Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 48 (1) (2019), 28 – 44
DOI : 10.15672/HJMS.2017.506

Conformal slant submersions

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Abstract

As a generalization of conformal holomorphic submersions and conformal anti-invariant submersions, we introduce a new conformal submersion from almost Hermitian manifolds onto Riemannian manifolds, namely conformal slant submersions. We give examples and find necessary and sufficient conditions for such maps to be harmonic morphism. We also investigate the geometry of foliations which are arisen from the definition of a conformal submersion and obtain a decomposition theorem on the total space of a conformal slant submersion. Moreover, we find necessary and sufficient conditions of a conformal slant submersion to be totally geodesic.

Mathematics Subject Classification (2010). Primary 53C15; Secondary 53C43

Keywords. Riemannian submersion, slant-submersion, conformal submersion, conformal slant-submersion

1. Introduction

A submanifold of a complex manifold is a complex (invariant) submanifold if the tangent space of the submanifold at each point is invariant with respect to the almost complex structure of the Kähler manifold. Besides complex submanifolds of a complex manifold, there is another important class of submanifolds called totally real submanifolds. A totally real submanifold of a complex manifold is a submanifold of such that the almost complex structure of ambient manifold carries the tangent space of the submanifold at each point into its normal space. Many authors have studied totally real submanifolds in various ambient manifolds and many interesting results were obtained, see ([45], page: 199) for a survey on all these results. As a generalization of holomorphic and totally real submanifolds, slant submanifolds were introduced by Chen in [13]. We recall that the submanifold M is called slant [14] if for any $p \in M$ and any $X \in T_pM$, the angle between JX and T_pM is a constant $\theta(X) \in [0, \frac{\pi}{2}]$, i.e, it does not depend on the choice of $p \in M$ and $X \in T_pM$. It follows that invariant and totally real immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively.

On the other hand, Riemannian submersions between Riemannian manifolds were studied by O'Neill [37] and Gray [25]. Since then Riemannian submersions have been an effective tool to obtain new manifolds and compare certain manifolds within differential

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Received: 20.06.2017; Accepted: 03.08.2017

geometry, see [8], [12] and [21]. It is also known that Riemannian submersions have many applications in different areas such as Kaluza-Klein theory [22], [10], statistical machine learning processes [46], medical imaging [36], statistical analysis on manifolds [9] and the theory of robotics [3]. As analogue of holomorphic submanifolds, holomorphic submersions were introduced by Watson [44] in seventies by using the notion of almost complex map. This notion has been extended to other manifolds, see [21] for holomorphic submersions and their extensions to other manifolds. Although holomorphic submersions have been studied widely, however this research area is still an active research area, see a recent paper [43]. The main property of such maps is that the horizontal distribution and the vertical distribution of holomorphic submersions are invariant with respect to the almost complex map of the total manifold. Thus holomorphic submersions include only those submersions whose vertical distribution is invariant under the almost complex structure of the total manifold. Therefore, the second author of the present paper considered a new submersion defined on an almost Hermitian manifold such that the vertical distribution is anti-invariant with respect to almost complex structure [41]. He showed that such submersions have rich geometric properties and they are useful for investigating the geometry of the total space. This new class of submersions called anti-invariant submersions can be seen as an analogue of totally real submanifolds in the submersion theory. As a generalization of anti-invariant submersions, slant submersions were defined in [42] and it is shown that such maps are useful for obtaining new conditions for harmonicity, see also [4,5,7,18–20,24,28–32,34,35,38] and [40] for new submersions in other total spaces.

As a generalization of Riemannian submersions, horizontally conformal submersions are defined as follows [6]: Suppose that (M, g_M) and (B, g_B) are Riemannian manifolds and $F : M \longrightarrow B$ is a smooth submersion, then F is called a horizontally conformal submersion, if there is a positive function λ such that

$$\lambda^2 g_M(X, Y) = g_B(F_*X, F_*Y)$$

for every $X, Y \in \Gamma((kerF_*)^\perp)$. It is obvious that every Riemannian submersion is a particular horizontally conformal submersion with $\lambda = 1$. One can see that Riemannian submersions are very special maps comparing with conformal submersions. We note that horizontally conformal submersions are special horizontally conformal maps which were introduced independently by Fuglede [23] and Ishihara [33]. We also note that a horizontally conformal submersion $F : M \longrightarrow B$ is said to be horizontally homothetic if the gradient of its dilation λ is vertical, i.e.,

$$\mathcal{H}(grad\lambda) = 0 \tag{1.1}$$

at $p \in M$, where $\mathcal{H}$ is the projection on the horizontal space $(kerF_*)^\perp$. Although conformal maps do not preserve distance between points contrary to isometries, they preserve angles between vector fields. This property enables one to transfer certain properties of a manifold to another manifold by deforming such properties.

