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Suborbital graphs for the group Γ^2

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Abstract

In this paper, we investigate suborbital graphs formed by the action of Γ^2 which is the group generated by the second powers of the elements of the modular group Γ on $\hat{\mathbb{Q}}$. Firstly, conditions for being an edge, self-paired and paired graphs are provided, then we give necessary and sufficient conditions for the suborbital graphs to contain a circuit and to be a forest. Finally, we examine the connectivity of the subgraph $F_{u,N}$ and show that it is connected if and only if $N \leq 2$.

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1. Introduction

Let PSL(2,$\mathbb{R}$) denote the group of all linear fractional transformations

$$T : z \to \frac{az+b}{cz+d}\ \text{, where } a, b, c \text{ and } d \text{ are real and } ad - bc = 1.$$

In terms of matrix representation, the elements of PSL(2,$\mathbb{R}$) correspond to the matrices

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1.$$

This is the automorphism group of the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im(z)} > 0\}$.

The modular group Γ=PSL$(2, \mathbb{Z})$, is the subgroup of PSL(2,$\mathbb{R}$) such that a, b, c and d are integers. It is generated by the matrices

$$U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad ; \quad V = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

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with defining relationships $U^2 = V^3 = I$, where I is the identity matrix. Γ is a Fuchsian group of signature $(0; 2, 3, \infty)$, so it is isomorphic to a free product $C_2 * C_3$.

Define Γ^m as the subgroup of Γ generated by the m^{th} powers of all elements of Γ. Especially, Γ^2 and Γ^3 have been studied extensively by Newman [13,14,15]. It turns out that,

$$\Gamma^2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : ab + bc + cd \equiv 0 \ (mod\ 2) \right\},$$

by Rankin [Eq. 1.7.1, 16]. From the equation $ab + bc + cd \equiv 0\ (mod\ 2)$, we see that at least one of the integers a, b, c, d must be even. Suppose first that $a = 2a_0$. Then using the determinant, we have that b and c are odd. So, d must be odd as well. Hence, we get the elements of Γ^2 as the matrices $\begin{pmatrix} 2a & b \\ c & d \end{pmatrix}$. Similarly, supposing $d = 2d_0$, we can get the elements of the form $\begin{pmatrix} a & b \\ c & 2d \end{pmatrix}$. Lastly, if a or d is not even, then both b and c will be even. To sum up, Γ^2 has three types of elements

$$\begin{pmatrix} 2a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & 2b \\ 2c & d \end{pmatrix}, \begin{pmatrix} a & b \\ c & 2d \end{pmatrix}$$

where b, c and d of the first, a and d of the second and a, b, c of the third matrix are odd.

1.1. Theorem. [13] *The group Γ^2 is the free product of two cyclic groups of order 3, and*

$$|\Gamma : \Gamma^2| = 2\ ,\ \ \Gamma = \Gamma^2 + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Gamma^2.$$

The elements of Γ^2 may be characterized by the requirement that the sum of the exponents of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ be divisible by 2.

The idea of a suborbital graph has been used mainly by finite group theorists. In [7], Jones, Singerman and Wicks showed that this idea is also useful in the study of the modular group, where they proved that the well-known Farey Graph is an example of a suborbital graph. Furthermore, they proved the following result:

Theorem A. The suborbital graph $G_{u,n}$ of Γ contains directed triangles if and only if $u^2 \pm u + 1 \equiv 0\ (mod\ n)$.

Morever they posed the conjecture: $G_{u,n}$ is a forest if and only if it contains no triangles, that is, if and only if $u^2 \pm u + 1 \not\equiv 0\ (mod\ n)$. Akbas proved in [2] that this conjecture is true. By similar arguments, we concern with suborbital graphs of Picard group $\mathbf{P}$, which is the subgroup of $\mathrm{PSL}(2, \mathbb{C})$ with entries coming from $\mathbb{Z}[i]$ in [3]. Since $\mathbb{Z}[i]$ is a unique factorization domain with finitely many units, our expectation was to find similar formulas. Consequently, theorem A was improved as

Theorem B. The suborbital graph $G_{u,N}$ of $\mathbf{P}$ contains directed triangles if and only if $\varepsilon^2 u^2 \mp \varepsilon u \pm 1 \equiv 0\ (mod\ N)$.

In this study, we will continue to investigate the combinatorial properties of these graphs for the group Γ^2. It is an important subgroup of Γ since all the groups Γ^m can be expressed in the terms of $\Gamma, \Gamma^2, \Gamma^3$. The purpose of this paper is to characterize all circuits in the suborbital graph and connectedness for Γ^2. As it can be seen from Section 3, we show that the main difference is in connectedness of related graphs.

2. The action of Γ^2 on $\hat{\mathbb{Q}}$

Every element of $\hat{\mathbb{Q}}$ can be represented as a reduced fraction $\frac{x}{y}$ with $x, y \in \mathbb{Z}$ and $(x, y) = 1$. This representation is not unique, because $\frac{x}{y} = \frac{-x}{-y}$. We represent ∞ as $\frac{1}{0} = \frac{-1}{0}$. The action of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on $\frac{x}{y}$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{x}{y} \longrightarrow \frac{ax+by}{cx+dy}.$$

Hence, the actions of a matrix on $\frac{x}{y}$ and on $\frac{-x}{-y}$ are identical. If the determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is 1 and $(x, y) = 1$, then $(ax + by, cx + dy) = 1$.

2.1. *Transitive Action.*

2.1. Lemma. (i) *The action of Γ^2 on $\hat{\mathbb{Q}}$ is transitive.*
(ii) *The stabilizer of a point is an infinite cyclic group.*

Proof. (i) Here we only prove the case that any element of the form $\frac{a}{2b}$ of $\hat{\mathbb{Q}}$ is sent ∞ by an element of Γ^2. The rest are similar. Let $\frac{a}{2b} \in \hat{\mathbb{Q}}$, $(a, 2b) = 1$. There exist integers x_0 and y_0 such that $ay_0 - 2bx_0 = 1$ (known as Bezout's identity [8]). Hence, we have that $T := \begin{pmatrix} a & x_0 \\ 2b & y_0 \end{pmatrix} \in \Gamma$. All solutions of the equation $ay - 2bx = 1$ are $x = x_0 + an$, $y = y_0 + 2bn$ for $n \in \mathbb{Z}$. If x_0 is odd, x would be even by taking n-odd. So, x_0 can be chosen as an even number. Hence, $T \in \Gamma^2$ and $T(\infty) = \frac{a}{2b}$ means that $\frac{a}{2b}$ is in the orbit of ∞.

