Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 59 – 74

Some relationships between intrinsic and extrinsic invariants of submanifolds in generalized S-space-forms

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Abstract

We establish some inequalities of Chen's type between certain intrinsic invariants (involving sectional, Ricci and scalar curvatures) and the squared mean curvature of submanifolds tangent to the structure vector fields of a generalized S-space-form and we discuss the equality cases of them. We apply the obtained results to slant submanifolds.

2000 AMS Classification: Primary 53C25, Secondary 53C40

Keywords: Generalized S-space-form, intrinsic invariant, slant submanifold.

Received 13/12/2011 : Accepted 14/12/2013 Doi : 10.15672/HJMS.201544993

1. Introduction

Intrinsic and extrinsic invariants are very powerful tools to study submanifolds of Riemannian manifolds and to establish relationships between them is one of the most fundamental problems in submanifolds theory. In this context, B.-Y. Chen [10, 11, 12] proved some basic inequalities for submanifolds of a real space-form. Corresponding inequalities have been obtained for different kinds of submanifolds (invariant, anti-invariant, slant) in ambient manifolds endowed with different kinds of structures (mainly, real, complex and Sasakian space-forms).

Moreover, it is well known that the sectional curvatures of a Riemannian manifold determine the curvature tensor field completely. So, if (M, g) is a connected Riemannian manifold with dimension greater than 2 and its curvature tensor field R has the pointwise expression

$$R(X,Y)Z = \lambda \{g(X,Z)Y - g(Y,Z)X\},$$

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where λ is a differentiable function on M, then M is a space of constant sectional curvature, that is, a real-space-form and λ is a constant function.

Further, when the manifold is equipped with some additional structure, it is sometimes possible to obtain conclusions from a special form of the curvature tensor field for this structure too. Thus, for almost-Hermitian manifolds, F. Tricerri and L. Vanhecke [25] introduced *generalized complex-space-forms* and, for almost contact metric manifolds, P.Alegre, D.E. Blair and A. Carriazo [1] defined and studied *generalized Sasakian-space-forms*, which generalize complex and Sasakian space-forms, respectively.

More in general, K. Yano [27] introduced the notion of f-structure on a $(2m+s)$-dimensional manifold as a tensor field f of type (1,1) and rank $2m$ satisfying $f^3+f=0$. Almost complex ($s=0$) and almost contact ($s=1$) structures are well-known examples of f-structures. The case $s=2$ appeared in the study of hypersurfaces in almost contact manifolds [3, 17]. A Riemannian manifold endowed with an f-structure compatible with the Riemannian metric is called a metric f-manifold. For $s=0$ we have almost Hermitian manifolds and for $s=1$, metric almost contact manifolds. In this context, D.E. Blair [2] defined K-manifolds (and particular cases of S-manifolds and C-manifolds) as the analogue of Kaehlerian manifolds in the almost complex geometry and of quasi-Sasakian manifolds in the almost contact geometry and he showed that the curvature of either S-manifolds or C-manifolds is completely determined by their f-sectional curvatures. Later, M. Kobayashi and S. Tsuchiya [21] got expressions for the curvature tensor field of S-manifolds and C-manifolds when their f-sectional curvature is constant depending on such a constant. Such spaces are called S-space-forms and C-space-forms and they generalize complex and Sasakian space-forms and cosymplectic space-forms, respectively.

For metric f-manifolds, the authors and A. Carriazo [6] and, independently, M. Falcitelli and A.M. Pastore [15], have introduced a notion of *generalized S-space-form*. The first ones limited their research to the case $s=2$, even though their definition is easily adaptable to any $s>2$ [24], giving some non-trivial examples [5, 6]. Consequently, generalized S-space-forms make a more general framework to study the geometry of certain metric f-manifolds. Moreover, their definition generalized those ones of generalized complex-space-forms ($s=0$) and Sasakian-space-forms ($s=1$). Actually, a generalized S-space-form is a $(2m+2)$-dimensional metric f-manifold $(\widetilde{M}, f, \xi_1, \xi_2, \eta_1, \eta_2, g)$ with two structure vector fields in which there exist seven differentiable functions F_1, F_2, F_3 and $F_{11}, F_{12}, F_{21}, F_{22}$ on $\widetilde{M}$ such that the curvature tensor field of $\widetilde{M}$ is given by

$$\widetilde{R} = \sum_{i=1}^{3} F_i \widetilde{R}_i + \sum_{i,j=1}^{2} F_{ij} \widetilde{R}_{ij},$$

where $\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3, \widetilde{R}_{11}, \widetilde{R}_{12}, \widetilde{R}_{21}, \widetilde{R}_{22}$ are basic constant sectional curvature tensorial parts. It easy to show that S-space-forms and C-space-forms become particular cases of generalized S-space-forms (see [6])

For these reasons and since some inequalities of Chen's type, involving sectional, scalar and Ricci curvatures and squared mean curvature, have been proved for different kinds of submanifolds in S-space-forms [7, 16, 19, 20], the purpose of this paper is to establish them for generalized S-space-forms with two structure vector fields. To this end, after a preliminaries section containing basic notions of Riemannian submanifolds theory, in Section 3 we present some definitions and formulas concerning metric f-manifolds for later use and in Section 4 we bound the Ricci curvature of submanifolds in generalized S-space-form by terms involving the mean curvature and the second fundamental form of the submanifold plus certain quantities involving the set of functions $\{F_i, F_{ij}\}$. In fact, we prove that for a submanifold M of dimension $(n+2)$, tangent to both structure

vector fields,

$$\mathrm{Ric(U)} \leq \frac{(n+2)^2}{4}\|H\|^2 + (n+1)F_1 + 3\|TU\|^2 F_2 - (F_{11} + F_{22}),$$

for any unit vector field U orthogonal to the structure vector fields, being TU the tangential component of fU (Theorem 4.1). Furthermore, if the submanifold is a mininimal one, a unit vector field U orthogonal to both structure vector fields satisfies the equality case of the above expression if and only if it lies in the relative null space of M (Theorem 4.2). We also present some examples to illustrate these results including a compact one and we particularize them in the case of C-manifolds (Theorem 4.6) and in the case of being the submanifold a slant one (Theorem 4.7).

Finally, since in 1995 B.-Y. Chen [10, 11] introduced a well-defined Riemannian invariant $\delta_{\widetilde{M}}$ of a Riemannian manifold $\widetilde{M}$ whose definition is given by $\delta_{\widetilde{M}}(p) = \tau(p) - (\inf K)(p)$, for any $p \in \widetilde{M}$ (where τ is the scalar curvature and $(\inf K)(p) = \inf\{K(\pi) : \text{plane sections } \pi \subset T_p(\widetilde{M})\}$, $K(\pi)$ denoting the sectional curvature of $\widetilde{M}$ associated with the plane section π) and he proved, for submanifolds M in real-space-forms, a basic inequality involving the intrinsic invariant δ_M and the squared mean curvature of the immersion, in Section 5 we investigate a similar inequality for a submanifold M of a generalized S-space-form, tangent to the structure vector fields, studying the equality case (Theorem 5.2).

