the frame of pullback constructions [10, Section 4]. This point of view allows the authors in [10, 11] to provide an ample description of various properties of $A \bowtie^f J$, in connection with the properties of A , J and f . Namely, in [10], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. Moreover, in [11], they pursue the investigation on the structure of the rings of the form $A \bowtie^f J$, with particular attention to the prime spectrum, to the chain properties and to the Krull dimension.

Various authors have studied clean rings and related conditions (cf. [2, 3, 16]). Recall that a ring R is called (*uniquely*) *clean* if each element in R can be written (uniquely) as the sum of a unit and an idempotent. The concept of clean rings was introduced by Nicholson [18]. Examples of clean rings (uniquely clean rings) include all commutative von Neumann regular rings (Boolean rings), local rings (with residue field \mathbb{Z}_2). A basic property of clean rings is that any homomorphic image of a clean ring is again clean. This leads to the definition of a neat rings [15]. A ring R is called *neat* if every proper homomorphic image of R is clean.

In this paper, we give a characterization for $A \bowtie^f J$ to be (uniquely) clean.

2. Main Results

We begin with the following result:

2.1. Proposition. *Let $f : A \rightarrow B$ be a ring homomorphism and J an ideal of B .*

- (1) *If $A \bowtie^f J$ is a clean (resp., uniquely clean) ring then A is a clean (resp., uniquely clean) ring and $f(A) + J$ is a clean ring.*
- (2) *Assume that $\frac{f(A) + J}{J}$ is uniquely clean. Then $A \bowtie^f J$ is a clean ring if and only if A and $f(A) + J$ are clean rings.*

Proof. (1) The cases $J = (0)$ and $f^{-1}(J) = (0)$ follow easily from [10, Proposition 5.1 (3)]. Otherwise, by the same reference, the rings A and $f(A) + J$ are proper homomorphic images of $A \bowtie^f J$. Then, they are clean. Assume now that $A \bowtie^f J$ is uniquely clean and consider $u + e = u' + e'$ where $u, u' \in U(A)$ and $e, e' \in \text{Idem}(A)$. Then, $(u, f(u)) + (e, f(e)) = (u', f(u')) + (e', f(e'))$ and clearly $(u, f(u)), (u', f(u')) \in U(A \bowtie^f J)$ and $(e, f(e)), (e', f(e')) \in \text{Idem}(A \bowtie^f J)$. Then, $(u, f(u)) = (u', f(u'))$ and $(e, f(e)) = (e', f(e'))$. Hence, $u = u'$ and $e = e'$. Consequently, A is uniquely clean.

(2) If $A \bowtie^f J$ is a clean ring, then so A and $f(A) + J$ (by 1). Conversely, assume that A and $f(A) + J$ are clean rings and consider $(a, j) \in A \times J$. Since A is clean, we can write $a = u + e$ with $(u, e) \in U(A) \times \text{Idem}(A)$. On the other hand, since $f(A) + J$ is clean, $f(a) + j = f(x) + j_1 + f(y) + j_2$ with $f(x) + j_1$ and $f(y) + j_2$ are respectively unit and idempotent element of $f(A) + J$. It is clear that $\overline{f(x)} = \overline{f(x) + j_1}$ (resp. $\overline{f(y)} = \overline{f(y) + j_2}$) (resp. $\overline{f(e)} = \overline{f(e)}$) are respectively unit and idempotent element of $\frac{f(A) + J}{J}$, and we have $\overline{f(a)} = \overline{f(u)} + \overline{f(e)} = \overline{f(x)} + \overline{f(y)}$. Thus, $\overline{f(u)} = \overline{f(x)}$ and $\overline{f(e)} = \overline{f(y)}$ since $\frac{f(A) + J}{J}$ is uniquely clean. Consider $j'_1, j'_2 \in J$ such that $f(x) = f(u) + j'_1$ and $f(y) = f(e) + j'_2$. We have, $(a, f(a) + j) = (u, f(u) + j'_1 + j_1) + (e, f(e) + j'_2 + j_2)$, and it is clear that $(e, f(e) + j'_2 + j_2)$ is an idempotent element of $A \bowtie^f J$. Hence, we have only to prove that $(u, f(u) + j'_1 + j_1)$ is invertible in $A \bowtie^f J$. Since $f(u) + j'_1 + j_1$ is invertible in $f(A) + J$, there exists an element $f(\alpha) + j_0$ such that $(f(u) + j'_1 + j_1)(f(\alpha) + j_0) = 1$. Thus, $\overline{f(u)f(\alpha)} = \overline{1}$. Then, $\overline{f(\alpha)} = \overline{f(u^{-1})}$. So, there exists $j'_0 \in J$ such that $f(\alpha) = f(u^{-1}) + j'_0$. Hence, $(u, f(u) + j'_1 + j_1)(u^{-1}, f(u^{-1}) + j'_0 + j_0) = (u, f(u) + j'_1 + j_1)(u^{-1}, f(\alpha) + j_0) = (1, 1)$. Accordingly, $(u, f(u) + j'_1 + j_1)$ is invertible in $A \bowtie^f J$. Consequently, $A \bowtie^f J$ is clean.