As a generalization of holomorphic submersions, conformal holomorphic submersions were studied by Gudmundsson and Wood [27]. They obtained necessary and sufficient conditions for conformal holomorphic submersions to be a harmonic morphism, see also [15–17] for the harmonicity of conformal holomorphic submersions.

Recently, we introduced conformal anti-invariant submersions [2] from almost Hermitian manifolds onto Riemannian manifolds, as a generalization of anti-invariant submersions, and investigated the geometry of such submersions. (see also: [1]) We showed that the geometry of such submersions are different from their counterparts anti-invariant submersions and semi-invariant submersions. In this paper, we study conformal slant submersions as a generalization of both conformal holomorphic submersions and conformal anti-invariant submersions and investigate the geometry of the total space and the base space for the existence of such submersions.

The paper is organized as follows. In the second section, we present the basic information needed for this paper. In section 3, we give definition of conformal slant submersions from almost Hermitian manifolds onto Riemannian manifolds, provide examples and give a sufficient condition for conformal slant submersions to be harmonic. We also investigate the geometry of leaves of $(kerF_*)^\perp$ and $(kerF_*)$. Moreover we obtain a decomposition theorem on the total space of a conformal slant submersion. Finally, we give necessary and sufficient conditions for a conformal slant submersion to be totally geodesic.

2. Preliminaries

In this section, we define almost Hermitian manifolds, recall the notion of (horizontally) conformal submersions between Riemannian manifolds and give a brief review of basic facts of (horizontally) conformal submersions.

Let (M, g_M) be an almost Hermitian manifold. This means [45] that M admits a tensor field J of type (1,1) on M such that, $\forall X, Y \in \Gamma(TM)$, we have

$$J^2 = -I, \quad g_M(X, Y) = g_M(JX, JY). \tag{2.1}$$

An almost Hermitian manifold M is called Kähler manifold if

$$(\nabla_X J)Y = 0, \ \forall X, Y \in \Gamma(TM), \tag{2.2}$$

where ∇ is the Levi-Civita connection on M. Conformal submersions belong to a wide class of conformal maps that we are going to recall their definition, but we will not study such maps in this paper.

Definition 2.1. ([6]). Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds, and let $x \in M$. Then φ is called horizontally weakly conformal or semi conformal at x if either

(i) $d\varphi_x = 0$, or
(ii) $d\varphi_x$ maps the horizontal space $\mathcal{H}_x = (ker(d\varphi_x))^\perp$ conformally onto $T_{\varphi(x)}N$, i.e., $d\varphi_x$ is surjective and there exists a number $\Lambda(x) \neq 0$ such that

$$h(d\varphi_x X, d\varphi_x Y) = \Lambda(x) g(X, Y) \ (X, Y \in \mathcal{H}_x). \tag{2.3}$$

We shall call a point x of type (i) in Definition 2.1 *critical point*. Also we shall call a point of type (ii) a *regular point*. At a critical point, $d\varphi_x$ has rank 0; at a regular point, $d\varphi_x$ has rank n and φ is submersion. The number $\Lambda(x)$ is called the *square dilation* (of φ at x); it is necessarily non-negative; its square root $\lambda(x) = \sqrt{\Lambda(x)}$ is called the dilation (of φ at x). The map φ is called *horizontally weakly conformal* or *semi conformal* (on M) if it is horizontally weakly conformal at every point of M. It is clear that if φ has no critical points, then we call it a (*horizontally*) conformal submersion.

Next, we recall the following definition from [26]. Let $\pi : M \longrightarrow N$ be a submersion. A vector field E on M is said to be projectable if there exists a vector field $\check{E}$ on N, such that $d\pi(E_x) = \check{E}_{\pi(x)}$ for all $x \in M$. In this case E and $\check{E}$ are called $\pi-$ related. A horizontal vector field Y on (M, g) is called basic, if it is projectable. It is well known fact, that if $\check{Z}$ is a vector field on N, then there exists a unique basic vector field Z on M, such that Z and $\check{Z}$ are $\pi-$ related. The vector field Z is called the horizontal lift of $\check{Z}$.