Moreover, we consider the well-defined function on M, $(\inf_{\mathcal{L}} K)(p) = \inf\{K(\pi) : \text{plane sections } \pi \subset \mathcal{L}_p\}$, where $\mathcal{L}$ denotes the complementary orthogonal distribution to that one spanned by the structure vector fields (on M). If $\delta_M^{\mathcal{L}}$ is the difference between the scalar curvature and $\inf_{\mathcal{L}} K$, it is clear that $\delta_M^{\mathcal{L}} \leq \delta_M$ and then, we obtain some general pinching results for $\delta_M^{\mathcal{L}}$, depending on the sign of F_2 (Theorems 5.9 and 5.10), studying the equality cases and presenting some examples to illustrate them. We conclude the paper applying the obtained theorems to slant submanifolds, specially in the case of the smallest possible dimension.

We should like to point out here that all the results of the paper improve those ones proved for S-space-forms in [7, 16]

2. Preliminaries.

Let M be a Riemannian manifold isometrically immersed in a Riemannian manifold $\widetilde{M}$. Let g denote the metric tensor of $\widetilde{M}$ as well as the induced metric tensor on M. If ∇ and $\widetilde{\nabla}$ denote the Riemannian connections of M and $\widetilde{M}$, respectively, the Gauss-Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y),\ \widetilde{\nabla}_X V = -A_V X + D_X V, \tag{2.1}$$

for any vector fields X, Y (resp., V) tangent (resp., normal) to M, where D is the normal connection, σ is the second fundamental form of the immersion and A_V is the Weingarten endomorphism associated with V. Then, A_V and σ are related by $g(A_V X, Y) = g(\sigma(X,Y), V)$.

The curvature tensor fields R and $\widetilde{R}$ of ∇ and $\widetilde{\nabla}$, respectively, satisfies the Gauss equation

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(\sigma(X,Z),\sigma(Y,W)) - g(\sigma(X,W),\sigma(Y,Z)), \tag{2.2}$$

for any X, Y, Z, W tangent to M.

The mean curvature vector H is defined by

$$H = \frac{1}{m}\mathrm{trace}\ \sigma = \frac{1}{m}\sum_{i=1}^{m} \sigma(e_i, e_i),$$

where $\dim M = m$ and $\{e_1, \ldots, e_m\}$ is a local orthonormal basis of tangent vector fields to M. In this context, M is said to be *minimal* if H vanishes identically or, equivalently, if $\mathrm{trace} A_V = 0$, for any vector field V normal to M. Moreover, M is said to be *totally geodesic* in $\widetilde{M}$ if $\sigma \equiv 0$ and *totally umbilical* if $\sigma(X,Y) = g(X,Y)H$, for any X, Y tangent to M. Moreover, the *relative null space* of M is defined by:

$$\mathcal{N} = \{X \text{ tangent to } M : \ \sigma(X,Y) = 0, \ \text{for all } Y \text{ tangent to } M\}.$$

3. Submanifolds of metric f-manifolds.

A $(2m+s)$-dimensional Riemannian manifold $(\widetilde{M}, g)$ with an f-structure f (that is, a tensor field f of type (1,1) and rank $2m$ satisfying $f^3 + f = 0$ [27]) is said to be a *metric f-manifold* if, moreover, there exist s global vector fields $\xi_1, \ldots, \xi_s$ on $\widetilde{M}$ (called *structure vector fields*) such that, if $\eta_1, \ldots, \eta_s$ are their dual 1-forms, then

$$f\xi_\alpha = 0; \ \eta_\alpha \circ f = 0; \ f^2 = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha; \tag{3.1}$$

$$g(X,Y) = g(fX, fY) + \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y),$$

for any X, Y tangent to $\widetilde{M}$. Let F be the 2-form on $\widetilde{M}$ defined by $F(X,Y) = g(X, fY)$. Since f is of rank $2m$, then $\eta_1 \wedge \cdots \wedge \eta_s \wedge F^m \neq 0$ and, particularly, $\widetilde{M}$ is orientable. The f-structure f is said to be *normal* if

$$[f,f] + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes d\eta_\alpha = 0,$$

where $[f,f]$ denotes the Nijenhuis tensor of f.

A metric f-manifold is said to be a K-manifold [2] if it is normal and $\mathrm{d}F = 0$. In a K-manifold $\widetilde{M}$, the structure vector fields are Killing vector fields [2]. Furthermore, a K-manifold is called an S-manifold if $F = \mathrm{d}\eta_\alpha$ and a C-manifold if $\mathrm{d}\eta_\alpha = 0$, for any $\alpha = 1, \ldots, s$. Note that, for $s = 0$, a K-manifold is a Kaehlerian manifold and, for $s = 1$, a K-manifold is a quasi-Sasakian manifold, an S-manifold is a Sasakian manifold and a C-manifold is a cosymplectic manifold. When $s \geq 2$, non-trivial examples can be found in [2, 18]. Moreover, a K-manifold $\widetilde{M}$ is an C-manifold if and only if

$$\widetilde{\nabla}_X \xi_\alpha = 0, \ \alpha = 1, \ldots, s, \tag{3.2}$$

for any tangent vector field X.

A plane section π on a metric f-manifold $\widetilde{M}$ is said to be an *f-section* if it is determined by a unit vector X, orthogonal to the structure vector fields and fX. The sectional curvature of π is called an *f-sectional curvature*. An S-manifold (resp., a C-manifold) is said to be an *S-space-form* (resp., a *C-space-form*) if it has constant f-sectional curvature (see [2, 21] for more details).

Next, let M be a isometrically immersed submanifold of a metric f-manifold $\widetilde{M}$. For any vector field X tangent to M we write

$$fX = TX + NX, \tag{3.3}$$

where TX and NX are the tangential and normal components of fX, respectively. The submanifold M is said to be *invariant* if N is identically zero, that is, if fX is tangent to M, for any vector field X tangent to M. On the other hand, M is said to be an *anti-invariant* submanifold if T is identically zero, that is, if fX is normal to M, for any X tangent to M.

From now on, we suppose that all the structure vector fields are tangent to the submanifold M. Then, the distribution on M spanned by the structure vector fields is denoted by $\mathcal{M}$ and its complementary orthogonal distribution is denoted by $\mathcal{L}$. Consequently, if $X \in \mathcal{L}$, then $\eta_\alpha(X) = 0$, for any $\alpha = 1, \ldots, s$ and if $X \in \mathcal{M}$, then $fX = 0$.