2.2. Remarks. Let $f : A \rightarrow B$ be a ring homomorphism and J an ideal of B .

- (1) If $B = J$ then, $A \bowtie^f B$ is clean if and only if A and B are clean since $A \bowtie^f B = A \times B$.
- (2) If $f^{-1}(J) = \{0\}$ then, $A \bowtie^f J$ is clean if and only if $f(A) + J$ is clean (by [10, Proposition 5.1(3)]).

2.3. Corollary. Let A be a ring and I an ideal such that A/I is uniquely clean. Then, $A \bowtie I$ is clean if and only if A is clean.

Contrary to the previous proposition, in what follows we show that, if $J \subseteq \text{Rad}(B)$, the characterization for $A \bowtie^f J$ to be clean does not depend to the choice of f .

2.4. Theorem. Let $f : A \rightarrow B$ be a ring homomorphism and J an ideal of B such that $f(u) + j$ is invertible (in B) for each $u \in U(A)$ and $j \in J$. Then, $A \bowtie^f J$ is clean (resp., uniquely clean) if and only if A is clean (resp., uniquely clean).

More generally, if $J \cap \text{Idem}(B) = 0$ then, the following are equivalent:

- (1) $A \bowtie^f J$ is clean (resp., uniquely clean).
- (2) A is clean (resp., uniquely clean) and $J \subseteq \text{Rad}(B)$.

We need the following lemma.

2.5. Lemma. Let $f : A \rightarrow B$ be a ring homomorphism and let J be an ideal of B such that $J \cap \text{Idem}(B) = 0$. Then $\text{Idem}(A \bowtie^f J) = \{(e, f(e)) \mid e \in \text{Idem}(A)\}$.

Proof. Let $(e, f(e) + j)$ be an idempotent element of $A \bowtie^f J$. It is clear that e must be an idempotent element of A . On the other hand, $(f(e) + j)^2 = f(e) + j$. Thus, $j - j^2 = 2f(e)j$. Then, $f(e)(j - j^2) = 2f(e)^2j = 2f(e)j$. Therefore, $-f(e)j^2 = f(e)j$. Thus, $(f(e)j)^4 = (f(e)^2j^2)^2 = (f(e)j^2)^2 = (-f(e)j)^2 = (f(e)j)^2$. Consequently, $f(e)j^2 = (f(e)j)^2 \in J \cap \text{Idem}(B) = \{0\}$. Hence, $f(e)j = -f(e)j^2 = 0$. Thus, $j - j^2 = 2f(e)j = 0$. Then, $j \in J \cap \text{Idem}(B) = \{0\}$. Consequently, $j = 0$. Accordingly, $\text{Idem}(A \bowtie^f J) \subseteq \{(e, f(e)) \mid e \in \text{Idem}(A)\}$. The converse is clear.

