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Continuous dependence of solutions to fourth-order nonlinear wave equation

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Abstract

This paper gives a priori estimates and continuous dependence of the solutions to fourth-order nonlinear wave equation.

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1. Introduction

We consider the following initial boundary value problem

(1.1) $u_{tt} - \alpha\Delta u - \beta\Delta u_t - \gamma\Delta u_{tt} = f(u)$

(1.2) $u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x),\ x \in \Omega$

(1.3) $u = 0,\ x \in \partial\Omega, t > 0,$

where $\Omega \subset \mathbb{R}^n$ is bounded region with smooth boundary $\partial\Omega$; α,β and γ are positive constants. $f(u)$ is a given nonlinear function which satisfies

(1.4) $f \in C^1(R), \left|f'(u)\right| \leq c\left(1 + |u|^{p-1}\right), p \geq 1, (n-2)p \leq n$

and

(1.5) $\limsup\limits_{u\to\infty} \dfrac{f(u)}{u} < \alpha\lambda_1$

where λ_1 is the first eigenvalue of the Laplace operator with the homogeneous Dirichlet boundary condition.

Continuous dependence of solutions on coefficients of equations is a type of structural stability, which reflects the effect of small changes in coefficients of equations on the solutions. Many results of this type can be found in [1].

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In [2], authors studied asymptotic behaviour of solution to initial value problem of fourth order wave equation with dispersive and dissipative terms by taking coefficients $\alpha = \beta = \gamma = 1$ in (1). They proved that the global strong solution of the problem decays to zero exponentially as $t \to \infty$. The authors Guo-wang Chen and Chang-Shun Hou, in article [3], studied the following initial value problem for a class of fourth order nonlinear wave equations,

$$v_{tt} - a_1 v_{xx} - a_2 v_{xxt} - a_3 v_{xxtt} = f(v_x)_x \quad , x \in R, t > 0$$

$$v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),\ x \in R$$

where a_1, a_2, a_3 are positive constants. They gave also the blow up results for this problem.

In [4], Shang studied the initial boundary value problem

(1') $$u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt} = f(u), x \in \Omega, t > 0$$

(2') $$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x),\ x \in \Omega$$

(3') $$u = 0,\ x \in \partial\Omega, t > 0,$$

Under the assumptions that $n = 1,2,3; f \in C^1, f'(u)$ is bounded above and satisfies (i) $|f'(u)| \le A|u|^p + B, 0 < p < \infty$ if $n = 2$; $0 < p \le \frac{2}{n-2}$ if $n = 3; u_i(x) \in H^2(\Omega) \cap H_0^1(\Omega)\,(i = 0,1)$, it was proven that problem (1')-(3') admits unique global strong solution u such that $\forall T > 0, u \in W^{2,\infty}\left(0,T; H^2(\Omega) \cap H_0^1(\Omega)\right)$.

In [5], problem (1')-(3') were studied again for all $n \ge 1$. By supposing that $f \in C^1$ and $f'(u)$ is bounded above satisfying (ii)$|f'(u)| \le A|u|^p + B, 0 < p < \infty$ if $n = 2$; $0 < p \le \frac{4}{n-2}$ if $n \ge 3, u_i(x) \in H^2(\Omega) \cap H_0^1(\Omega)\,(i = 0,1)$, it was proven that problem (1')-(3') admits unique global strong solution u such that for all $T > 0, u \in W^{2,\infty}\left(0,T; H^2(\Omega) \cap H_0^1(\Omega)\right)$.

In [6], authors studied the spatial behavior of a coupled system of wave-plate type . They got the alternative results of Phragmen-Lindelof type in terms of an area measure of the amplitude in question based on a first-order differential inequality. They also got the spatial decay estimates based on a second-order differential inequality.

The aim of this paper is to prove the continuous dependence of solutions to the problem (1)-(3) on coefficients α, β and γ.

Throughout this paper , we use the notation $\|.\|_p$ for the norm in $L^P(\Omega)$. We use $\|.\|$ instead of $\|.\|_2$.

2. A Priori Estimates

In this section, we obtain a priori estimates for the problem (1)-(3).

2.1. Theorem. *Assume that the conditions (4) and (5) hold. Then for $u_0, u_1 \in H_0^1(\Omega)$ the solution u of problem (1)-(3) satisfies the following estimates:*

(2.1) $$\|\nabla u\|^2 + \|\nabla u_t\|^2 \le D_1$$

and

(2.2) $$\int_0^t \|\nabla u_{ss}\|^2 \, ds \le D_2 t$$

for any $t > 0$. Here $D_1 > 0$ and $D_2 > 0$ depend on initial data and the parameters of (1).