The submanifold M is said to be a *slant* submanifold if, for any $p \in M$ and any $X \in T_pM$, linearly independent on $(\xi_1)_p, \ldots, (\xi_s)_p$, the angle between fX and T_pM is a constant $\theta \in [0, \pi/2]$, called the slant angle of M in $\widetilde{M}$. Note that this definition generalizes that one given by B.-Y. Chen [13] for complex geometry and that one given by A. Lotta [22] for contact geometry. Moreover, invariant and anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively (for a general view about slant submanifolds, the survey written by A. Carriazo [4] can be consulted). A slant immersion which is neither invariant nor anti-invariant is called a *proper* slant immersion. In [9], it is proved that, a θ-slant submanifold M of a metric f-manifold $\widetilde{M}$ satisfies

$$g(NX, NY) = \sin^2\theta\, g(fX, fY), \tag{3.4}$$

for any vector fields X, Y tangent to M. Moreover, if we denote by $n + s$ the dimension of M, given a local orthonormal basis $\{e_1, \ldots, e_{n+s}\}$ of tangent vector fields to M, it is easy to show that

$$\sum_{j=1}^{n+s} g^2(e_i, fe_j) = cos^2\theta(1 - \sum_{\alpha=1}^{s} \eta_\alpha^2(e_i)), \tag{3.5}$$

for any $i = 1, \ldots, n$.

Concerning the behavior of the second fundamental of a submanifold in a metric f-manifold, we know that the study of totally geodesic or totally umbilical slant submanifolds of S-manifolds reduces to the study of invariant submanifolds [9]. It is necessary, then, to use a variation of these concepts, more related to the structure, namely *totally f-geodesic* and *totally f-umbilical* submanifolds, introduced by Ornea [23]. Thus, a submanifold of a metric f-manifold, tangent to the structure vector fields, is said to be a totally f-geodesic submanifold (resp., totally f-umbilical) if the distribution $\mathcal{L}$ is totally geodesic (resp., totally umbilical), that is, if $\sigma(X, Y) = 0$ (resp., if there exist a normal vector field V such that $\sigma(X, Y) = g(X, Y)V$), for any $X, Y \in \mathcal{L}$.

Denoting by $n + s$ (resp. $2m + s$) the dimension of M (resp. $\widetilde{M}$) and given a local orthonormal basis

$$\{e_1, \ldots, e_n, e_{n+1} = \xi_1, \ldots, e_{n+s} = \xi_s, e_{n+s+1}, \ldots, e_{2m+s}\}$$

of tangent vector fields to $\widetilde{M}$, such that $\{e_1, \ldots, e_n\}$ is a local orthonormal basis of $\mathcal{L}$, the squared norms of T and N are defined by

$$\|T\|^2 = \sum_{i,j=1}^{n} g^2(e_i, Te_j),\ \|N\|^2 = \sum_{i=1}^{n} \|Ne_1\|^2, \tag{3.6}$$

respectively, being independent of the choice of the above orthonormal basis. Moreover, we put $\sigma_{ij}^r = g(\sigma(e_i, e_j), e_r)$, for any $i, j = 1, \ldots, n + s$ and $r = n + s + 1, \ldots, 2m + s$. Then, the mean curvature vector H and the squared norm of σ can be written as:

$$H = \frac{1}{n+s} \sum_{r=n+1}^{2m} \sum_{i=1}^{n+s} \sigma_{ii}^r e_r, \tag{3.7}$$

$$\|\sigma\|^2 = \sum_{r=n+1}^{2m} \left\{ \sum_{i=1}^{n+s} (\sigma_{ii}^r)^2 + 2 \sum_{1 \le i < j \le n+s} (\sigma_{ij}^r)^2 \right\}. \tag{3.8}$$

4. Slant submanifolds of generalized S-space-forms with two structure vector fields.

The notion of generalized S-space-forms was introduced by the authors and A. Carriazo in [6], considering the case of two structure vector fields which appeared in the study of hypersurfaces in almost contact manifolds [3, 17] and which was the first motivation to investigate metric f-manifolds but, in fact, their definition is easily adaptable to any $s > 2$. Independently, M. Falcitelli and A.M. Pastore gave a slightly different definition [15]. From it, one can deduce that the distribution spanned by the structure vector fields must be flat which is the case, for instance, of S-manifolds and C-manifolds. However, in [5, 6] some non-trivial examples of generalized S-space-forms with non-flat distribution spanned by the structure vector fields are provided. Moreover, it is easy to show that both definitions coincide for metric f-manifolds such that either $\widetilde{\nabla}\xi_\alpha = -f$ or $\nabla\xi_\alpha = 0$, for any $\alpha = 1, \ldots, s$. Thus and for the purpose of this paper, we shall use the definition of [6].

Consequently, from now on, we consider a $(2m+2)$-dimensional metric f-manifold $(\widetilde{M}, f, \xi_1, \xi_2, \eta_1, \eta_2, g)$ with two structure vector fields. Then, $\widetilde{M}$ is said to be a *generalized S-space-form* [6, 26] if there exist seven differentiable functions F_1, F_2, F_3 and $F_{11}, F_{12}, F_{21}, F_{22}$ on $\widetilde{M}$ such that the curvature tensor field of $\widetilde{M}$ is given by

$$\widetilde{R} = \sum_{i=1}^{3} F_i \widetilde{R}_i + \sum_{i,j=1}^{2} F_{ij} \widetilde{R}_{ij}, \tag{4.1}$$

where

$$\begin{aligned}
\widetilde{R}_1(X,Y)Z &= g(Y,Z)X - g(X,Z)Y,\\
\widetilde{R}_2(X,Y)Z &= g(X,fZ)fY - g(Y,fZ)fX + 2g(X,fY)fZ,\\
\widetilde{R}_3(X,Y)Z &= \eta_1(X)\eta_2(Y)\eta_2(Z)\xi_1 - \eta_2(X)\eta_1(Y)\eta_2(Z)\xi_1\\
&\quad +\eta_2(X)\eta_1(Y)\eta_1(Z)\xi_2 - \eta_1(X)\eta_2(Y)\eta_1(Z)\xi_2,\\
\widetilde{R}_{ij}(X,Y)Z &= \eta_i(X)\eta_j(Z)Y - \eta_i(Y)\eta_j(Z)X\\
&\quad +g(X,Z)\eta_i(Y)\xi_j - g(Y,Z)\eta_i(X)\xi_j,\ i,j = 1,2,
\end{aligned}$$

for any X, Y, Z tangent to M. Some examples of generalized S-space-forms are given in [5, 6]. In particular, S-space-forms and C-space-forms are generalized S-space-forms.