Proof of Theorem 2.4. Note in first that if $f(u) + j$ is invertible (in B) for each $u \in U(A)$ and $j \in J$ then $J \cap \text{Idem}(B) = (0)$. Indeed, if $j \in J \cap \text{Idem}(B)$ then, $1 - j = -(1 + j) \in \text{Idem}(B) \cap U(B) = 1$. Thus, $j = 0$. Moreover, if $A \bowtie^f J$ is (uniquely) clean then so is A (by Proposition 2.1 (1)).

Assume that A is clean and $f(u) + j$ is invertible (in B) for each $u \in U(A)$ and $j \in J$. Consider $(a, j) \in A \times J$. Since A is clean, $a = u + e$ where u and e are

respectively unit and idempotent in A . Moreover, $f(u) + j$ is invertible in B . Then, there exists $v \in B$ such that $(f(u) + j)v = 1$. Hence, $(f(u) + j)(f(u^{-1}) - vf(u^{-1})j) = f(u)f(u^{-1}) + jf(u^{-1}) - (f(u) + j)vf(u^{-1})j = 1 + jf(u^{-1}) - f(u^{-1})j = 1$. Thus, $(u, f(u) + j)$ is invertible in $A \bowtie^f J$ (since $(u, f(u) + j)(u^{-1}, f(u^{-1}) - vf(u^{-1})j) = (1, 1)$). Hence, $(a, f(a) + j) = (u, f(u) + j) + (e, f(e))$ is the sum of a unit and an idempotent element in $A \bowtie^f J$. Consequently, $A \bowtie^f J$ is clean.

Assume moreover that A is uniquely clean. Since $J \cap \text{Idem}(B) = (0)$ (as we see in the first lines of the proof) and by lemma 2.5, $\text{Idem}(A \bowtie^f J) = \{(e, f(e)) \mid e \in \text{Idem}(A)\}$. Suppose now that we have $(u, f(u) + j) + (e, f(e)) = (u', f(u') + j') + (e', f(e'))$ where $(u, f(u) + j), (u', f(u') + j') \in U(A \bowtie^f J)$ and $e, e' \in \text{Idem}(A)$. Clearly, $u, u' \in U(A)$, and $u + e = u' + e'$. Since A is uniquely clean $u = u'$ and $e = e'$. Thus, $(e, f(e)) = (e', f(e'))$ and $(u, f(u) + j) = (u', f(u') + j')$. Consequently, $A \bowtie^f J$ is uniquely clean, as desired.

Assume that $J \cap \text{Idem}(B) = (0)$.

(1) \Rightarrow (2) We have only to prove that $J \subseteq \text{Rad}(B)$. Consider $j \in J$ and $x \in B$. Since $A \bowtie^f J$ is clean and by (1) above, $(0, xj) = (u, f(u) + xj) + (e, f(e))$ where $(u, f(u) + xj)$ is unit in $A \bowtie^f J$ and $e \in \text{Idem}(A)$. We have $0 = u + e$ and so $u = -1$ and $e = 1$. Therefore, $1 - xj = -(f(-1) + xj)$ is invertible in B and then $j \in \text{Rad}(B)$.

(2) \Rightarrow (1) Follows from above since for each $J \subseteq \text{Rad}(B)$, it is clear that $f(u) + j = f(u)(1 + f(u^{-1})j)$ is invertible in B .

2.6. Corollary. *Let $f : A \rightarrow B$ be a ring homomorphism and J an ideal of B such that $J \subseteq \text{Rad}(B)$. Then $A \bowtie^f J$ is clean (resp., uniquely clean) if and only if A is clean (resp., uniquely clean).*

Proof. Let $x \in J \cap \text{Idem}(B)$. Since $J \subseteq \text{Rad}(B)$, there exists a positive integer n such that $x^n = 0$. On the other hand, x is an idempotent element, and then $x = x^n = 0$. Thus, $J \cap \text{Idem}(B) = (0)$. Consequently, the result follows directly from Theorem 2.4.