(ii) By (i), since the stabilizers of any two points in $\hat{\mathbb{Q}}$ are conjugate in Γ^2, it is sufficient to consider the stabilizer Γ^2_∞ of ∞. It is clear that $\Gamma^2_\infty = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\rangle$. ∎

We remark that Lemma 2.1 (i) can be proven by using the signature of Γ^2 as well. There is a homomorphism $\theta : \Gamma \longrightarrow C_2 = \{e, \alpha\}$ defined by $\theta(U) = \alpha$, and $\theta(V) = e$. The kernel is Γ^2. By the permutation theorem [19], Γ^2 has signature $(0; 3, 3, \infty)$. It means that there is only one orbit, so the action is transitive.

2.2. *Imprimitive Action.* We now discuss the imprimitivity of the action of Γ^2 on $\hat{\mathbb{Q}}$. For this, let (G, Ω) be a transitive permutation group, consisting of a group G acting on a set Ω transitively. An equivalence relation $\approx$ on Ω is called *G-invariant* if, whenever $\alpha, \beta \in \Omega$ satisfy $\alpha \approx \beta$, then $g(\alpha) \approx g(\beta)$ for all $g \in G$. The equivalence classes are called blocks.

We call (G, Ω) *imprimitive* if Ω admits some G-invariant equivalence relation different from

(i) the identity relation, $\alpha \approx \beta$ if and only if $\alpha = \beta$;

(ii) the universal relation, $\alpha \approx \beta$ for all $\alpha, \beta \in \Omega$.

Otherwise, (G, Ω) is called *primitive*. These two relations are supposed to be trivial relations.

2.2. Lemma. [4] *Let (G, Ω) be a transitive permutation group. (G, Ω) is primitive if and only if G_α, the stabilizer of $\alpha \in \Omega$, is a maximal subgroup of G for each $\alpha \in \Omega$.*

From the above lemma we see that whenever, for some α, $G_\alpha \lneqq H \lneqq G$, then Ω admits some G-invariant equivalence relation other than the trivial one and the universal one.

Because of the transitivity, every element of Ω has the form $g(\alpha)$ for some $g \in G$. Thus one of the non-trivial G-invariant equivalence relations on Ω by H is given as follows:

$$g(\alpha) \approx g'(\alpha) \text{ if and only if } g' \in gH.$$

The number of blocks (equivalence classes) is the index $|G : H|$ and the block containing α is just the orbit $H(\alpha)$.

Let $N \in \mathbb{N}$ and let $\Gamma_0^2(N)$ be defined by

$$\Gamma_0^2(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^2 : c \equiv 0 \ (mod\ N) \right\}.$$

Then $\Gamma_0^2(N)$ is a subgroup of Γ^2. It is clear that $\Gamma_\infty^2 \leq \Gamma_0^2(N) \leq \Gamma^2$ for $N \in \mathbb{N}$ and $\Gamma_\infty^2 \lneq \Gamma_0^2(N) \lneq \Gamma^2$ for $N > 1$.

2.3. Lemma. $|\Gamma_0(N) : \Gamma_0^2(N)| = 2$. *In fact,*

$$\Gamma_0(N) = \begin{cases} \Gamma_0^2(N) \cup \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \Gamma_0^2(N), & N \text{ is odd} \\ \Gamma_0^2(N) \cup \begin{pmatrix} N+1 & -1 \\ N & 1 \end{pmatrix} \Gamma_0^2(N), & N \text{ is even} \end{cases} \blacksquare$$

Proof. First, we suppose that N is even. Let's show that $\Gamma_0^2(N) \cup \begin{pmatrix} N+1 & -1 \\ N & 1 \end{pmatrix} \Gamma_0^2(N) = \Gamma_0(N)$. We have that $T := \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$ with $ad - bcN = 1$. Here, a and d are odd. If b is even, T would be an element of $\Gamma_0^2(N)$. We suppose that b is odd. Hence, it can be written as $T = \begin{pmatrix} N+1 & -1 \\ N & 1 \end{pmatrix} \begin{pmatrix} x & y \\ cN & z \end{pmatrix}$. Then, we have that $\begin{pmatrix} 1 & 1 \\ -N & N+1 \end{pmatrix} \begin{pmatrix} a & b \\ cN & d \end{pmatrix} = \begin{pmatrix} x & y \\ cN & z \end{pmatrix}$. Let's say that

$$\underbrace{\begin{pmatrix} a + cN & b + d \\ -aN + cN(N+1) & -bN + dN(N+1) \end{pmatrix}}_{A} = \begin{pmatrix} x & y \\ cN & z \end{pmatrix}.$$

As $b + d$ is even, $A \in \Gamma_0^2(N)$.

Now, let N be odd. In this case, assume that b and c are even in T. Then a and d are odd. Hence, T is an element of $\Gamma_0^2(N)$. Moreover, it can be written as $T = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \begin{pmatrix} x & y \\ cN & z \end{pmatrix}$. As above, let's say that $\underbrace{\begin{pmatrix} a & b \\ (c-a)N & d - bN \end{pmatrix}}_{B} = \begin{pmatrix} x & y \\ cN & z \end{pmatrix}$. Since $d - bN$ is even, $B \in \Gamma_0^2(N)$. In the case: b-even and c-odd, it is clear that $B \in \Gamma_0^2(N)$. If a and d are even in the equation $ad - bcN = 1$, $B \in \Gamma_0^2(N)$ again. Finally if a is odd and d is even (or vice versa), the result is the same. Consequently, we obtain that $|\Gamma_0(N) : \Gamma_0^2(N)| = 2$. $\blacksquare$

Therefore, from the above constructed equivalence relation "$\approx$", we get Γ^2-invariant equivalence relation on $\hat{\mathbb{Q}}$ by $\Gamma_0^2(N)$. It is clear that, by Lemma 2.3, Γ^2 acts imprimitively on $\hat{\mathbb{Q}}$.