Let M be a submanifold isometrically immersed in $\widetilde{M}$, tangent to both structure vector fields and suppose that $\dim(M) = n+2$. As above, let us consider a local orthonormal basis

$$\{e_1, \ldots, e_n, e_{n+1} = \xi_1, e_{n+2} = \xi_2, e_{n+3}, \ldots, e_{2m+2}\} \tag{4.2}$$

of tangent vector fields to $\widetilde{M}$, such that $\{e_1, \ldots, e_n\}$ is a local orthonormal basis of $\mathcal{L}$. The scalar curvature τ of M is defined by

$$\tau = \frac{1}{2}\sum_{i\neq j} K(e_i \wedge e_j), \tag{4.3}$$

where K denotes the sectional curvature of M. From (2.2), (3.7)-(3.8) and (4.1), we obtain the following relation between the scalar curvature and the mean curvature of M:

$$\begin{aligned}
2\tau =& (n+1)(n+2)F_1 - 2(n+1)(F_{11}+F_{22}) + 2F_3\\
&+ 3F_2\|T\|^2 + (n+2)^2\|H\|^2 - \|\sigma\|^2.
\end{aligned} \tag{4.4}$$

Now, from (3.7), (3.8) and (4.4), a straightforward computation gives:

$$\begin{aligned}\tau =&\frac{(n+2)^2}{4}\|H\|^2+\frac{(n+1)(n+2)}{2}F_1\\ &+\frac{3\|T\|^2}{2}F_2+F_3-(n+1)(F_{11}+F_{22})\\ &-\sum_{r=n+3}^{2m+2}\left\{\sum_{1\le i<j\le n+2}(\sigma_{ij}^r)^2-\frac{1}{4}\sum_{i=1}^{n+2}(\sigma_{ii}^r)^2+\frac{1}{2}\sum_{1\le i<j\le n+2}\sigma_{ii}^r\sigma_{jj}^r\right\}.\end{aligned}\tag{4.5}$$

By using the above formula, we can prove the following general result:

4.1. Theorem. *Let M be an $(n+2)$-dimensional submanifold of a generalized S-space-form $\widetilde{M}$, tangent to both structure vector fields. Then,*

$$\mathrm{Ric(U)}\le\frac{(n+2)^2}{4}\|H\|^2+(n+1)F_1+3\|TU\|^2F_2-(F_{11}+F_{22}),\tag{4.6}$$

for any unit vector field $U\in\mathcal{L}$.

Proof. We choose a local orthonormal basis of tangent vector fields to $\widetilde{M}$ as in (4.2) and such that $e_1=U$. Then, from (4.3):

$$\tau=\mathrm{Ric(U)}+\sum_{2\le i<j\le n}K(e_i\wedge e_j)+\sum_{i=2}^{n}\sum_{\alpha=1}^{2}K(e_i\wedge\xi_\alpha)+K(\xi_1\wedge\xi_2).\tag{4.7}$$

Now, by using (4.1), we get

$$\begin{aligned}\sum_{2\le i<j\le n}K(e_i\wedge e_j)=&\frac{(n-1)(n-2)}{2}F_1\\ &+\sum_{2\le i<j\le n}\left\{3F_2g(e_1,fe_j)^2+\sum_{r=n+3}^{2m+2}\left(\sigma_{ii}^r\sigma_{jj}^r-(\sigma_{ij}^r)^2\right)\right\},\\ \sum_{i=2}^{n}\sum_{\alpha=1}^{2}K(e_i\wedge\xi_\alpha)=&2(n-1)F_1-(n-1)(F_{11}+F_{22})\\ &+\sum_{r=n+3}^{2m+2}\sum_{i=2}^{n}\sum_{\alpha=1}^{2}\left(\sigma_{ii}^r\sigma_{n+\alpha n+\alpha}^r-(\sigma_{in+\alpha}^r)^2\right)\end{aligned}$$

and:

$$K(\xi_1\wedge\xi_2)=F_1+F_3-(F_{11}+F_{22})-\|\sigma(\xi_1,\xi_2)\|^2+g(\sigma(\xi_1,\xi_1),\sigma(\xi_2,\xi_2)).$$

Then, substituting into (4.7) and taking into account (4.5), we obtain,

$$\begin{aligned}\mathrm{Ric}(U)=&\frac{(n+2)^2}{4}\|H\|^2+(n+1)F_1+3\|TU\|^2F_2-(F_{11}+F_{22})\\ &-\sum_{r=n+3}^{2m+2}\left\{\frac{1}{4}\left(\sigma_{11}^r-\sum_{i=2}^{n+2}\sigma_{ii}^r\right)^2+\sum_{i=2}^{n+2}(\sigma_{1i}^r)^2\right\},\end{aligned}\tag{4.8}$$

which completes the proof. □

What about the equality case of (4.6)? If the submanifold is minimal, we can prove the following theorem.

4.2. Theorem. *Let M be a minimal $(n+2)$-dimensional submanifold of a generalized S-space-form $\widetilde{M}$, tangent to both structure vector fields. Then, a unit vector field U in $\mathcal{L}$ satisfies the equality case of (4.6) if and only if U lies in the relative null space $\mathcal{N}$ of M.*

Proof. If $U \in \mathcal{L}$ is a unit vector field satisfying the equality case of (4.6), then, choosing a local orthonormal basis of tangent vector fields to $\widetilde{M}$ as in (4.2) and such that $e_1 = U$, from (4.8) we get that $\sigma^r_{1n+\alpha} = 0$, for any $r = n+3, \ldots, 2m+2$ and $\alpha = 1, 2$. So, $\sigma(U, \xi_\alpha) = 0$, $\alpha = 1, 2$. Furthermore, by using (4.8) again, we obtain $\sigma^r_{1i} = 0$, for any $i = 2, \ldots, n$, $r = n+3, \ldots, 2m+2$ (that is, $\sigma(U, e_i) = 0$, for any $i = 2, \ldots, n$) and

$$\sigma^r_{11} = \sum_{i=2}^{n+2} \sigma^r_{ii},$$

for any $r = n+3, \ldots, 2m+2$. But, since $H = 0$,

$$\sigma^r_{11} = -\sum_{i=2}^{n+2} \sigma^r_{ii},$$

for any $r = n+3, \ldots, 2m+2$, thus $\sigma^r_{11} = 0$ and $\sigma(U, U) = 0$. Consequently, $U \in \mathcal{N}$.