2.7. Example. Let $A \subset B$ be an extension of commutative rings and $X := \{X_1, X_2, \dots, X_n\}$ a finite set of indeterminates over B . Set the subring $A + XB[[X]] := \{h \in B[[X]] \mid h(0) \in A\}$ of the ring of power series $B[[X]]$. Then, $A + XB[[X]]$ is clean if and only if A is clean.

Proof. By [10, Example 2.5], $A + XB[[X]]$ is isomorphic to $A \bowtie^\sigma J$, where $\sigma : A \hookrightarrow B[[X]]$ is the natural embedding and $J := XB[[X]]$. By [4, p. 11, Exercise 5], $\text{Rad}(B[[X]]) = \{g \in B[[X]] \mid g(0) \in \text{Rad}(A)\}$. Thus, $J \subseteq \text{Rad}(B[[X]])$. Hence, by Corollary 2.6, $A \bowtie^\sigma J$ is clean if and only if A is clean. Thus, we have the desired result.

2.8. Example. Let T be a ring and $J \subseteq \text{Rad}(T)$ an ideal of T and let D be a subring of T such that $J \cap D = (0)$. The ring $D + J$ is clean if and only if D is clean.

Proof. By [10, Proposition 5.1 (3)], $D + J$ is isomorphic to the ring $D \bowtie^\iota J$ where $\iota : D \hookrightarrow T$ is the natural embedding. Thus, by Corollary 2.6, $D + J$ is clean if and only if A is clean.

The next result shows that the characterization for $A \bowtie^f J$ to be (uniquely) clean can be reconducted to the case where A is a reduced ring and $J \cap \text{Nilp}(B) = (0)$.

2.9. Theorem. *Let $f : A \rightarrow B$ be a ring homomorphism and J an ideal of B . Set $\overline{A} = A/\text{Nilp}(A)$, $\overline{B} = B/\text{Nilp}(B)$, $\pi : B \rightarrow \overline{B}$ the canonical projection, and $\overline{J} = \pi(J)$. Consider the ring homomorphism $\overline{f} : \overline{A} \rightarrow \overline{B}$ defined by setting $\overline{f}(\overline{a}) = \overline{f(a)}$. Then, $A \bowtie^f J$ is clean (resp., uniquely clean) if and only if $\overline{A} \bowtie^{\overline{f}} \overline{J}$ is clean (resp., uniquely clean).*

To prove this theorem, we need the following lemma.

2.10. Lemma. *Let $f : A \rightarrow B$ be a ring homomorphism and J an ideal of B . Then,*

$$\text{Nilp}(A \bowtie^f J) = \{(a, f(a) + j) \mid a \in \text{Nilp}(A), j \in \text{Nilp}(B) \cap J\}.$$

Proof. Consider $(a, f(a) + j) \in \text{Nilp}(A \bowtie^f J)$. Then, there exists a positive integer n such that $(a, f(a) + j)^n = 0$. Thus, $a^n = 0$, and so $a \in \text{Nilp}(A)$ and $f(a) \in \text{Nilp}(B)$. On the other hand, $(f(a) + j)^n = 0$. Thus, $f(a) + j \in \text{Nilp}(B)$. Accordingly, $j \in \text{Nilp}(B)$ since $f(a) \in \text{Nilp}(B)$. Hence, $j \in \text{Nilp}(B) \cap J$.

Conversely, consider $a \in \text{Nilp}(A)$ and $j \in \text{Nilp}(B) \cap J$. It is clear that $f(a) \in \text{Nilp}(B)$. Then, $f(a) + j \in \text{Nilp}(B)$. Hence, $(a, f(a) + j)$ is a nilpotent element of $A \bowtie^f J$.

