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On a conjecture of double inequality for the tangent function

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Abstract

In this paper, we prove the conjecture posed by J.-L. Zhao, Q.-M. Luo, B.-N. Guo and F. Qi (Remarks on inequalities for the tangent function, Hacettepe J. Math. Stat., 41, no. 4, 499-506, 2012) about a sharp double inequality of the tangent function, which is a generalization of the Becker-Stark inequality. Also, the new double inequality is compared with the double inequality presented in [3]

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1. Introduction

In 1955, Stečkin [14] obtained the following one-side inequality for the tangent function

$$\frac{\tan x}{x} > \frac{4/\pi}{\pi - 2x}, \qquad 0 < x < \pi/2 \tag{1.1}$$

where the constant $4/\pi$ is the best possible.

Later in 1978, Becker and Stark [2] presented the following double inequality

$$\frac{8}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2}{\pi^2 - 4x^2}, \qquad 0 < x < \pi/2 \tag{1.2}$$

which is a generalization of the Stečkin's inequality (1.1).

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In 2003, C.-P. Chen and F. Qi [3] established a double inequality for remainder $r_n(x) = \tan x - S_n(x)$, where $S_n(x)$ is the n^{th} partial sum of the power series of $\tan x$. Their double inequality can be reformulated as [16]:

1.1. Theorem. *For $0 < x < \pi/2$ and $n \in \mathbb{N}$, we have*

$$\frac{2^{2(n+1)}(2^{2(n+1)}-1)|B_{2n+2}|}{(2n+2)!}x^{2n}\tan x < \tan x - S_n(x) < \left(\frac{2}{\pi}\right)^{2n} x^{2n}\tan x, \tag{1.3}$$

where

$$S_n(x) = \sum_{i=1}^{n} \frac{2^{2i}(2^{2i}-1)|B_i|}{(2i)!}x^{2i-1} \tag{1.4}$$

and B_j's are the Bernoulli numbers.

The inequality (1.3) for $n = 1$ and $0 < x < \frac{3}{2}\sqrt{\frac{5(\pi^2-8)}{38}}$ will give us a refinement of the left-hand side of the Becker-Stark inequality (1.2). Also, the inequality (1.3) for $n = 2$ is better than the Djokovic inequality [8]

$$x + \frac{1}{3}x^3 < \tan x < x + \frac{4}{9}x^3, \qquad 0 < x < \pi/6.$$

In 2010, Zhu and Hua [17] established the following general refinement of the Becker-Stark inequality

1.2. Theorem. *Let $0 < x < \pi/2$ and a natural number $n \geq 0$. Then*

$$\frac{P_{2n}(x) + \alpha_n x^{2n+2}}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{P_{2n}(x) + \beta_n x^{2n+2}}{\pi^2 - 4x^2}, \tag{1.5}$$

where $P_{2n}(x) = \sum_{i=0}^{n} a_i x^{2i}$ and

$$a_i = \pi^2 \mid B_{2i+2} \mid \frac{2^{2(i+1)}(2^{2(i+1)}-1)}{(2i+2)!} - 4 \mid B_{2i} \mid \frac{2^{2i}(2^{2i}-1)}{(2i)!}, \quad i = 0, 1, 2, \ldots .$$

Furthermore, $\alpha_n = \frac{8 - P_{2n}(\pi/2)}{(\pi/2)^{2n+2}}$ and $\beta_n = \alpha_{n+1}$ are the best constants in (1.5).

In 2012, Zhao, Luo, Guo and Qi [16] showed that the double inequalities (1.3) and (1.5) are not included in each other, reorganized the proof of (1.3) by using the usual definition of Bernoulli numbers and corrected some errors on [12]. Moreover, they propose a sharp double inequality as a conjecture. In this paper we will prove this conjecture. Further interesting generalizations and applications about inequalities of the tangent function can be found in [4]-[6], [9], [10], [15], [18]-[20] and the references therein.

In our present investigation, we will apply the following monotone form of L'Hôpital's rule [1] (see also, [7], [11], [13]).

1.3. Theorem. *Let $-\infty < a < b < \infty$, and let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, differentiable on (a, b). Let $g'(x) \neq 0$ on (a, b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b), then so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}. \tag{1.6}$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

2. Main Results.

Consider the following two functions for $n \in \mathbb{N}$ and $x \in \left(0, \frac{\pi}{2}\right)$

$$M_n(x) = \frac{\tan x}{S_n(x)} \left[1 - \left(\frac{2x}{\pi}\right)^{2n}\right] \tag{2.1}$$

and

$$h_n(x) = \frac{\tan x - S_n(x)}{x^{2n} \tan x}. \tag{2.2}$$

Then

$$\frac{1}{M_n(x)} = \frac{1 - x^{2n} h_n(x)}{1 - (2/\pi)^{2n} x^{2n}}. \tag{2.3}$$