Conversely, if $U \in \mathcal{N}$, choosing a local orthonormal basis of tangent vector fields to $\widetilde{M}$ as in (4.2) with $e_1 = U$, we have that $\sigma^r_{1i} = 0$, for any $i = 1, \ldots, n+2$ and $r = n+3, \ldots, 2m+2$. Again, since $H = 0$, we obtain that

$$\sum_{i=2}^{n+2} \sigma^r_{ii} = 0,$$

for any $r = n+3, \ldots, 2m+2$. Then, from (4.8) we deduce the equality case of (4.6). □

The following corollary is immediate:

4.3. Corollary. *Let M be a minimal $(n+2)$-dimensional submanifold of a generalized S-space-form $\widetilde{M}$, tangent to both structure vector fields. If the equality case of (4.6) holds for all unit vector fields in $\mathcal{L}$, then M is a totally f-geodesic manifold.*

4.4. Example. To illustrate the above results, consider $\mathbf{C}^2$ with its structure of complex-space-form with constant holomorphic curvature $c = 0$. Then, it is known ([6]) that the warped product $\widetilde{M} = \mathbf{R} \times_{h_2} (\mathbf{R} \times_{h_1} \mathbf{C}^2)$ is a generalized S-space-form ($h_1, h_2 > 0$ differentiable functions on $\mathbf{R}$) with functions:

$$F_1 = -\frac{(h_1')^2}{h_1^2 h_2^2} - \frac{(h_2')^2}{h_2^2};\ F_2 = F_{12} = F_{21} = 0;$$

$$F_3 = F_{11} = -\frac{(h_1')^2}{h_1^2 h_2^2} + \frac{h_1''}{h_1 h_2^2};$$

$$F_{22} = -\frac{(h_1')^2}{h_1^2 h_2^2} - \frac{(h_2'^2)}{h_2^2} + \frac{h_2''}{h_2}.$$

Moreover,

$$x(t_1, t_2, u, v) = (t_1, t_2, u\cos\theta, u\sin\theta, v, 0)$$

defines a 4-dimensional totally geodesic θ-slant submanifold M in $\widetilde{M}$ [5]. Consequently, for any unit vector field $U \in \mathcal{L}$, we deduce because the equality case of (4.6) that:

$$\mathrm{Ric(U)} = 3F_1 - F_{11} - F_{22} = -\frac{(h_1')^2}{h_1^2 h_2^2} - \frac{2(h_2')^2}{h_2^2} - \frac{h_1''}{h_1 h_2^2} - \frac{h_2''}{h_2}.$$

4.5. Example. Consider the sphere S^{2m+1} ($m \geq 2$) with its usual structure (ϕ, ξ, η, g) of Sasakian-space-form with constant ϕ-sectional curvature $c = 1$. Then, it is a generalized Sasakian-space-form with functions (see [1]) $f_1 = 1$ and $f_2 = f_3 = 0$. Let $\widetilde{M}$ be an isometrically immersed (orientable) hypersurface of $S^{2m+1}M$ such that the structure

vector field ξ of S^{2m+1} is always tangent to $\widetilde{M}$ and let N denote the unit normal vector field of $\widetilde{M}$ in S^{2m+1}. If we put

$$\xi_1 = \xi;\ \xi_2 = -\phi N;\ \eta_1 = \eta;\ \eta_2(X) = -g(X, \phi N);\ fX = \phi X - \eta_2(X)N,$$

for any vector field X tangent to $\widetilde{M}$, then $(\widetilde{M}, f, \xi_1, \xi_2, \eta_1, \eta_2, g)$ is a metric f-manifold [28]. Moreover, if $\widetilde{M}$ is a pseudo-umbilical hypersurface of S^{2m+1}, that is, if its shape operator A satisfies (see [28])

$$AX = \widetilde{f}_1 \left(X - \eta_1(X)\xi_1\right) + \widetilde{f}_2\eta_2(X)\xi_2 - \eta_1(X)\xi_2 - \eta_2(X)\xi_1,$$

for any X tangent to $\widetilde{M}$, $\widetilde{f}_1$ and $\widetilde{f}_2$ being differentiable functions on $\widetilde{M}$, we know that $\widetilde{M}$ is a generalized S-space-form with functions (see [6] for more details):

$$F_1 = 1 + \widetilde{f}_1^2;\ F_2 = 0;\ F_{11} = \widetilde{f}_1^2;\ F_{22} = -\widetilde{f}_1\widetilde{f}_2;$$

$$F_{12} = F_{21} = \widetilde{f}_1;\ F_3 = -1 - \widetilde{f}_1\widetilde{f}_2.$$

In this context, from Theorem 7.1 in [28], if $\widetilde{M}$ is also a compact hypersurface of S^{2m+1}, then it is congruent to $S^{2m-1}(r_1) \times S^1(r_2)$ with $r_1^2 + r_2^2 = 1$. Consequently, we have an example of a compact generalized S-space-form. Next, let M be a $(n+2)$-dimensional submanifold of $\widetilde{M}$ satisfying the conditions of Theorem 4.2 (for instance, a totally geodesic submanifold). Thus, if $X \in \mathcal{L}$ is a unit vector field, we deduce that:

$$Ric(U) = \frac{(n+2)^2}{4}\|H\|^2 + (n\widetilde{f}_1 + \widetilde{f}_2)\widetilde{f}_1 + n + 1.$$

Next, assume that $m \geq 2$. If the ambient generalized S-space-form $\widetilde{M}$ is an S-manifold, it is known (see Proposition 7 in [15] and Theorem 3.2 in [18]) that $\widetilde{M}$ is an S-space-form. Therefore [6],

$$F_1 = \frac{c+6}{4};\ F_2 = F_3 = \frac{c-2}{4};\ F_{11} = F_{22} = \frac{c+2}{4};\ F_{12} = F_{21} = -1, \tag{4.9}$$

where c is denoting the constant f-sectional curvature. In this case, a better (in the sense of lower) upper bound for $Ric(U)$ than the one obtained in (4.6) was got in [16]. In fact and in terms of the functions of (4.9), it was proved that:

$$\mathrm{Ric}(U) \leq \frac{(n+2)^2}{4}\|H\|^2 + (n-1)F_1 + (3F_1 - 4)\|TU\|^2. \tag{4.10}$$

It is easy to show that both upper bounds of (4.6) and (4.10) are equal if and only if $\|NU\| = 0$ and their common value is:

$$\frac{(n+2)^2}{4}\|H\|^2 + (n+2)F_1 - 4.$$

Conditions for the equality case of (4.10) have also been given in [16].