Proof of Theorem 2.9. It is easy to see that \bar{f} is well defined and it is clearly a ring homomorphism. Consider the map:

$$\begin{aligned} \psi : A \bowtie^f J / \text{Nilp}(A \bowtie^f J) &\rightarrow \overline{A} \bowtie^{\bar{f}} \overline{J} \\ \overline{(a, f(a) + j)} &\mapsto (\bar{a}, \bar{f}(\bar{a}) + \bar{j}) \end{aligned}$$

The map ψ is well defined. Indeed, if $\overline{(a, f(a) + j)} = \overline{(b, f(b) + j')}$ then, $(a - b, f(a - b) + j - j') \in \text{Nilp}(A \bowtie^f J)$. Hence, by Lemma 2.10, $a - b \in \text{Nilp}(A)$ and $j - j' \in \text{Nilp}(B)$. Then, $\bar{a} = \bar{b}$ and $\bar{j} = \bar{j}'$. It is also easy to check that ψ is a ring homomorphism. Moreover, $(\bar{a}, \bar{f}(\bar{a}) + \bar{j}) = (0, 0)$ implies that $a \in \text{Nilp}(A)$ and $j \in \text{Nilp}(B)$. Consequently, $(a, f(a) + j) \in \text{Nilp}(A \bowtie^f J)$, that is $\overline{(a, f(a) + j)} = (0, 0)$. Accordingly, ψ is injective. Clearly, ψ is surjective by construction. Thus, it is an isomorphism. Consequently, the desired result follows directly from [2, Theorems 9 and 23 (3)].

The next result is a consequence of the previous theorem and it is also a particular case of Corollary 2.6.

2.11. Corollary. *Let $f : A \rightarrow B$ be a ring homomorphism and J an ideal of B . If $J \subseteq \text{Nilp}(B)$ then, $A \bowtie^f J$ is clean (resp., uniquely clean) if and only if A is clean (resp., uniquely clean).*

Proof. With the notation of Theorem 2.9, $A \bowtie^f J$ is clean (resp., uniquely clean) if and only if $\overline{A} \bowtie^{\bar{f}} \overline{0}$ is clean (resp., uniquely clean) since $\overline{J} = (0)$. On the other hand, by [10, Proposition 5.1 (3)], $\overline{A} \bowtie^{\bar{f}} \overline{0} \cong \overline{A}$. Consequently, the desired result follows from [2, Theorems 9 and 23 (3)].

Let E be an A -module and set $B := A \times E$. Let $\iota : A \hookrightarrow B$ be the canonical embedding. After identifying E with $0 \times E$, E becomes an ideal of B . It is non straightforward but it is known that $A \times E$ coincides with $A \bowtie^{\iota} E$ (cf. [10, Remark 2.8]).

2.12. Corollary. *With the above notation, the ring $A \times E$ is clean (resp., uniquely clean) if and only if A is clean (resp., uniquely clean).*

Proof. This result follows immediately from Corollary 2.11 since $(0 \times E)^2 = (0)$.

The study of clean property over the ring $A \bowtie^f J$ allows us to provide a new proof of a characterization for $A \bowtie^f J$ to be local, already obtained in [10].

2.13. Theorem. *Let $f : A \rightarrow B$ be a ring homomorphism and J an ideal of B . Then, $A \bowtie^f J$ is a local ring if and only if A is a local ring and $J \subseteq \text{Rad}(B)$.*

Proof. Note in first that a commutative ring is local if and only if it is an indecomposable clean ring (that is a clean ring where $\{0, 1\}$ is the set of all idempotent elements), by [2, Theorem 3].