The fundamental tensors of a submersion were introduced in [37]. They play a similar role to that of the second fundamental form of an immersion. More precisely, O'Neill's tensors T and A defined for vector fields E, F on M by

$$A_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F,\ T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F \tag{2.4}$$

where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections (see [21]). On the other hand, from (2.4), we have

$$\nabla_V W = T_V W + \hat{\nabla}_V W \tag{2.5}$$

$$\nabla_V X = \mathcal{H}\nabla_V X + T_V X \tag{2.6}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + A_X Y \tag{2.7}$$

for $X, Y \in \Gamma((kerF_*)^{\perp})$ and $V, W \in \Gamma(kerF_*)$, where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. If X is basic, then $\mathcal{H}\nabla_V X = A_X V$. It is easily seen that for $x \in M$, $X \in \mathcal{H}_x$ and $\mathcal{V}_x$ the linear operators T_V, $A_X : T_x M \longrightarrow T_x M$ are skew-symmetric. We see that the restriction of T to the vertical distribution $T\mid_{V\times V}$ is exactly the second fundamental form of the fibres of π. Since T_V is skew-symmetric we get: π has totally geodesic fibres if and only if $T \equiv 0$. For the special case when π is horizontally conformal we have the following:

Proposition 2.2. ([26]). *Let $\pi : (M^m, g) \longrightarrow (N^n, h)$ be a horizontally conformal submersion with dilation λ and X, Y be horizontal vectors, then*

$$A_X Y = \frac{1}{2}\{\mathcal{V}[X,Y] - \lambda^2 g(X,Y) grad_{\mathcal{V}}(\frac{1}{\lambda^2})\}. \tag{2.8}$$

We now recall the notion of harmonic maps between Riemannian manifolds. Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\varphi : M \longrightarrow N$ is a smooth map between them. Then the differential φ_* of φ can be viewed a section of the bundle $Hom(TM, \varphi^{-1}TN) \longrightarrow M$, where $\varphi^{-1}TN$ is the pullback bundle which has fibres $(\varphi^{-1}TN)_p = T_{\varphi(p)}N$, $p \in M$. $Hom(TM, \varphi^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of φ is given by

$$(\nabla\varphi_*)(X,Y) = \nabla^{\varphi}_X \varphi_*(Y) - \varphi_*(\nabla^M_X Y) \tag{2.9}$$

for $X, Y \in \Gamma(TM)$, where ∇^{φ} is the pullback connection. It is known that the second fundamental form is symmetric. A smooth map $\varphi : (M, g_M) \longrightarrow (N, g_N)$ is said to be harmonic if $trace(\nabla\varphi_*) = 0$. On the other hand, the tension field of φ is the section $\tau(\varphi)$ of $\Gamma(\varphi^{-1}TN)$ defined by

$$\tau(\varphi) = div\varphi_* = \sum_{i=1}^{m} (\nabla\varphi_*)(e_i, e_i), \tag{2.10}$$

where $\{e_1, ..., e_m\}$ is an orthonormal frame on M. Then it follows that φ is harmonic if and only if $\tau(\varphi) = 0$, for details, see [6]. Now, we recall the following lemma from [6].

Lemma 2.3. *Suppose that $\varphi : M \longrightarrow N$ is a horizontally conformal submersion. Then, for any horizontal vector fields X, Y and vertical fields V, W we have*

(i) $\nabla d\varphi(X,Y) = X(ln\lambda)d\varphi(Y) + Y(ln\lambda)d\varphi(X) - g(X,Y)d\varphi(gradln\lambda)$;
(ii) $\nabla d\varphi(V,W) = -d\varphi(A^{\mathcal{V}}_V W)$;
(iii) $\nabla d\varphi(X,V) = -d\varphi(\nabla^M_X V) = d\varphi((A^{\mathcal{H}})^*_X V))$.