Let $v = \frac{r}{s}$ and $w = \frac{x}{y}$ be elements of $\hat{\mathbb{Q}}$. Because of the transitive action, we have that $v = g_1(\infty)$ and $w = g_2(\infty)$ for some elements $g_1, g_2 \in \Gamma^2$ of the form

$$g_1 := \begin{pmatrix} r & * \\ s & * \end{pmatrix}, \quad g_2 := \begin{pmatrix} x & * \\ y & * \end{pmatrix}.$$

From the relation

$$v \approx w \text{ if and only if } g_1^{-1} g_2 \in \Gamma_0^2(N),$$

we get

$$v \approx w \text{ if and only if } ry - sx \equiv 0 \ (mod\ N).$$

By our general discussion of imprimitivity, the number of blocks under $\approx$ is given by $\Psi(N) = \left|\Gamma^2 : \Gamma_0^2(N)\right|$. So the block of ∞ is obtained as

$$[\infty] := \left\{ \frac{x}{y} \in \hat{\mathbb{Q}} \mid y \equiv 0 \ (mod\ N) \right\}.$$

2.4. Lemma. $\Psi(N) = N \prod_{p|N} \left(1 + \dfrac{1}{p}\right)$ *where the product is over the distinct primes p dividing N.*

Proof. To calculate $\Psi(N)$ we use two decomposition of the index $\left|\Gamma : \Gamma_0^2(N)\right|$ as the following

$$|\Gamma : \Gamma^2||\Gamma^2 : \Gamma_0^2(N)| = |\Gamma : \Gamma_0(N)||\Gamma_0(N) : \Gamma_0^2(N)|.$$

Here, $|\Gamma : \Gamma^2| = 2$ and $|\Gamma : \Gamma_0(N)| = N \prod_{p|N} \left(1 + \frac{1}{p}\right)$ are well-known by [13,16] and [16,17] respectively. We prove that the index of $|\Gamma_0(N) : \Gamma_0^2(N)|$ is equal to 2 in Lemma 2.3. Writing these values in above equation, the result is obvious.

3. Suborbital Graphs for Γ^2 on $\hat{\mathbb{Q}}$

In[18], Sims introduced the idea of the suborbital graphs of a permutation group G acting on a set Δ , these are graphs with vertex-set Δ, on which G induces automorphisms. We summarise Sims'theory as follows:

Let (G, Δ) be transitive permutation group. Then G acts on $\Delta \times \Delta$ by $g(\alpha, \beta) = (g(\alpha), g(\beta))(g \in G, \alpha, \beta \in \Delta)$. The orbits of this action are called *suborbitals* of G. The orbit containing (α, β) is denoted by $O(\alpha, \beta)$. From $O(\alpha, \beta)$ we can form a *suborbital graph* $G(\alpha, \beta)$: its vertices are the elements of Δ, and there is a directed edge from γ to δ if $(\gamma, \delta) \in O(\alpha, \beta)$. A directed edge from γ to δ is denoted by $\gamma \to \delta$. If $(\gamma, \delta) \in O(\alpha, \beta)$, then we will say that there exists an edge $\gamma \to \delta$ in $G(\alpha, \beta)$. In this paper our calculation concerns Γ^2, so we can draw this edge as a hyperbolic geodesic in the upper half-plane $\mathbb{H}$, that is, as euclidean semi-circles or half-lines perpendicular to the real line.

The orbit $O(\beta, \alpha)$ is also a suborbital graph and it is either equal to or disjoint from $O(\alpha, \beta)$. In the latter case $G(\beta, \alpha)$ is just $G(\alpha, \beta)$ with the arrows reversed and we call, in this case, $G(\alpha, \beta)$ and $G(\beta, \alpha)$ *paired suborbital graphs.* In the former case $G(\alpha, \beta) = G(\beta, \alpha)$ and the graph consists of pairs of oppositely directed edges; it is convenient to replace each such pair by a single undirected edge, so that we have an undirected graph which we call *self paired.*

3.1. Definition. By a directed circuit in a graph we mean a sequence $v_1, v_2, \ldots, v_m$ of different vertices such that $v_1 \longrightarrow v_2 \longrightarrow \ldots \longrightarrow v_m \longrightarrow v_1$, where $m \geq 3$.

If $m = 3$, then the circuit, directed or not, is called a triangle.
If $m = 2$, then we will say the configuration $v_1 \longrightarrow v_2 \longrightarrow v_1$ is self paired.

A graph which contains no circuit is called a forest.

The above ideas are also described in a paper by Neumann [12] and in books by Tsuzuku [20] and by Biggs and White [4], the emphasis being on applications to finite groups. The reader is refereed to [1, 2, 3, 6, 7, 9, 10, 11] for some relevant previous work on suborbital graphs.

If $\alpha = \beta$, then $O(\alpha, \alpha) = \{(\gamma, \gamma) \mid \gamma \in \Delta\}$ is the diagonal of $\Delta \times \Delta$. The corresponding suborbital graph $G(\alpha, \alpha)$, called the trivial suborbital graph, is self-paired: it consists of a loop based at each vertex $\gamma \in \Delta$. We shall be mainly interested in the remaining non-trivial suborbital graphs.

Since Γ^2 acts transitively on $\hat{\mathbb{Q}}$, each suborbital contains a pair (∞, v) for some $v \in \hat{\mathbb{Q}}$; writing $v = \frac{u}{N}$, $(u, N) = 1$ and $N \geq 0$. We denote this suborbital by $O_{u,N}$ and the corresponding suborbital graph by $G_{u,N}$.