Assume that $A \bowtie^f J$ is a local ring. Then, $A \bowtie^f J$ is an indecomposable clean ring. Clearly, A must be clean. Moreover, if $e \in \text{Idem}(A)$ then $(e, f(e)) \in \text{Idem}(A \bowtie^f J) = \{(0, 0), (1, 1)\}$. Then, $\text{Idem}(A) = \{0, 1\}$. Accordingly, A is an indecomposable clean ring, and so local ring.

Consider $j \in J$ and $x \in B$. Since $A \bowtie^f J$ is clean and by above, $(0, xj) = (-1, -1 + xj) + (1, 1)$ where $(-1, -1 + xj)$ is unit in $A \bowtie^f J$ as $\text{Idem}(A \bowtie^f J) = \{(0, 0), (1, 1)\}$. Thus, $1 - xj = -(-1 + xj)$ is invertible in B . Thus, $j \in \text{Rad}(B)$.

Conversely, Assume that A is a local ring and $J \subseteq \text{Rad}(B)$. By Theorem 2.4, $A \bowtie^f J$ is a clean ring. On the other hand, by Lemma 2.5, $\text{Idem}(A \bowtie^f J) = \{(e, f(e)) \mid e \in \text{Idem}(A)\} = \{(0, 0), (1, 1)\}$. Thus, $A \bowtie^f J$ is an indecomposable clean ring. Consequently, $A \bowtie^f J$ is a local ring.

2.14. Example. Let $A \subset B$ be an extension of commutative rings and $X := \{X_1, X_2, \dots, X_n\}$ a finite set of indeterminates over B . Then, $A + XB[[X]]$ is local if and only if A is local.

Proof. By [10, Example 2.5], $A + XB[[X]]$ is isomorphic to $A \bowtie^\sigma J$, where $\sigma : A \hookrightarrow B[[X]]$ is the natural embedding and $J := XB[[X]]$. By [4, p. 11, Exercise 5], $\text{Rad}(B[[X]]) = \{g \in B[[X]] \mid g(0) \in \text{Rad}(A)\}$. Thus, $J \subseteq \text{Rad}(B[[X]])$. Hence, by Theorem 2.13, $A \bowtie^\sigma J$ is local if and only if A is local. Thus, we have the desired result.

2.15. Example. Let T be a ring, J an ideal of T , and D a subring of T such that $J \cap D = (0)$. The ring $D + J$ is local if and only if D is local and $J \subseteq \text{Rad}(T)$.

Proof. By [10, Proposition 5.1 (3)], $D + J$ is isomorphic to the ring $D \bowtie J$ where $\iota : D \hookrightarrow T$ is the natural embedding. Thus, by Theorem 2.13, $D + J$ is clean if and only if A is clean.

2.16. Corollary. Let $f : A \rightarrow B$ be a ring homomorphism and J an ideal of B . The following are equivalent:

- (1) $A \bowtie^f J$ is local and uniquely clean.
- (2) A is local, uniquely clean and $J \subseteq \text{Rad}(B)$.

In particular, if A is a ring and I an ideal of A then $A \bowtie I$ is local and uniquely clean if and only if A is local and uniquely clean.

Proof. (1) \Rightarrow (2) Follows from Proposition 2.1 (1) and Theorem 2.13.

(2) \Rightarrow (1) Follows from Theorem 2.13 and Corollary 2.6.

When the ideal J is generated by an idempotent element gives a different particular case since $J \cap \text{Idem}(B) \neq 0$. However, it allows a more easy study of the transfer of the clean property between A and $A \bowtie^f J$ more easily, with respect to f .

2.17. Proposition. Let $f : A \rightarrow B$ be a ring homomorphism and let (e) be an ideal of B generated by the idempotent element e . Then $A \bowtie^f (e)$ is clean if and only if A and $f(A) + (e)$ are clean.

In particular, if e is an idempotent element of A then, $A \bowtie (e)$ is clean if and only if A is clean.