Now, we suppose that the ambient generalized S-space-form $\widetilde{M}$ is a C-manifold. Then, from Proposition 8 and Remark 2 in [15] it is known that $\widetilde{M}$ is a C-space-form and so [6],

$$F_1 = F_2 = F_3 = F_{11} = F_{22} = \frac{c}{4};\ F_{12} = F_{21} = 0, \tag{4.11}$$

where c is denoting the constant f-sectional curvature. If M is a $(n+2)$-dimensional submanifold of $\widetilde{M}$, tangent to both structure vector fields, from (2.1), (3.2) and (3.3) it is easy to show that

$$\sigma(X, \xi_\alpha) = 0, \tag{4.12}$$

for any X tangent to M and $\alpha = 1, 2$. Then, because of the identities (4.11) the equation (4.8) becomes to

$$\begin{aligned}\mathrm{Ric}(U) =& \frac{(n+2)^2}{4}\|H\|^2 + \{(n-1) + 3\|TU\|^2\}F_1 \\ &- \sum_{r=n+3}^{2m+2}\left\{\frac{1}{4}\left(\sigma_{11}^r - \sum_{i=2}^{n}\sigma_{ii}^r\right)^2 + \sum_{i=2}^{n}(\sigma_{1i}^r)^2\right\}\end{aligned} \tag{4.13}$$

and so,

$$\mathrm{Ric}(U) \leq \frac{(n+2)^2}{4}\|H\|^2 + \{(n-1) + 3\|TU\|^2\}F_1, \tag{4.14}$$

for any unit vector field $U \in \mathcal{L}$. To study the equality case of the above equation, we prove:

4.6. Theorem. *Let M be a $(n+2)$-dimensional submanifold ($n \geq 2$) of a generalized S-space-form $\widetilde{M}$, tangent to both structure vector fields. If $\widetilde{M}$ is also an C-manifold, then the equality case of (4.14) holds for all unit vector field in $\mathcal{L}$ if and only if either M is a totally f-umbilical submanifold when $n = 2$ or M is a totally geodesic submanifold when $n > 2$.*

Proof. If the equality case of (4.14) is true for any unit vector field $U \in \mathcal{L}$, then, by choosing local orthonormal basis of tangent vector fields to $\widetilde{M}$ as in (4.2) and since e_1 can be chosen to be any arbitrary unit vector field in $\mathcal{L}$, from (4.13) we get

$$\sigma_{ii}^r = \sigma_{jj}^r = \frac{1}{2}(\sigma_{11}^r + \cdots + \sigma_{nn}^r),\ i, j = 1, \ldots, n,$$

$$\sigma_{ij}^r = 0,\ i \neq j,\ i, j = 1, \ldots, n,$$

for any $r = n+3, \ldots, 2m$. Thus, we have to consider two cases. Firstly, if $n = 2$, we deduce that $\sigma_{11}^r = \sigma_{22}^r$, for any r and M is a totally f-umbilical submanifold. Secondly, if $n > 2$ we obtain that $\sigma_{ii}^r = 0$, for any $i = 1, \ldots n$ and r and so, together with (4.12), we deduce that M is a totally geodesic submanifold. The converse part is obvious from (4.13). □

To illustrate the above result we can consider the warped product of Example 4.4 with functions $h_1 = h_2 = 1$. It is easy to show that, in this case, the generalized S-space-form $\widetilde{M}$ is a C-manifold.

The above results imply the following theorem for slant submanifolds of generalized S-space-forms.

4.7. Theorem. *Let M be a $(n+2)$-dimensional ($n \geq 2$) θ-slant submanifold of a generalized S-space-form $\widetilde{M}$ and $U \in \mathcal{L}$ be any unit vector field. Then:*

(i) We have that:

$$\mathrm{Ric(U)} \leq \frac{1}{4}(n+2)^2\|H\|^2 + (n+1)F_1 + 3\cos^2\theta F_2 - (F_{11} + F_{22}). \tag{4.15}$$

(ii) If $\widetilde{M}$ is also an S-manifold, we have

$$\mathrm{Ric}(U) \leq \frac{(n+2)^2}{4}\|H\|^2 + (n-1)F_1 + (3F_1 - 4)\cos^2\theta$$

and the equality holds for all unit vector field in $\mathcal{L}$ if and only if either M is a totally f-geodesic submanifold when $n > 2$ or M is a totally f-umbilical submanifold when $n = 2$.

(iii) If $\widetilde{M}$ is also a C-manifold, we have

$$\mathrm{Ric}(U) \leq \frac{(n+2)^2}{4}\|H\|^2 + \{(n-1) + 3\cos^2\theta\}F_1$$

and the equality holds for all unit vector field in $\mathcal{L}$ if and only if either M is a totally f-umbilical submanifold when $n = 2$ or M is a totally geodesic submanifold when $n > 2$.

Proof. For any unit vector field $U \in \mathcal{L}$, by using a local orthonormal basis of tangent vector fields to $\widetilde{M}$ as in (4.2), such that $e_1 = U$, we get from (3.5) and (3.6) that $\|TU\|^2 = \cos^2\theta$ and so, from (4.6) we have (4.15). For the rest of the proof we only have to consider the results of [16] for S-manifolds and Theorem 4.6 for C-manifolds. □

5. The scalar curvature.

In the articles [10, 11], B.-Y. Chen introduced a well-defined Riemannian invariant $\delta_{\widetilde{M}}$ of a Riemannian manifold $\widetilde{M}$ whose definition is given by

$$\delta_{\widetilde{M}}(p) = \tau(p) - (\inf K)(p),$$

for any $p \in \widetilde{M}$, where τ is the scalar curvature and

$$(\inf K)(p) = \inf\{K(\pi) : \text{plane sections } \pi \subset T_p(\widetilde{M})\},$$

with $K(\pi)$ denoting the sectional curvature of $\widetilde{M}$ associated with the plane section π. Moreover, for submanifolds M in a real-space form of constant sectional curvature c, Chen gave the following basic inequality involving the intrinsic invariant δ_M and the squared mean curvature of the immersion

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{(n+1)(n-2)}{2}c,$$

where n denotes the dimension of M. A similar inequality for S-space-forms, conditions for the equality case and some applications have been established in [7]. In this section, we want to study the more general case of generalized S-space-forms.

Let $\widetilde{M}$ be a generalized S-space-form with two structure vectors ξ_1, ξ_2 and M a $(n+2)$-dimensional submanifold of $\widetilde{M}$, tangent to both structure vector fields. Let $\pi \subset \mathcal{L}_p$ a plane section at $p \in M$. Then,

$$F^2(\pi) = g^2(e_1, fe_2) \tag{5.1}$$

is a real number in $[0, 1]$ which is independent on the choice of the orthonormal basis $\{e_1, e_2\}$ of π. First, we recall an algebraic lemma from [14]:

5.1. Lemma. *Let $a_1, \ldots, a_k, c$ be $k+1$ $(k \geq 2)$ real numbers such that:*

$$\left(\sum_{i=1}^{k} a_i\right)^2 = (k-1)\left(\sum_{i=1}^{k} {a_i}^2 + c\right).$$

Then, $2a_1a_2 \geq c$, with the equality holding if and only if:

$$a_1 + a_2 = a_3 = \cdots = a_k.$$

Now, we can prove the following theorem.