3.1. *Graph* $G_{u,N}$. If $v = \infty$, we would have the simplest suborbital graph, namely $G_{1,0} = G_{-1,0}$. Therefore, we can take $v \in \mathbb{Q}$. Let $v' = \frac{u'}{N'} \in \mathbb{Q}$. The necessary and sufficient condition for $O(\infty, v) = O(\infty, v')$ is that v and v' are in the same orbit of Γ^2_∞. Since Γ^2_∞ is generated by $z : v \to v + 2$, then $z\left(\frac{u}{N}\right) = \frac{u+2N}{N} = \frac{u'}{N'}$. Therefore, we have that $N = N'$ and $u \equiv u'$ $(mod\ 2N)$. Hence, $G_{u,N} = G_{u',N'}$ if and only if $N = N'$ and $u \equiv u'$ $(mod\ 2N)$. Consequently, for a fixed N there are $2\varphi(N)$ distinct suborbital graphs $G_{u,N}$ where $\varphi(N)$ is Euler's phi function.

3.2. Theorem. $\frac{r}{s} \to \frac{x}{y} \in G_{u,N}$ *if and only if*

(i) *If r is even, then $x \equiv \pm ur$ $(mod\ N)$, $y \equiv \pm us$ $(mod\ N)$, $y \not\equiv \pm us$ $(mod\ 2N)$ and $ry - sx = \mp N$.*
(ii) *If s is even, then $x \equiv \pm ur$ $(mod\ 2N)$, $y \equiv \pm us$ $(mod\ N)$ and $ry - sx = \mp N$.*
(iii) *If r and s are odd, then $x \equiv \pm ur$ $(mod\ N)$, $y \equiv \pm us$ $(mod\ 2N)$ and $ry - sx = \mp N$.*

Proof. (i) Let r be even. By the transitivity of Γ^2, without loss of generality, we assume that $\frac{r}{s} < \frac{x}{y}$ where all letters are positive integers. Thus, we have that $ry - sx < 0$. Since $\frac{r}{s} \to \frac{x}{y} \in G_{u,N}$, there exist some $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^2$ such that $T\left(\frac{1}{0}, \frac{u}{N}\right) = \left(\frac{r}{s}, \frac{x}{y}\right)$. As $ry - sx < 0$, the multiplication of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & N \end{pmatrix}$ is equal to $\begin{pmatrix} -r & x \\ -s & y \end{pmatrix}$ or $\begin{pmatrix} r & -x \\ s & -y \end{pmatrix}$. If the first case is valid, we have that $a = -r$, $c = -s$, $au + bN = x$ and $cu + dN = y$. That is, $x \equiv -ur$ $(mod\ N)$ and $y \equiv -us$ $(mod\ N)$. Since r is even, then a is also even. To have $T \in \Gamma^2$, d must be odd. From $-us + dN = y$, we have that $y \not\equiv \pm us$ $(mod\ 2N)$.

(ii) Suppose s is even. In a similar way, we see that b and c must be even because $T\left(\frac{1}{0}\right) = \frac{-r}{-s} = \frac{a}{c}$. As in (i), we may assume that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & N \end{pmatrix} = \begin{pmatrix} -r & x \\ -s & y \end{pmatrix}$. Hence, we have that $a = -r$, $c = -s$, $au+bN = x$, $cu+dN = y$ and $ry-sx = -N$. That is, $-ur + bN = x$ and $-us + dN = y$. Since b is even, we have that $x \equiv -ur$ $(mod\ 2N)$ and $y \equiv -us$ $(mod\ N)$.

(iii) Let r and s be odd. With similar argument, it can be seen that d must be even. From the same matrix equation in (ii), we obtain that $x \equiv -ur$ $(mod\ N)$ and $y \equiv -us$ $(mod\ 2N)$.

In the opposite direction, we shall prove (i) for minus sign. Suppose that r is even, $x \equiv -ur\ (mod\ N)$, $y \equiv -us\ (mod\ N)$, $y \not\equiv -us\ (mod\ 2N)$ and $ry - sx = -N$. In this case, there exist integers b, d such that $x = -ur - bN$, $y = -us - dN$. So, it is clear that $\begin{pmatrix} -r & -b \\ -s & -d \end{pmatrix} \in \Gamma^2$ which means $\frac{r}{s} \to \frac{x}{y} \in G_{u,N}$. Because $-N = ry - sx = r(-us - dN) - s(-ur - bN)$. This implies $-rd + sb = 1$. As r is even, d must be even. Otherwise, it contradicts our hypothesis. With similar argument, we obtain the elements of Γ^2 which are $\begin{pmatrix} -r & 2b \\ -s & d \end{pmatrix}$ and $\begin{pmatrix} -r & -b \\ -s & -2d \end{pmatrix}$ for (ii) and (iii) respectively. ∎

3.3. Theorem. *All suborbital graphs for Γ^2 on $\hat{\mathbb{Q}}$ are paired.*

Proof. Because of the transitivity of Γ^2, it is sufficient to show that $G\left(\infty, \frac{u}{N}\right) \neq G\left(\frac{u}{N}, \infty\right)$. It means that there is no $T \in \Gamma^2$ which sends the pair $\left(\infty, \frac{u}{N}\right)$ to the pair $\left(\frac{u}{N}, \infty\right)$. On the contrary, assume that $T(\infty) = \frac{u}{N}$ and $T\left(\frac{u}{N}\right) = \infty$. By comparing the determinants, we have that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & N \end{pmatrix} = \begin{pmatrix} -u & 1 \\ -N & 0 \end{pmatrix} \text{ or } \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & N \end{pmatrix} = \begin{pmatrix} u & -1 \\ N & 0 \end{pmatrix}.$$

In the first case, we obtain $a = -u$, $c = -N$, $au + bN = 1$ and $cu + dN = 0$. That is, $d = u$ and $u^2 = -1 + bN$. Taking $T = \begin{pmatrix} -u & b \\ -N & u \end{pmatrix}$ we see that the only case for T to be an element of Γ^2 is that N and b must be even. Since $u^2 = -1 + bN$, then $u^2 \equiv -1\ (mod\ bN)$. As N and b are even, $u^2 \equiv -1\ (mod\ 4)$ which has no solution. For the second case, taking $T = \begin{pmatrix} u & b \\ N & -u \end{pmatrix}$, similar contradiction is obtained.