Here $(A^{\mathcal{H}})^*_X$ is the adjoint of $A^{\mathcal{H}}_X$ characterized by

$$\langle (A^{\mathcal{H}})^*_X E, F\rangle = \langle E, A^{\mathcal{H}}_X F\rangle\ (E, F \in \Gamma(TM)).$$

Let g_B be a Riemannian metric tensor on the manifold $B = B_1 \times B_2$ and assume that the canonical foliations D_{B_1} and D_{B_2} intersect perpendicularly everywhere. Then g_B is the metric tensor of a usual product of Riemannian manifolds if and only if D_{B_1} and D_{B_2} are totally geodesic foliations [39].

3. Conformal Slant submersions

In this section, we define conformal slant submersions from an almost Hermitian manifold onto a Riemannian manifold, investigate the effect of the existence of conformal slant submersions on the source manifold and the target manifold. But we first present the following notion.

Definition 3.1. Let F be a horizontally conformal submersion from an almost Hermitian manifold (M_1, g_1, J_1) onto a Riemannian manifold (M_2, g_2). If for any non-zero vector $X \in \Gamma(kerF_{*p}); p \in M_1$, the angle $\theta(X)$ between JX and the space $(kerF_{*p})$ is a constant, i.e. it is independent of the choice of the point $p \in M_1$ and choice of the tangent vector X in $(kerF_{*p})$, then we say that F is a conformal slant submersion. In this case, the angle θ is called the slant angle of the conformal slant submersion.

We note that it is known that the distribution $kerF_*$ is integrable. In fact, its leaves are $F^{-1}(q)$, $q \in M_2$, i.e., fibers. Thus it follows from above definition that the fibres of a conformal slant submersion are slant submanifolds of M_1, for slant submanifolds, see [13]. We now give some examples of conformal slant submersions.

Example 3.2. Every Hermitian submersion from an almost Hermitian manifold onto an almost Hermitian manifold is a conformal slant submersion with $\lambda = 1$ and $\theta = 0$.

Example 3.3. Every conformal anti-invariant submersion from an almost Hermitian manifold to a Riemannian manifold is a conformal slant submersion with $\lambda = 1$ and $\theta = \frac{\pi}{2}$.

Example 3.4. Every slant submersion from an almost Hermitian manifold onto Riemannian manifold is a conformal slant submersion with $\lambda = 1$.

A conformal slant submersion is said to be proper if it is neither Hermitian nor conformal anti- invariant submersion. We now present two examples of a proper conformal slant submersion. We denote by J_α the compatible almost complex structure on R^4 defined by

$$J_\alpha(a,b,c,d) = (\cos\alpha)(-c,-d,a,b) + (\sin\alpha)(-b,a,d,-c), \quad 0 < \alpha \leq \frac{\pi}{2}$$

Example 3.5. Consider the following submersion given by

$$\begin{array}{cccc} F: & R^4 & \longrightarrow & R^2 \\ & (x_1,x_2,x_3,x_4) & & (e^{x_1}\sin x_2, e^{x_1}\cos x_2), \end{array}$$

where $x_2 \in \mathbb{R} - \{k\frac{\pi}{2}, k\pi\}$, $k \in \mathbb{Z}$. Then it follows that

$$kerF_* = span\{V_1 = \partial x_3,\ V_2 = \partial x_4\}$$

and

$$\begin{aligned}(kerF_*)^\perp = span\{X_1 &= e^{x_1}\sin x_2 \partial x_1 + e^{x_1}\cos x_2 \partial x_2, \\ X_2 &= e^{x_1}\cos x_2 \partial x_1 - e^{x_1}\sin x_2 \partial x_2\}.\end{aligned}$$

Then by direct computations for any $0 < \theta \leq \frac{\pi}{2}$, F is a slant submersion with slant angle θ. On the other hand,

$$F_*X_1 = (e^{x_1})^2 \partial y_1,\ F_*X_2 = (e^{x_1})^2 \partial y_2.$$

Hence, we have

$$g_2(F_*X_1, F_*X_1) = (e^{x_1})^2 g_1(X_1, X_1), \quad g_2(F_*X_2, F_*X_2) = (e^{x_1})^2 g_1(X_2, X_2),$$

where g_1 and g_2 denote the standard metrics (inner products) of R^4 and R^2. Thus F is a conformal slant submersion with $\lambda = e^{x_1}$.