Let

$$f_n(x) = \begin{cases} 1 - x^{2n} h_n(x) & 0 < x \leq \pi/2, \\ 1 & x = 0 \end{cases}$$

and

$$g_n(x) = 1 - (2/\pi)^{2n} x^{2n}, \quad 0 \leq x \leq \pi/2.$$

The functions $f_n, g_n : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ are continuous on $\left[0, \frac{\pi}{2}\right]$, differentiable on $\left(0, \frac{\pi}{2}\right)$. Also, $g_n'(x) = -2n(2/\pi)^{2n} x^{2n-1} \neq 0$ on $\left(0, \frac{\pi}{2}\right)$. Now, consider the function

$$G_n(x) = \frac{f_n'(x)}{g_n'(x)} = (\pi/2)^{2n} \left[\frac{x}{2n} h_n'(x) + h_n(x)\right],$$

then

$$G_n'(x) = \frac{(\pi/2)^{2n}}{2n} \left[(2n+1) h_n'(x) + x h_n''(x)\right].$$

But the function $h_n(x)$ is absolutely monotonic on $\left(0, \frac{\pi}{2}\right)$ [16], that is

$$(h_n(x))^{(i)} \geq 0, \quad \forall\, i \in \mathbb{N};\ x \in \left(0, \frac{\pi}{2}\right).$$

Then

$$G_n'(x) > 0, \qquad x \in \left(0, \frac{\pi}{2}\right)$$

and hence the function $G_n = \frac{f_n'}{g_n'}$ is increasing function on $\left(0, \frac{\pi}{2}\right)$. Using Theorem 1.3, we get that

$$\frac{f_n(x) - f_n(\pi/2)}{g_n(x) - g_n(\pi/2)}$$

is also increasing on $\left(0, \frac{\pi}{2}\right)$. But $f_n(\pi/2) = g_n(\pi/2) = 0$ and hence $\frac{1}{M_n(x)} = \frac{f_n(x)}{g_n(x)}$ is increasing on $\left(0, \frac{\pi}{2}\right)$. Then $M_n(x)$ is decreasing on $\left(0, \frac{\pi}{2}\right)$. Then

$$\lim_{x \to \frac{\pi}{2}^-} M_n(x) < M_n(x) < \lim_{x \to 0^+} M_n(x).$$

Using

$$\lim_{x \to 0^+} M_n(x) = \lim_{x \to 0^+} \frac{\tan x}{S_n(x)} = \lim_{x \to 0^+} \frac{\tan x / x}{1 + \sum_{i=2}^{n} \frac{2^{2i}(2^{2i}-1)|B_i|}{(2i)!} x^{2(i-1)}} = 1$$

and

$$\lim_{x \to \frac{\pi}{2}^-} M_n(x) = \frac{1}{S_n(\pi/2)} \lim_{x \to \frac{\pi}{2}^-} \frac{1 - \left(\frac{2x}{\pi}\right)^{2n}}{\cot x} = \frac{4n/\pi}{S_n(\pi/2)},$$

we get the following result

2.1. Theorem ([16], Conjecture 1)**.** *For $0 < x < \pi/2$ and $n \in \mathbb{N}$, we have*

$$\frac{4n/\pi}{S_n(\pi/2)} < \frac{\tan x}{S_n(x)}\left[1 - \left(\frac{2x}{\pi}\right)^{2n}\right] < 1. \tag{2.4}$$

Furthermore, 1 and $\frac{4n/\pi}{S_n(\pi/2)}$ are the best possible constants in (2.4).

2.2. Remark. If we set $n = 1$ in the inequality (2.4), then we obtain the inequality (1.2) and hence the inequality (2.4) is an extension of Becker-Stark inequality (1.2).

Now we will study the concavity of the function $M_n(x)$. Let us recall that, a function φ is concave if every chord lies below the graph of φ. Let $y_n(x)$ be the line segment with the endpoints $(0, 1)$ and $\left(\frac{\pi}{2}, \frac{4n/\pi}{S_n(\pi/2)}\right)$. Then

$$y_n(x) = \left[\frac{4n/\pi}{S_n(\pi/2)} - 1\right]\frac{2x}{\pi} + 1, \qquad 0 \le x \le \pi/2$$

and let

$$H_n(x) = \begin{cases} 1 & x = 0, \\ M_n(x) & 0 < x < \pi/2, \\ \frac{4n/\pi}{S_n(\pi/2)} & x = \pi/2. \end{cases}$$

The functions $H_n, y_n : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ are continuous on $\left[0, \frac{\pi}{2}\right]$, differentiable on $\left(0, \frac{\pi}{2}\right)$. Also, $y_n'(x) = \left[\frac{4n/\pi}{S_n(\pi/2)} - 1\right] \neq 0\ \forall x$. Now consider the function

$$T_n(x) = \frac{H_n'(x)}{y_n'(x)} = \frac{\pi}{2\left(\frac{4n/\pi}{S_n(\pi/2)} - 1\right)}H_n'(x).$$