3.4. Corollary. *There are no self-paired suborbital graphs for Γ^2 on $\hat{\mathbb{Q}}$.*

In section 2 we introduced, for each integer N, a Γ^2-invariant equivalence relation $\underset{N}{\approx}$ on $\hat{\mathbb{Q}}$, with $\frac{r}{s} \underset{N}{\approx} \frac{x}{y}$ if and only if $ry - sx \equiv 0\ (mod\ N)$. If $\frac{r}{s} \to \frac{x}{y}$ in $G_{u,N}$, then Theorem 3.2 implies that $ry - sx = \pm N$, so $\frac{r}{s} \underset{N}{\approx} \frac{x}{y}$. Thus, each connected component of $G_{u,N}$ lies in a single block for $\underset{N}{\approx}$, of which there are $\Psi(N)$, so we have:

3.5. Corollary. *The graph $G_{u,N}$ is a disjoint union of $\Psi(N)$ subgraphs.*

3.2. *Subgraph* $F_{u,N}$. We represent the subgraph of $G_{u,N}$ whose vertices form the block $[\infty] = \{x/y \in \hat{\mathbb{Q}} \mid y \equiv 0\ (mod\ N)\}$ by $F_{u,N}$.

3.6. Corollary. *The graph $G_{u,N}$ consists of $\Psi(N)$ disjoint copies of $F_{u,N}$.*

Proof. The vertices of each subgraph form a single block with respect to the Γ^2-invariant equivalence relation $\underset{N}{\approx}$ defined by $ry - sx \equiv 0\ (mod\ N)$. Therefore, if $x_1 \to x_2$ is an edge in the subgraph $F_{u,N}$, $T(x_1) \to T(x_2)$ is also an edge in any other subgraph with $T \in \Gamma^2$ because of the transitivity of Γ^2 on $\hat{\mathbb{Q}}$.

Now, Theorem 3.2 immediately gives:

3.7. Theorem. $\frac{r}{s} \to \frac{x}{y} \in F_{u,N}$ *if and only if*

(i) *If r is even, then $x \equiv \pm ur\ (mod\ N)$, $y \equiv \pm us\ (mod\ N)$, $y \not\equiv \pm us\ (mod\ 2N)$ and $ry - sx = \mp N$.*

(ii) *If s is even, then $x \equiv \pm ur\ (mod\ 2N)$, $y \equiv \pm us\ (mod\ N)$ and $ry - sx = \mp N$.*

(iii) *If r and s are odd, then $x \equiv \pm ur\ (mod\ N)$, $y \equiv \pm us\ (mod\ 2N)$ and $ry - sx = \mp N$.*

3.8. Theorem. *$\Gamma_0^2(N)$ permutes the vertices and the edges of $F_{u,N}$ transitively.*

Proof. Let v, w be any vertices of $F_{u,N}$. Since Γ^2 acts on $\hat{\mathbb{Q}}$ transitively, there exist $T \in \Gamma^2$ such that $T(v) = w$. Taking $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $v = \frac{k_1}{l_1 N}$ and $w = \frac{k_2}{l_2 N}$ we see that $N|c$. It means that $\Gamma_0^2(N)$ permutes the vertices of $F_{u,N}$.

Let $\frac{x_1}{y_1 N} \xrightarrow{e_1} b_1$ and $\frac{x_2}{y_2 N} \xrightarrow{e_2} b_2$ be any edges of $F_{u,N}$. We can give following diagram:

$$\begin{array}{ccc} \left(\frac{1}{0}, \frac{u}{N}\right) & \xrightarrow{T_2} & \left(\frac{x_2}{y_2 N}, b_2\right) \\ \downarrow T_1 & \nearrow T_2 \circ T_1^{-1} & \\ \left(\frac{x_1}{y_1 N}, b_1\right) & & \end{array}$$

By this representation, we have $T_1 = \begin{pmatrix} x_1 & * \\ y_1 N & * \end{pmatrix}$ and $T_2 = \begin{pmatrix} x_2 & * \\ y_2 N & * \end{pmatrix}$. Since $T_2 \circ T_1^{-1}$ has the form $\begin{pmatrix} * & * \\ kN & * \end{pmatrix}$ for some integer k, then $T := T_2 \circ T_1^{-1} \in \Gamma_0^2(N)$. It is clear that $T\left(\frac{x_1}{y_1 N}\right) = \frac{x_2}{y_2 N}$ and $T(b_1) = b_2$. Since T is an element of a group of hyperbolic isometries of $\mathbb{H}$, geodesics are sent to geodesics under its action. So, T transform the edges e_1 to e_2. Consequently, $\Gamma_0^2(N)$ permutes the edges of $F_{u,N}$.

3.9. Lemma. *There is an isomorphism $F_{u,N} \longrightarrow F_{-u,N}$ given by $v \longrightarrow -v$.*

Proof. It is clear that $v \longrightarrow -v$ is one-to-one and onto. Let's show that the structure is preserved. Here, it means that if $a \to b \in F_{u,N}$, then $-a \to -b \in F_{-u,N}$. Suppose that $\frac{r}{s} \to \frac{x}{y} \in F_{u,N}$ and r is even. By Theorem 3.7(i), taking $\frac{r}{s} < \frac{x}{y}$, we have that $x \equiv -ur \ (mod\ N)$, $y \equiv -us \ (mod\ N)$, $y \not\equiv -us \ (mod\ 2N)$ and $ry - sx = -N$. Since $\frac{r}{s} < \frac{x}{y}$, then $\frac{-r}{s} > \frac{-x}{y}$. Taking $-x \equiv (-u)(-r) \ (mod\ N)$, $y \equiv (-u)s \ (mod\ N)$, $y \not\equiv (-u)s \ (mod\ 2N)$ and $-ry + sx = N$, we have that $\frac{-r}{s} \to \frac{-x}{y} \in F_{u,N}$. For other conditions, the rest are similar.