Proof. From Proposition 2.1(1), we have only to show that $A \bowtie^f (e)$ is clean provided A and $f(A) + (e)$ are clean. Let $(a, f(a) + re)$ be an element of $A \bowtie^f (e)$ (with $a \in A$ and $r \in B$). Since A and $f(A) + (e)$ are clean, there exists u and v (resp. u' and v') in A (resp. $f(A) + (e)$) which are respectively unit and idempotent such that $a = u + v$ and

$f(a) + re = u' + v'$. We have

$$(a, f(a) + re) = (u, f(u) + (u' - f(u))e) + (v, f(v) + (v' - f(v))e)$$

On the other hand,

$$[f(u) + (u' - f(u))e][f(u^{-1}) + (u'^{-1} - f(u^{-1}))e] = [f(u)(1-e) + u'e][f(u^{-1})(1-e) + u'^{-1}e] = 1$$

and

$$\begin{aligned} [f(v) + (v' - f(v))e]^2 &= [f(v)(1-e) + v'e]^2 \\ &= f(v)(1-e) + v'e \\ &= f(v) + (v' - f(v))e \end{aligned}$$

Then, $(u, f(u) + (u' - f(u))e)$ and $(v, f(v) + (v' - f(v))e)$ are respectively unit and idempotent in $A \bowtie^f(e)$. Consequently, $A \bowtie^f(e)$ is clean, as desired.

Finally, if $A = B$ and $f = \text{id}_A$ then $A \bowtie^f(e) = A \bowtie(e)$ and $f(A) + (e) = A$. Thus, the particular case is a direct consequence of what is above.

2.18. Example. For each ring homomorphism $f : \mathbb{Z}/6\mathbb{Z} \rightarrow \mathbb{Z}/6\mathbb{Z}$, the ring $\mathbb{Z}/6\mathbb{Z} \bowtie^f(\bar{4})$ is clean and reduced since $\mathbb{Z}/6\mathbb{Z}$ is clean, reduced and $\bar{4}$ is an idempotent element of $\mathbb{Z}/6\mathbb{Z}$ and since $(\bar{4}) \cap \text{Nilp}(\mathbb{Z}/6\mathbb{Z}) = \{0\}$ (by Proposition 2.17 and [10, Proposition 5.4]).

Recall that a Boolean ring R is a ring (with identity) for which $x^2 = x$ for all x in R ; that is, R consists only of idempotent elements. It is proved that a ring R is Boolean if and only if it is uniquely clean with $\text{Rad}(R) = (0)$ if and only if R is clean with characteristic equal to 2 and 1 is the only unit in R ; cf. [19, Theorem 19].

The next result gives a characterization for $A \bowtie^f J$ to be a Boolean ring.

2.19. Proposition. *Let $f : A \rightarrow B$ be a ring homomorphism and let J be an ideal of B .*

- (1) *If $J \subseteq \text{Idem}(B)$ then, $A \bowtie^f J$ is clean if and only if A is clean.*
- (2) *The ring $A \bowtie^f J$ is Boolean if and only if A is Boolean and $J \subseteq \text{Idem}(B)$.*

Proof. Note first that if $J \subseteq \text{Idem}(B)$ then $2J = (0)$. Indeed, let $j \in J$. Clearly, $2j \in J \subseteq \text{Idem}(B)$. Then, $j + j = (j + j)^2 = j^2 + 2j^2 + j^2 = j + 2j + j$. Hence, $2j = 0$.

(1) Let $(a, f(a) + j)$ be an element of $A \bowtie^f J$ (with $a \in A$ and $j \in J$). We have $a = u + e$ where u and e are respectively unit and idempotent in A . We have $(f(e) + j)^2 = f(e)^2 + j^2 + 2f(e)j = f(e) + j$ since $2j = 0$. Hence, $(u, f(u))$ and $(e, f(e) + j)$ are respectively unit and idempotent in $A \bowtie^f J$, and we have $(a, f(a) + j) = (u, f(u)) + (e, f(e) + j)$. Consequently, $A \bowtie^f J$ is clean.