Proof. First, by taking the inner product of (1) by u_t in $L^2(\Omega)$ and integrating by parts, we get

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2+\frac{\alpha}{2}\|\nabla u\|^2+\frac{\gamma}{2}\|\nabla u_t\|^2-\int_\Omega F(u)dx\right]+\beta\|\nabla u_t\|^2=0 \tag{2.3}$$

and

$$E(t)\le E(0) \tag{2.4}$$

where $F(u)=\int_0^u f(s)\,ds$ and $E(t)=\frac{1}{2}\|u_t\|^2+\frac{\alpha}{2}\|\nabla u\|^2+\frac{\gamma}{2}\|\nabla u_t\|^2-\int_\Omega F(u)dx$. From (5) and definition of limsup we obtain

$$F(u)\le c+\frac{\alpha\lambda_1}{2}u^2-\frac{\varepsilon}{2}u^2 \tag{2.5}$$

Using (10) and Poincare's inequality from (9) we find (6).
Next we multiply (1) by u_{tt} in $L^2(\Omega)$ to get

$$\frac{d}{dt}\frac{\beta}{2}\|\nabla u_t\|^2+\gamma\|\nabla u_{tt}\|^2+\|u_{tt}\|^2+\alpha\int_\Omega\nabla u\nabla u_{tt}dx=\int_\Omega f(u)u_{tt}dx \tag{2.6}$$

Using Cauchy-Schwarz inequality, ε-Cauchy inequality and from (4), we take,

$$(\gamma\ -\frac{\varepsilon}{2})\|\nabla u_{tt}\|^2+\frac{d}{dt}\frac{\beta}{2}\|\nabla u_t\|^2\le c_2+\frac{|\alpha|^2}{2\varepsilon}\|\nabla u\|^2+\frac{c_1^2}{2}\int_\Omega|u|^{2p}\,dx \tag{2.7}$$

where c_1, c_2 are constants and ε is sufficiently small and positive. Using Sobolev inequality and (6) we have

$$\int_\Omega|u|^{2p}\,dx=\|u\|_{2p}^{2p}\le c_3\|\nabla u\|^{2p}\le c_4 \tag{2.8}$$

where c_3 is a Sobolev constant and $c_4=c_4(\alpha,\gamma,u_0,u_1)$. From (12) and (13) we obtain

$$(\gamma\ -\frac{\varepsilon}{2})\|\nabla u_{tt}\|^2+\frac{d}{dt}\frac{\beta}{2}\|\nabla u_t\|^2\le c_5 \tag{2.9}$$

where c_5 depends on initial data and the parameters of (1). Now, we integrate (14) from (0,t), then we obtain

$$\int_0^t\|\nabla u_{ss}\|^2\,ds\le c_6 t \tag{2.10}$$

where c_6 depends on initial data and the parameters of (1). Hence, (7) follows from (15).

3. Continuous Dependence on the Coefficients

In this section, we prove that the solution of the problem (1)-(3) depends continuously on the coefficients α,β and γ in $H^1(\Omega)$.

We consider the problem

$$u_{tt}-\alpha_1\Delta u-\beta_1\Delta u_t-\gamma_1\Delta u_{tt}=f(u) \tag{3.1}$$

$$u(x,0)=0,u_t(x,0)=0 \tag{3.2}$$

$$u|_{\partial\Omega}=0 \tag{3.3}$$

and

$$v_{tt}-\alpha_2\Delta v-\beta_2\Delta v_t-\gamma_2\Delta v_{tt}=f(v) \tag{3.4}$$

(3.5) $\quad v(x,0)=0, v_t(x,0)=0$

(3.6) $\quad v|_{\partial\Omega}=0$

Let us define the difference variables w, α, β and γ by w=u-v, $\alpha=\alpha_1-\alpha_2$, $\beta=\beta_1-\beta_2$ and $\gamma=\gamma_1-\gamma_2$ then w satisfy the following the initial boundary value problem:

(3.7) $\quad w_{tt}-\alpha_1\Delta w-\alpha\Delta v-\beta_1\Delta w_t-\beta\Delta v_t-\gamma_1\Delta w_{tt}-\gamma\Delta v_{tt}=f(u)-f(v)$

(3.8) $\quad w(x,0)=0, w_t(x,0)=0$

(3.9) $\quad w|_{\partial\Omega}=0$

The main result of this section is the following theorem.

3.1. Theorem. *Let w be the solution of the problem (22)-(24). If*

(3.10) $\quad |f(u)-f(v)|\le c_7\left(1+|u|^{p-1}+|v|^{p-1}\right)|u-v|$

holds, then w satisfies the estimate

$$\|w_t\|^2+\|\nabla w\|^2+\|\nabla w_t\|^2\le e^{Mt}K\left[(\alpha_1-\alpha_2)^2+(\beta_1-\beta_2)^2+(\gamma_1-\gamma_2)^2\right]t$$

where M and K are positive constants depending on initial data and the parameters of (1).