3.10. Lemma. *If $M|N$, then there is a homomorphism $F_{u,N} \longrightarrow F_{-u,M}$ given by $v \longrightarrow -Nv/M$.*

Proof. We suppose that $\frac{r}{sN}$, $\frac{x}{yN}$ are adjacent vertices in $F_{u,N}$ and $\frac{r}{sN} < \frac{x}{yN}$ and that is written as $\frac{r}{sN} \xrightarrow{<} \frac{x}{yN} \in F_{u,N}$. If r is even, then $x \equiv -ur \ (mod\ N)$, $yN \equiv -usN \ (mod\ N)$, $yN \not\equiv -us \ (mod\ 2N)$ and $ry - sx = -1$. Since $M|N$, $x \equiv -ur \ (mod\ M)$, $yM \equiv -usM \ (mod\ M)$, $yM \not\equiv -us \ (mod\ 2M)$. $ry - sx = -1$ is also true for M. For other conditions, the rest are similar.

3.11. Theorem. *$F_{u,N}$ contains directed triangles if and only if $u^2 \mp u + 1 \equiv 0 \ (mod\ N)$.*

Proof. Suppose that $F_{u,N}$ contains a directed triangle. Because of the transitive action, the form of directed triangle can be taken as $\infty \to \frac{u}{N} \xrightarrow{<} \frac{r}{N} \to \infty$ for some integer r. First, let u be even. From the second edge, we have $u - r = -1$ and $r \equiv -u^2 \ (mod\ N)$ by Theorem 3.2. So, we obtain $u^2 + u + 1 \equiv 0 \ (mod\ N)$. Similarly, if $\frac{u}{N} \xrightarrow{>} \frac{r}{N}$, then we see that $u^2 - u + 1 \equiv 0 \ (mod\ N)$. Now, N is even. By applying Theorem 3.2 to the second edge, we have $u - r = -1$ and $r \equiv -u^2 \ (mod\ 2N)$, giving $u^2 + u + 1 \equiv 0 \ (mod\ 2N)$. It is impossible, because there is no solution for this equivalence. Finally, suppose that u, N are odd. Again, from the second edge, we have $u - r = -1$ and $r \equiv -u^2 \ (mod\ N)$, giving $u^2 + u + 1 \equiv 0 \ (mod\ N)$. If $\frac{u}{N} \xrightarrow{>} \frac{r}{N}$, it would be $u^2 - u + 1 \equiv 0 \ (mod\ N)$. Combining all of the equivalences, we obtain $u^2 \mp u + 1 \equiv 0 \ (mod\ N)$.

Conversely, if $u^2 \mp u + 1 \equiv 0 \ (mod\ N)$, we see that either $u + 1 \equiv -u^2 \ (mod\ N)$ or $-u + 1 \equiv -u^2 \ (mod\ N)$. Theorem 3.2. implies that there is an edge $\frac{u}{N} \to \frac{u+1}{N}$ with

$\frac{u}{N} < \frac{u+1}{N}$ in $F_{u,N}$ or $\frac{u}{N} \to \frac{u-1}{N}$ with $\frac{u}{N} > \frac{u+1}{N}$ in $F_{u,N}$. Consequently, there is a directed triangle $\infty \to \frac{u}{N} \to \frac{u \pm 1}{N} \to \infty$ in $F_{u,N}$. ■

Let us give some examples. For u, N-odd, $\frac{1}{0} \to \frac{3}{13} \to \frac{4}{13} \to \frac{1}{0}$ or $\frac{1}{13} \to \frac{10}{9 \cdot 13} \to \frac{9}{8 \cdot 13} \to \frac{1}{13}$ is a directed triangle in $F_{3,13}$. For u-even and N-odd, $\frac{1}{0} \to \frac{2}{7} \to \frac{3}{7} \to \frac{1}{0}$ or $\frac{1}{7} \to \frac{5}{4 \cdot 7} \to \frac{4}{3 \cdot 7} \to \frac{1}{7}$ is a directed triangle in $F_{2,7}$. For N-even, we know that there is no triangle.

Observation. We know that there is no triangle in $F_{u,2N_0}$ for N-even by Theorem 3.11. Because of the relationships between elliptic elements with circuits, our expectation is that there is no elliptic element of order 3 of the form $\begin{pmatrix} u & 2b \\ 2N_0 & d \end{pmatrix} \in \Gamma^2$. Indeed, being an elliptic element of order 3, it is well-known that $u + d = \pm 1$. Taking determinant of $\begin{pmatrix} 1-d & 2b \\ 2N_0 & d \end{pmatrix}$, we have $d - d^2 - 4bN_0 = 1$. It is clear that there is no solution for $d - d^2 \equiv 1 \ (mod\ 4)$. ■

On the other hand, we know that the suborbital graph for modular group is a forest if and only if it contains no triangles [2]. Using this fact, we can give the following result;

3.12. Corollary. *The graph $G_{u,N}$ is a forest if and only if $u^2 \pm u + 1 \not\equiv 0 \ (mod\ N)$.*

3.3. *Connectedness*. In this last section, we examine the connectedness of $F_{u,N}$.

3.13. Definition. A subgraph K of $G_{u,N}$ is called connected if any pair of its vertices can be joined by a path in K.

3.14. Theorem. *The subgraphs $F_{0,1}$ and $F_{1,1}$ are connected.*

Proof. Here, to see the situation better, we write the edge conditions for $F_{0,1}$ and $F_{1,1}$ by Theorem 3.2 explicitly.

Case $F_{0,1}$: $\frac{r}{s} \to \frac{x}{y} \in F_{0,1}$ if and only if

(i) If r-even, then y-odd and $ry - sx = \mp 1$.
(ii) If s-even, then x-even and $ry - sx = \mp 1$.
(iii) If r, s-odd, then y-even and $ry - sx = \mp 1$.

We will show that each vertex $\frac{a}{b}$ of $F_{0,1}$ can be joined to ∞ by a path in $F_{0,1}$. It is clear for $b = 1$. Since $(a, b) = 1$, we can write the equation $ad - bc = -1$ by Bezout's identity. For this pair (c, d) satisfying the equation we claim that $\frac{a}{b}$ can be joined with $\frac{c}{d}$ by above edge condition.