5.2. Theorem. *Let M be a $(n+2)$-dimensional submanifold of a generalized S-space-form $\widetilde{M}$, tangent to both structure vector fields. Then, for any point $p \in M$ and any plane section $\pi \subset \mathcal{L}_p$, we have:*

$$\begin{aligned}\tau - K(\pi) \leq \frac{n(n+2)^2}{2(n+1)}\|H\|^2 + \frac{n(n+3)}{2}F_1 + F_3 - (n+1)(F_{11}+F_{22}) \\ + 3F_2\left(\frac{\|T\|^2}{2} - F^2(\pi)\right)\end{aligned} \tag{5.2}$$

The equality in (5.2) holds at $p \in M$ if and only if there exist orthonormal bases $\{e_1, \ldots, e_{n+2}\}$ and $\{e_{n+3}, \ldots, e_{2m+2}\}$ of T_pM and $T_p^\perp M$, respectively, such that:

(i) $e_{n+j} = (\xi_j)_p$, *for* $j = 1, 2$.

(ii) π *is spanned by* e_1 *and* e_2.

(iii) The shape operators $A_r = A_{e_r}$, $r = n+3, \ldots, 2m+2$, *take the following forms at* p:

$$A_{n+3} = \begin{pmatrix} a & b & 0 & 0 & 0 \\ b & c-a & 0 & 0 & 0 \\ 0 & 0 & c & \cdots & 0 \\ & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}, \tag{5.3}$$

$$A_r = \begin{pmatrix} a_r & b_r & 0 \\ b_r & -a_r & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.4}$$

where $a, b, c, a_r, b_r \in \mathbf{R}$, *for any* $r = n+4, \ldots, 2m+2$.

Proof. Let $\pi \subset \mathcal{L}_p$ be a plane section and choose orthonormal bases $\{e_1, \ldots, e_{n+2}\}$ of T_pM and $\{e_{n+2}, \ldots, e_{2m+2}\}$ of $T_p^\perp M$ such that $e_{n+j} = (\xi_j)_p$, for $j = 1, 2$, π is spanned by e_1, e_2 and e_{n+3} is in the direction of the mean curvature vector H. Then, from (4.1)

$$K(\pi) = \sigma_{11}^{n+3}\sigma_{22}^{n+3} - (\sigma_{12}^{n+3})^2 + \sum_{r=n+4}^{2m+2}(\sigma_{11}^r\sigma_{22}^r - (\sigma_{12}^r)^2) + F_1 + 3F_2F^2(\pi). \tag{5.5}$$

Now, put:

$$\varepsilon = 2\tau - \frac{n(n+2)^2}{n+1}|H|^2 - n(n+3)F_1 + 2(n+1)(F_{11}+F_{22}) - 3F_2\|T\|^2 - 2F_3. \tag{5.6}$$

Hence, (4.4) and (5.6) imply:

$$(n+2)^2\|H\|^2 = (n+1)\{\|\sigma\|^2 + \varepsilon - 2F_1\}$$

that is, respect to the above orthonormal bases:

$$\begin{aligned}\left(\sum_{i=1}^{n+2}\sigma_{ii}^{n+3}\right)^2 =& (n+1)\left\{\sum_{i=1}^{n+2}(\sigma_{ii}^{n+3})^2 + \sum_{i\neq j}^{n+2}(\sigma_{ij}^{n+3})^2\right. \\ &\left. + \sum_{r=n+4}^{2m+2}\sum_{i,j=1}^{n+2}(\sigma_{ij}^r)^2 + \varepsilon - 2F_1\right\}.\end{aligned}$$

Therefore, by applying Lemma 5.1, we get:

$$2\sigma_{11}^{n+3}\sigma_{22}^{n+3} \geq \sum_{i\neq j}^{n+2}(\sigma_{ij}^{n+3})^2 + \sum_{r=n+4}^{2m+2}\sum_{i,j}^{n+2}(\sigma_{ij}^r)^2 + \varepsilon - 2F_1. \tag{5.7}$$

Thus, from (5.5) and (5.7), we obtain:

$$
\begin{aligned}
K(\pi) \geq & \sum_{r=n+3}^{2m+2}\sum_{j>2}^{n+2}\{(\sigma_{1j}^r)^2+(\sigma_{2j}^r)^2\}+\frac{1}{2}\sum_{i\neq j>2}^{n+2}(\sigma_{ij}^{n+3})^2+\frac{1}{2}\sum_{r=n+4}^{2m+2}\sum_{i,j>2}^{n+2}(\sigma_{ij}^r)^2 \\
& +\frac{1}{2}\sum_{r=n+4}^{2m+2}(\sigma_{11}^r+\sigma_{22}^r)^2+\frac{\varepsilon}{2}+3F_2F^2(\pi)\geq\frac{\varepsilon}{2}+3F_2F^2(\pi).
\end{aligned}
\tag{5.8}
$$

Consequently, combining (5.6) and (5.8), we get (5.2). If the equality in (5.2) holds, then the inequalities in (5.7) and (5.8) become equalities. So, we have:

$$\sigma_{1j}^r=\sigma_{2j}^r=0,\ r=n+2,\dots,2m+2, j>2;$$

$$\sigma_{ij}^{n+3}=0,\ i\neq j>2;$$

$$\sigma_{ij}^r=0,\ r=n+4,\dots,2m+2; i,j>2;$$

$$\sigma_{11}^r+\sigma_{22}^r=0,\ r=n+4,\dots,2m+2;$$

$$\sigma_{11}^{n+3}+\sigma_{22}^{n+3}=\sigma_{ii}^{n+3},\ i=3,\dots,n+2.$$

Thus, with respect to the chosen orthonormal basis $\{e_1,\dots,e_{2m+2}\}$, the shape operators of M take the forms (5.3) and (5.4).

The converse follows from a direct calculation. □

Now, we consider:

$$(\inf\nolimits_{\mathcal{L}}K)(p)=\inf\{K(\pi):\text{plane sections }\pi\subset\mathcal{L}_p\}.$$

Then, $\inf_{\mathcal{L}}K$ is a well-defined function on M. Let $\delta_M^{\mathcal{L}}$ denote the difference between the scalar curvature and $\inf_{\mathcal{L}}K$, that is:

$$\delta_M^{\mathcal{L}}(p)=\tau(p)-(\inf\nolimits_{\mathcal{L}}K)(p).$$

It is clear that $\delta_M^{\mathcal{L}}\leq\delta_M$.

It is obvious that if the submanifold M is anti-invariant, then $\|T\|^2=F^2(\pi)=0$, for any plane section in $\mathcal{L}$. Consequently, from (5.2) we obtain:

5.3. Corollary. *Let M be a $(n+2)$-dimensional submanifold of a generalized S-space-form $\widetilde{M}$, tangent to both structure vector fields. If either $F_2=0$ or M is an anti-invariant submanifold, then we have:*

$$\delta_M^{\mathcal{L}}\leq\frac{n(n+2)^2}{2(n+1)}\|H\|^2+\frac{n(n+3)}{2}F_1-(n+1)(F_{11}+F_{22})+F_3.$$

By using Theorem 5.2 we can obtain some general pinching results for $\delta_M^{\mathcal{L}}$ if either $F_2\geq 0$ or $F_2<0$.