If we assume that $H_n''(x) > 0$, then we get $T_n(x)$ is decreasing function. Using Theorem 1.3, we get that

$$F(x) = \frac{H_n(x) - H_n(0)}{y_n(x) - y_n(0)} = \frac{H_n(x) - 1}{y_n(x) - 1}$$

is also decreasing function on $\left(0, \frac{\pi}{2}\right)$. But

$$\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} \frac{\pi}{2\left(\frac{4n/\pi}{S_n(\pi/2)} - 1\right)}H_n'(x) = 0$$

and

$$\lim_{x\to \frac{\pi}{2}^-} F(x) = \frac{H_n(\pi/2) - 1}{y_n(\pi/2) - 1} = 1,$$

which is a contradiction since $F(x)$ is decreasing. Then we get the following result

2.3. Lemma ([16], Conjecture 1)**.** *For $0 < x < \pi/2$ and $n \in \mathbb{N}$, the function*

$$M_n(x) = \frac{\tan x}{S_n(x)}\left[1 - \left(\frac{2x}{\pi}\right)^{2n}\right] \tag{2.5}$$

is concave.

3. Comparison of Theorems 1.1 and 2.1.

The inequalities in Theorems 1.1 and 1.2 are not included in each other [16]. Now, we will compare the Theorems 1.1 and 2.1 .

The inequality (1.3) can be rewritten in the form

$$\frac{1}{1-\left(\frac{2^{2(n+1)}(2^{2(n+1)}-1)|B_{2(n+1)}|}{(2n+2)!}\right)x^{2n}} < \frac{\tan x}{S_n(x)} < \frac{1}{1-\left(\frac{2}{\pi}\right)^{2n}x^{2n}} \tag{3.1}$$

and the inequality (2.4) can be rewritten in the form

$$\frac{4n}{\pi S_n(\pi/2)\left(1-\left(\frac{2}{\pi}\right)^{2n}x^{2n}\right)} < \frac{\tan x}{S_n(x)} < \frac{1}{1-\left(\frac{2}{\pi}\right)^{2n}x^{2n}}. \tag{3.2}$$

So, the two inequalities (3.1) and (3.2) have the same upper bound. To compare the lower bounds of (3.1) and (3.2), take $n = 1, 2, 3$ in the left-hand side of (3.1), to obtain

$$\begin{aligned} n &= 1 \qquad L_1(x) = \tfrac{1}{1-\frac{x^2}{3}} \\ n &= 2 \qquad L_2(x) = \tfrac{1}{1-\frac{2x^4}{15}} \\ n &= 3 \qquad L_3(x) = \tfrac{1}{1-\frac{17x^6}{315}} \end{aligned}$$

and set $n = 1, 2, 3$ in the left-hand side of (3.2), to get

$$\begin{aligned} n &= 1 \qquad K_1(x) = \tfrac{8}{\pi^2(1-\frac{4x^2}{\pi^2})} \\ n &= 2 \qquad K_2(x) = \tfrac{8}{\pi(\pi/2+\pi^3/24)\left(1-\frac{16x^4}{\pi^4}\right)} \\ n &= 3 \qquad K_3(x) = \tfrac{12}{\pi(\pi/2+\pi^3/24+\pi^5/240)\left(1-\frac{64x^6}{\pi^6}\right)}. \end{aligned}$$

Then

$$\begin{aligned} &L_1(x) > K_1(x) && if\ 0 < x < \tfrac{1}{2}\sqrt{3(\pi^2-8)} \\ &L_1(x) < K_1(x) && if\ \tfrac{1}{2}\sqrt{3(\pi^2-8)} < x < \pi/2 \\ &L_2(x) > K_2(x) && if\ 0 < x < \tfrac{1}{2}\sqrt[4]{\tfrac{5(-192\pi^2+12\pi^4+\pi^6)}{3(20-\pi^2)}} \\ &L_2(x) < K_2(x) && if\ \tfrac{1}{2}\sqrt[4]{\tfrac{5(-192\pi^2+12\pi^4+\pi^6)}{3(20-\pi^2)}} < x < \pi/2 \\ &L_3(x) > K_3(x) && if\ 0 < x < \tfrac{1}{2}\sqrt[6]{\tfrac{7(-2880\pi^4+120\pi^6+10\pi^8+\pi^{10})}{10(84+7\pi^2-\pi^4)}} \\ &L_3(x) < K_3(x) && if\ \tfrac{1}{2}\sqrt[6]{\tfrac{7(-2880\pi^4+120\pi^6+10\pi^8+\pi^{10})}{10(84+7\pi^2-\pi^4)}} < x < \pi/2. \end{aligned}$$

Hence, the lower bounds of (3.1) and (3.2) are not included in each other. Also, we can conclude that inequality (3.1) is better than inequality (3.2) near the origin and that inequality (3.2) is better than inequality (3.1) near $\pi/2$.

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