Subcase1. Suppose a-even. By the equation we have that b, c must be odd and there are two possibilities for d. If d-odd, then $\frac{a}{b} \xrightarrow{i} \frac{c}{d}$ (means that we have $\frac{c}{d} \to \frac{a}{b}$ by (i)). If d-even, then $\frac{c}{d} \xrightarrow{ii} \frac{a}{b}$.

Subcase2. Let b-even. By the equation we have that a, d must be odd and there are two possibilities for c. If c-odd, then $\frac{c}{d} \xrightarrow{iii} \frac{a}{b}$. If d-even, then $\frac{a}{b} \xrightarrow{ii} \frac{c}{d}$.

Subcase3. Assume that a-odd and b-odd. By the equation it is impossible that c, d are odd or even at once, so there are two possibilities. If c-odd and d-even, then $\frac{a}{b} \xrightarrow{iii} \frac{c}{d}$. If c-even and d-odd, then $\frac{c}{d} \xrightarrow{i} \frac{a}{b}$.

Consequently $F_{0,1}$ is connected.

Case $F_{1,1}$: $\frac{r}{s} \to \frac{x}{y} \in F_{1,1}$ if and only if

(i) If r-even, then y-even and $ry - sx = \mp 1$.

(ii) If s-even, then x-odd and $ry - sx = \mp 1$.
(iii) If r, s-odd, then y-odd and $ry - sx = \mp 1$.

Taking a vertex $\frac{a}{b}$ in $F_{1,1}$, there exists the equation $ad - bc = -1$ by Bezout's identity. We shall show that $\frac{a}{b}$ is adjacent to vertex $\frac{c}{d}$ in $F_{1,1}$.

Subcase1. Suppose a-even. By the equation we have that b, c must be odd and there are two possibilities for d. If d-odd, then $\frac{c}{d} \xrightarrow{iii} \frac{a}{b}$. If d-even, then $\frac{a}{b} \xrightarrow{i} \frac{c}{d}$.

Subcase2. Let b-even. By the equation we have that a, d must be odd and there are two possibilities for c. If c-odd, then $\frac{a}{b} \xrightarrow{ii} \frac{c}{d}$. If c-even, then $\frac{c}{d} \xrightarrow{i} \frac{a}{b}$.

Subcase3. Assume that a-odd and b-odd. By the equation it is impossible that c, d are odd or even at once, so there are two possibilities. If c-odd and d-even, then $\frac{c}{d} \xrightarrow{ii} \frac{a}{b}$. If c-even and d-odd, then $\frac{a}{b} \xrightarrow{iii} \frac{c}{d}$.

Consequently, $F_{1,1}$ is connected.

3.15. Theorem. *The subgraphs $F_{1,2}$ and $F_{3,2}$ are connected.*

Proof. We shall show that each vertex $v = \frac{a}{2b}$ $(b \geq 1)$ of $F_{1,2}$ is joined to ∞ by a path in $F_{1,2}$. Since the pattern is periodic with period 2, we can show by induction on b. If $b = 1$, then $v = \frac{a}{2}$ can be joined with ∞. If $a = 1$, it is clear that $\frac{1}{0} \rightarrow \frac{1}{2}$. If $a = 1$, then $\frac{3}{2} \rightarrow \frac{1}{0}$ because $1 \equiv -3$ $(mod\ 4)$ and $3 \cdot 0 - 2 \cdot 1 = -2$. If $a = 5$, then $\frac{1}{0} \rightarrow \frac{5}{2}$. The same holds for the rest periodically. So we can assume that $b \geq 2$.

To complete the proof, we show that v is adjacent to a vertex $w = \frac{a}{2b_1}$ with $b_1 < b$. It means that, w is connected by a path to ∞, and hence so is v. As $(a, b) = 1$, there exist integers c, d such that $ad - bc = 1$. For some $k \in \mathbb{Z}$, replacing c and d by $c + ka$ and $d + kb$, we can suppose $0 < d < b$.

(i) If c is odd, then $w = \frac{c}{2d}$ can be joined with $\frac{a}{2b}$. Indeed, $\frac{a}{2b} \xrightarrow{>} \frac{c}{2d}$ gives that $a \cdot 2d - c \cdot 2b = 2$ and $c \equiv a$ $(mod\ 4)$. If $c \not\equiv a$ $(mod\ 4)$, taking $c \equiv -a$ $(mod\ 4)$ we obtain $\frac{a}{2b} \xleftarrow{<} \frac{c}{2d}$ by $2bc - 2ad = -2$. Hence, if c is odd, $\frac{a}{2b}$ is adjacent to $\frac{c}{2d}$ in $F_{1,2}$.

(ii) If c is even, then $a - c$ is odd. As $0 < b - d < b$, we can take $w = \frac{a-c}{2(b-d)}$, adjacent to $\frac{a}{2b}$ because $2(bc - cd) = -2$. Here, if $2a - c \not\equiv 0$ $(mod\ 4)$, then we have that $a - c \equiv a$ $(mod\ 4)$ and $2(ad - bc) = 2$.

Consequently, $F_{1,2}$ is connected. By the isomorphism $\underset{v}{F_{1,2}} \underset{\rightarrow}{\longrightarrow} \underset{-v}{F_{-1,2}} = F_{3,2}$, $F_{3,2}$ is also connected.