5.4. Theorem. *Let M be a $(n+2)$-dimensional submanifold of a generalized S-space-form $\widetilde{M}$, tangent to both structure vector fields. If $F_2\geq 0$, then we have:*

$$\delta_M^{\mathcal{L}}\leq\frac{n(n+2)^2}{2(n+1)}\|H\|^2+\frac{n(n+3)}{2}F_1+F_3-(n+1)(F_{11}+F_{22})+\frac{3n}{2}F_2. \tag{5.9}$$

The equality in (5.9) holds identically if and only if n is even and M is an invariant submanifold.

Proof. Since $F_2 \geq 0$, from (5.2) we deduce

$$\delta_M^{\mathcal{L}} \leq \frac{n(n+2)^2}{2(n+1)}\|H\|^2 + \frac{n(n+3)}{2}F_1 + F_3 - (n+1)(F_{11}+F_{22}) + 3F_2\frac{\|T\|^2}{2}$$

and, by using that $\|T\|^2 \leq n$, we get (5.9). Moreover, the equality holds if and only if $\|T\|^2 = n$, that is, if and only if M is invariant and so, n is even. □

5.5. Theorem. *Let M be a $(n+2)$-dimensional submanifold of a generalized S-space-form $\widetilde{M}$, tangent to both structure vector fields. If $F_2 < 0$, then we have:*

$$\delta_M^{\mathcal{L}} \leq \frac{n(n+2)^2}{2(n+1)}\|H\|^2 + \frac{n(n+3)}{2}F_1 + F_3 - (n+1)(F_{11}+F_{22}). \tag{5.10}$$

The equality in (5.10) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \ldots, e_n, (\xi_1)_p, (\xi_2)_p\}$ of $T_p(M)$ such that the subspace spanned by $e_3, \ldots, e_n$ is anti-invariant, that is, $Te_j = 0$, for any $j = 3, \ldots, n$.

Proof. From Theorem 5.2, we have (5.2) which implies:

$$\begin{aligned}\delta_M^{\mathcal{L}} &\leq \frac{n(n+2)^2}{2(n+1)}\|H\|^2 + \frac{n(n+3)}{2}F_1 + F_3 - (n+1)(F_{11}+F_{22}) \\ &\quad + 3F_2\left\{\sum_{j=3}^{n}\left(g^2(e_1,Te_j) + g^2(e_2,Te_j)\right) + \frac{1}{2}\sum_{i,j=3}^{n} g^2(e_i,Te_j)\right\} \\ &\leq \frac{n(n+2)^2}{2(n+1)}\|H\|^2 + \frac{n(n+3)}{2}F_1 + F_3 - (n+1)(F_{11}+F_{22}).\end{aligned} \tag{5.11}$$

If the equality in (5.10) holds, then both inequalities in (5.11) become equalities. Thus, we complete the proof. □

We observe that examples of generalized S-space-forms with $F_2 = 0$ are given in Examples 4.4 and 4.5. On the other hand, $\mathbf{R}^{2m+2}$ with its structure of S-space-form with constant f-sectional curvature $c = -6$ (see [18] for more details) is a generalized S-space-form with $F_2 = -2 < 0$ ([6]). Moreover, the warped product $\widetilde{M} = \mathbf{R} \times_{h_2} (\mathbf{R} \times_{h_1} N(c))$, $h_1, h_2 > 0$ differentiable functions on $\mathbf{R}$ and $N(c)$ a complex-space-form of constant holomorphic curvature $c > 0$, is a generalized S-space-form with ([6]):

$$F_2 = \frac{c}{4h_1^2h_2^2} > 0.$$

Finally, we are going to study inequality (5.2) when M is a slant submanifold. First, we observe that, if M is a $(n+2)$-dimensional θ-slant submanifold of a metric f-manifold, then, from (3.4), (3.5) and (3.6):

$$\|T\|^2 = n\cos^2\theta;\ \|N\|^2 = n\sin^2\theta. \tag{5.12}$$

Now, by using (5.2) and (5.12), we obtain:

5.6. Theorem. *Let M be a $(n+2)$-dimensional θ-slant submanifold of a generalized S-space-form $\widetilde{M}$. Then, for any point $p \in M$ and any plane section $\pi \subset \mathcal{L}_p$, we have:*

$$\begin{aligned}\tau - K(\pi) \leq{}& \frac{n(n+2)^2}{2(n+1)}\|H\|^2 + \frac{n(n+3)}{2}F_1 + F_3 \\ &- (n+1)(F_{11}+F_{22}) + 3F_2\left(\frac{n}{2}\cos^2\theta - F^2(\pi)\right).\end{aligned} \tag{5.13}$$

It is well known [8] that there are no proper slant submanifolds of metric f-manifolds of dimension lower than $2+s$, being s the number of structure vector fields. Then, for $(2+2)$-dimensional slant submanifolds, we can state the following result:

5.7. Corollary. *Let M be a 4-dimensional θ-slant submanifold of a generalized S-space-form $\widetilde{M}$. Then, we have:*

$$\delta_M^{\mathcal{L}} \leq \frac{16}{3}\|H\|^2 + 5F_1 - 3(F_{11} + F_{22}) + F_3. \tag{5.14}$$

Moreover, the equality holds if and only if M is minimal.

Proof. Since $n = 2$, then it is clear that

$$\delta_M^{\mathcal{L}} = \tau - K(\mathcal{L}) \tag{5.15}$$

and $F^2(\mathcal{L}) = \cos^2\theta$. Thus, (5.14) follows directly from (5.13). On the other hand, by using (4.1) and (4.3), it easy to show that:

$$\tau - K(\mathcal{L}) = 5F_1 - 3(F_{11} + F_{22}) + F_3. \tag{5.16}$$

Hence, (5.15) and (5.16) imply the condition for the equality case in (5.14). □

This result improves that one obtained for S-space-forms in [7].

To illustrate the above result, we can consider $\mathbf{R}^{4+2}$ with it structure of S-space-form of constant f-sectional curvature $c = -6$. Then, it is a generalized S-space-form with $F_1 = 0$, $F_3 = -2$ and $F_{11} = F_{22} = -1$ ([6]). Moreover, for any $\theta \in [0, \pi/2]$,

$$x(u, v, t_1, t_2) = 2(u\cos\theta, u\sin t, v, 0, t_1, t_2)$$

defines a minimal (2+2)-dimensional θ-slant submanifold M (see [9] for more details). Consequently, from (5.14), we deduce that $\delta_M^{\mathcal{L}} = 4$.

Acknowledgements: The authors wish to express their gratitude to the referee of this paper for his/her valuable comments in order to improve it.

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