The converse implication is clear.

(2) If $A \bowtie^f J$ is Boolean, for each $a \in A$, $(a, f(a)) = (a, f(a))^2 = (a^2, f(a)^2)$. Then, $a = a^2$. Hence, A is Boolean. Moreover, for each $j \in J$, $(0, j) = (0, j)^2 = (0, j^2)$. Thus, $j = j^2$. Hence, $J \subseteq \text{Idem}(B)$.

Now, assume that A is Boolean and $J \subseteq \text{Idem}(B)$. We have just proved that $2J = (0)$. Hence, for each $a \in A$ and $j \in J$, $(a, f(a) + j)^2 = (a^2, f(a)^2 + j^2 + 2f(a)j) = (a, f(a) + j)$. Thus, $A \bowtie^f J$ is Boolean.

2.20. Remark. Given a ring R , there are two cases where $R = U(R) \cup \text{Idem}(R)$. Namely, a field or a Boolean ring; cf. [2, Theorem 14]. In the previous proposition, we provide a characterization of when $A \bowtie^f J$ is Boolean. On the other hand, it is easy to prove that $A \bowtie^f J$ is a field if and only if A is a field and $J \in \{(0), B\}$ where B is a field.

It is well known that von Neumann rings are particular cases of clean rings. The following result can be easily obtained from the characterization of the reduced property and from the evaluation of the dimension given in [9] and [10].

2.21. Proposition. *Let $f : A \rightarrow B$ be a ring homomorphism and let J be an ideal of B .*

- (1) If A and $f(A) + J$ are von Neumann regulars then so is $A \bowtie^f J$.
- (2) If $A \bowtie^f J$ is von Neumann regular then A is von Neumann regular and $J \cap \text{Nilp}(B) = (0)$ where the equivalence holds if f is surjective.

2.22. Corollary. *Let R be a commutative ring and let I be a proper ideal of R . Then R is a von Neumann regular ring if and only if $R \bowtie I$ is a von Neumann regular ring.*

A ring R is called neat if every proper homomorphic image of R is clean. For instance, any clean ring is neat but the converse is false (for example, the ring of integers is neat but not clean).

Now, we construct a class of rings such that the neat and clean properties coincides. Recall that a ring R is said to be indecomposable when the only idempotents of A are 0 and 1. Otherwise, the ring is called decomposable.

2.23. Proposition. *Let $f : A \rightarrow B$ be a ring homomorphism and let J be a proper ideal of B . Consider the following conditions:*

- (1) A is a decomposable ring.
- (2) $J \cap \text{Idem}(B) \neq (0)$.
- (3) $f(u) + j$ is invertible (in B) for each $u \in U(A)$ and $j \in J$.

If one of the three above conditions is satisfied then, $A \bowtie^f J$ is a clean if and only if $A \bowtie^f J$ is neat.

Proof. Assume that A is a decomposable ring, and consider $e \in \text{Idem}(A) \setminus \{0, 1\}$. Then, $(e, f(e)) \in \text{Idem}(A \bowtie^f J) \setminus \{(0, 0), (1, 1)\}$.

Also, if $j \in J \cap \text{Idem}(B) \neq (0)$, then, $(0, j) \in \text{Idem}(A \bowtie^f J) \setminus \{(0, 0), (1, 1)\}$. Hence, if one of the conditions (1) or (2) is satisfied, the ring $A \bowtie^f J$ is decomposable. Thus, by [15, Proposition 2.3], $A \bowtie^f J$ is clean if and only if it is neat.

Suppose that (3) is satisfied and that $A \bowtie^f J$ is neat. By the definition of neat rings, $A \cong A \bowtie^f J / (\{0\} \times J)$ is clean. Thus, by Theorem 2.4, $A \bowtie^f J$ is clean. The opposite implication is clear.

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