3.16. Corollary. *All graphs $F_{u,2}$ are connected.*

3.17. Corollary. *The graph $G_{u,2}$ has $2 \cdot \psi(2) = 6$ connected components. Its blocks are $[\infty], [1], [0]$. The connected components of $[\infty]$ are $F_{1,2}$ and $F_{3,2}$.*

3.18. Theorem. *The subgraphs $F_{1,3}$, $F_{2,3}$, $F_{4,3}$ and $F_{5,3}$ are not connected.*

Proof. It is sufficient to study with $F_{1,3}$ and $F_{2,3}$. Because there is an isomorphism from $F_{1,3}(F_{2,3})$ to $F_{5,3}(F_{4,3})$ respectively.
Case $F_{1,3}$: If $F_{1,3}$ is connected, then each vertex $v = \frac{a}{3b}$ would be joined to ∞. We shall show that no vertices of $F_{1,3}$ where $1 < v < 2$ are adjacent to ∞. Further, we assert that there is no such a vertex v adjacent to vertices outside this interval. Of course, there is at least some vertex of $F_{1,3}$ in this strip. Suppose $\frac{2}{3} \leq \frac{c}{3d} < 1 < \frac{a}{3b} < 2$. Then we have $\frac{c}{d} < 3 < \frac{a}{b}$. This is impossible because $cd - ad = -1$. Similarly, if $1 < \frac{k}{3l} < \frac{f}{3e} \leq \frac{7}{3}$, then $\frac{k}{l} < 4 < \frac{f}{e}$ contradicts $ke - lf = -1$. It means that no vertices of $F_{1,3}$ between 1 and 2 are adjacent to ∞ and that $F_{1,3}$ is not connected.

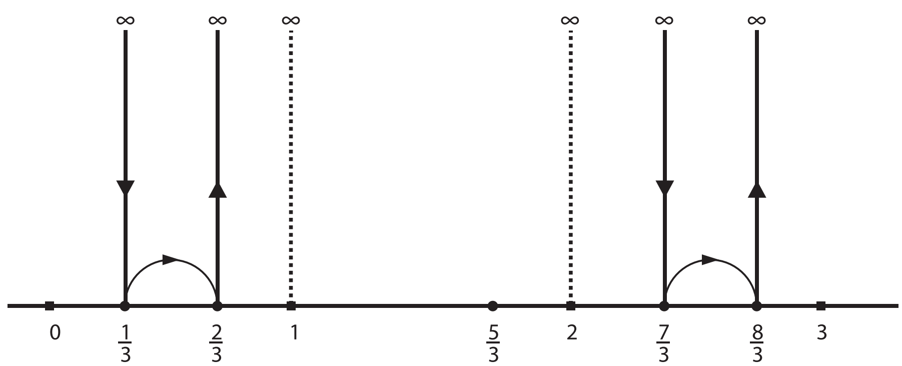

Figure 1. The subgraph $F_{1,3}$

Case $F_{2,3}$: As above, let's show that no vertices of $F_{2,3}$ between $\frac{3}{2}$ and 2 are adjacent to vertices outside this interval. Suppose that $1 \leq \frac{x}{3y} < \frac{3}{2} < \frac{a}{3b} < 2$ and $\frac{x}{3y} \overset{<}{\longrightarrow} \frac{a}{3b} \in F_{2,3}$. Then we have that $\frac{x}{y} < \frac{9}{2} < \frac{a}{b}$ and $xb - ay = -1$. By [7], we obtain that $x = 4$, $y = 1$, $a = 5$ and $b = 1$. But $\frac{4}{3} \to \frac{5}{3}$ is not in $F_{2,3}$. If $\frac{2}{3} < \frac{x}{3y} < 2 < \frac{a}{3b} < \frac{8}{3}$ and $\frac{x}{3y} \overset{<}{\longrightarrow} \frac{a}{3b} \in F_{2,3}$, then we would have $\frac{x}{y} < 6 < \frac{a}{b}$ and $xb - ay = -1$. It is impossible because of well-known Farey sequence. Consequently, $F_{2,3}$ is not connected.

3.19. Corollary. *All graphs $F_{u,3}$ are not connected.*

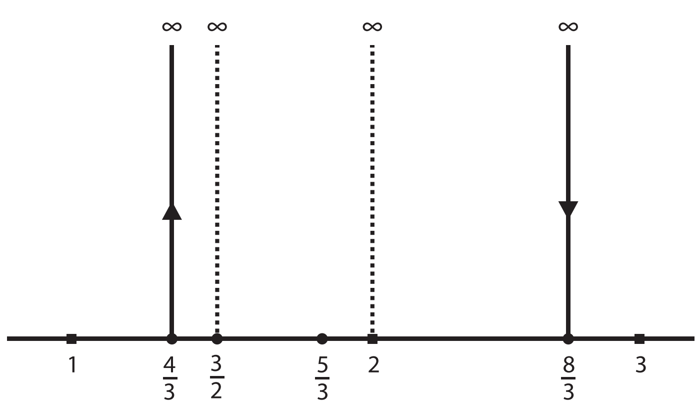

Figure 2. The subgraph $F_{2,3}$

3.20. Theorem. *The subgraphs $F_{1,4}$, $F_{3,4}$, $F_{5,4}$ and $F_{7,4}$ are not connected.*

Proof. As remarked in the proof of Theorem 3.18, it is sufficient to study with $F_{1,4}$ and $F_{3,4}$.

Case $F_{1,4}$: We will show that no vertices in $F_{1,3}$ between $\frac{1}{2}$ and 1 are adjacent to vertices outside this interval. Suppose $\frac{1}{4} \leq \frac{a}{4b} < \frac{1}{2} < \frac{x}{4y} < 1$. Then we have $\frac{a}{b} < 2 < \frac{x}{y}$. This is impossible because $ay - bx = -1$. Similarly, if $\frac{a}{4b} < 1 < \frac{x}{4y} \leq \frac{7}{4}$, then $\frac{a}{b} < 4 < \frac{x}{y} < 7$ is a contradiction. So $F_{1,4}$ is not connected.

Case $F_{3,4}$: As above, it is seen that no vertices of $F_{3,4}$ between 1 and 2 are adjacent to vertices outside this interval. Consequently, $F_{3,4}$ is not connected.

3.21. Theorem. *The subgraph $F_{u,N}$ is connected if and only if $N \leq 2$.*

Proof. If $F_{u,N}$ is connected, we know that $N \leq 4$ by [7]. For $N = 3, 4$, we proved that $F_{u,N}$ is not connected by Theorem 3.18 and 3.20. Conversely, if $N \leq 2$, the result immediately follows from Theorem 3.14 and 3.15. ∎

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