Proof. Let us take the inner product of (22) with w_t in $L^2(\Omega)$; we have

$$\frac{d}{dt}\left[\frac{1}{2}\|w_t\|^2+\frac{\alpha_1}{2}\|\nabla w\|^2+\frac{\gamma_1}{2}\|\nabla w_t\|^2\right]+\beta_1\|\nabla w_t\|^2+$$

(3.11) $$\alpha\int_\Omega\nabla v\nabla w_t dx+\beta\int_\Omega\nabla v_t\nabla w_t dx+\gamma\int_\Omega\nabla v_{tt}\nabla w_t dx=\int_\Omega|f(u)-f(v)|\,w_t dx$$

From (26) we obtain

$$\frac{d}{dt}E_1(t)+\beta_1\|\nabla w_t\|^2\le|\alpha|\,\|\nabla w_t\|\,\|\nabla v\|+|\beta|\,\|\nabla w_t\|\,\|\nabla v_t\|+$$

(3.12) $$|\gamma|\,\|\nabla w_t\|\,\|\nabla v_{tt}\|+\int_\Omega|f(u)-f(v)|\,w_t dx$$

where $E_1(t)=\frac{1}{2}\|w_t\|^2+\frac{\alpha_1}{2}\|\nabla w\|^2+\frac{\gamma_1}{2}\|\nabla w_t\|^2$.
Using the Holder, Sobolev, Cauchy-Schwarz inequalities and (25) we obtain the estimate

$$\int_\Omega|f(u)-f(v)|\,w_t dx\le c_7\int_\Omega\left(1+|u|^{p-1}+|v|^{p-1}\right)|w|\,w_t dx$$

$$\le c_8\left(1+\|\nabla u\|^{p-1}+\|\nabla v\|^{p-1}\right)\|w\|_{\frac{2n}{n-2}}\|w_t\|$$

(3.13) $$\le C\left(\|\nabla w\|^2+\|w_t\|^2\right)$$

where c_7, c_8 are constants and $C=C(c_7,c_8)$.Using Cauchy-Schwarz inequality and (28), from (27), we get

$$\frac{d}{dt}E_1(t)+(\beta_1-\varepsilon)\|\nabla w_t\|^2\le\frac{3}{4\varepsilon}|\alpha|^2\|\nabla v\|^2+\frac{3}{4\varepsilon}|\beta|^2\|\nabla v_t\|^2+$$

(3.14) $$\frac{3}{4\varepsilon}|\gamma|^2\|\nabla v_{tt}\|^2+c_9\left(\|\nabla w\|^2+\|w_t\|^2\right)$$

and from (29) we can write

(3.15) $$\frac{d}{dt}E_1(t)\le\frac{3}{4\varepsilon}\left(|\alpha|^2\|\nabla v\|^2+|\beta|^2\|\nabla v_t\|^2+|\gamma|^2\|\nabla v_{tt}\|^2\right)+ME_1(t)$$

where $M=\frac{2C(1+\alpha_1)}{\alpha_1}$. Applying Gronwall's inequality with (6) and (7), we get

(3.16) $$E_1(t)\le e^{Mt}K\left(|\alpha|^2+|\beta|^2+|\gamma|^2\right)t$$

Hence proof is completed.

References

[1] K.A. Ames, B. Straughan, *Non-Standard and Improperly Posed Problems*, in: Mathematics in science and Engineering series, vol.194, Academic Press, San Diego, 1997.

[2] Xu Run-zhang ,Zhao Xi-ren, Shen Ji-hong, *Asymptotic behaviour of solution for fourth order wave equation with dispersive and dissipative terms*, Appl. Math. Mech.-Engl. Ed., 29(2), 259-262, 2008.

[3] Guo-wang Chen, Chang-shun Hou, *Initial value problem for a class of fourth-order nonlinear wave equations*, Appl. Math. Mech.-Engl. Ed. 30(3), 391-401, 2009.

[4] Shang Yadong, *Initial boundary value problem of equation* $u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt} = f(u)$, Acta Mathematicae Applicate Sinica 23(3), 385-393, 2000.

[5] Liu Yacheng, Li Xiaoyuan, *Some remarks on the equation* $u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt} = f(u)$, Journal of Natural Science of Heilongjiang University 21(3),1-6, 2004.

[6] Gusheng Tang, yan Liu and Wenhui Liao, *Spatial Behavior of a Coupled System of Wave-Plate Type*, Abstract and Applied Analysis, Volume 2014, Article ID 853693, 13 pages.