Example 3.6. Let F be a submersion defined by

$$F: \begin{array}{ccc} R^4 & \longrightarrow & R^2 \\ (x_1, x_2, x_3, x_4) & & (\cosh x_1 \sin x_3, \sinh x_1 \cos x_3), \end{array}$$

where $x_3 \in \mathbb{R} - \{k\frac{\pi}{2}, k\pi\}$, $k \in \mathbb{Z}$. Then it follows that

$$kerF_* = span\{V_1 = \partial x_2,\ V_2 = \partial x_4\}$$

and

$$(kerF_*)^\perp = span\{X_1 = \sinh x_1 \sin x_3 \partial x_1 + \cosh x_1 \cos x_3 \partial x_2, \\ X_2 = \cosh x_1 \cos x_3 \partial x_1 - \sinh x_1 \sin x_3 \partial x_2\}.$$

Then by direct computations for any $0 < \theta \leq \frac{\pi}{2}$, F is a slant submersion with slant angle θ. On the other hand,

$$F_* X_1 = (\sinh^2 x_1 \sin^2 x_3 + \cosh^2 x_1 \cos^2 x_3) \partial y_1$$

and

$$F_* X_2 = (\sinh^2 x_1 \sin^2 x_3 + \cosh^2 x_1 \cos^2 x_3) \partial y_2.$$

Hence, we have

$$g_2(F_* X_1, F_* X_1) = (\sinh^2 x_1 \sin^2 x_3 + \cosh^2 x_1 \cos^2 x_3) g_1(X_1, X_1)$$

and

$$g_2(F_* X_2, F_* X_2) = (\sinh^2 x_1 \sin^2 x_3 + \cosh^2 x_1 \cos^2 x_3) g_1(X_2, X_2),$$

where g_1 and g_2 denote the standard metrics (inner products) of R^4 and R^2. Thus F is a conformal slant submersion with $\lambda^2 = \sinh^2 x_1 \sin^2 x_3 + \cosh^2 x_1 \cos^2 x_3$.

Let F be a conformal slant submersion from an almost Hermitian manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then for $U \in \Gamma(kerF_*)$, we write

$$JU = \phi U + \omega U \tag{3.1}$$

where ϕU and ωU are vertical and horizontal parts of JU. Also for $X \in \Gamma((kerF_*)^\perp)$, we have

$$JX = \mathcal{B}X + \mathcal{C}X, \tag{3.2}$$

where $\mathcal{B}X$ and $\mathcal{C}X$ are vertical and horizontal components. Using (2.5), (2.6), (3.1) and (3.2) we obtain

$$(\nabla_U \omega)V = \mathcal{C}T_U V - T_U \phi V \tag{3.3}$$

$$(\nabla_U \phi)V = \mathcal{B}T_U V - T_U \omega V, \tag{3.4}$$

where ∇ is the Levi-Civita connection on M_1 and

$$(\nabla_U \omega)V = \mathcal{H}\nabla_U \omega V - \omega \hat{\nabla}_U V$$

$$(\nabla_U \phi)V = \hat{\nabla}_U \phi V - \phi \hat{\nabla}_U V$$

for $U, V \in \Gamma(kerF_*)$. Let F be a proper conformal slant submersion from an almost Hermitian manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2), then we say that ω is parallel with respect to the Levi-Civita connection ∇ on $(kerF_*)$ if its covariant derivative with respect to ∇ vanishes, i.e., we have

$$(\nabla_U \omega)V = \nabla_U \omega V - \phi \hat{\nabla}_U V$$

for $U, V \in \Gamma(kerF_*)$. The proof of the following result is exactly same with slant immersions (see [11] and [13]), therefore we omit its proof.

Theorem 3.7. *Let F be a conformal slant submersion from an almost Hermitian manifold (M_1, g_1, J_1) onto a Riemannian manifold (M_2, g_2). Then F is a proper conformal slant submersion if and only if there exists a constant $\lambda_1 \in [-1, 0]$ such that*

$$\phi^2 U = \lambda_1 U$$

for $U \in \Gamma(kerF_*)$. If F is a proper conformal slant submersion, then $\lambda_1 = -\cos^2\theta$.

By using above theorem, it is easy to see the following lemma.

Lemma 3.8. *Let F be a proper conformal slant submersion from an almost Hermitian manifold (M_1, g_1, J_1) onto a Riemannian manifold (M_2, g_2) with slant angle θ. Then, for any $U, V \in \Gamma(kerF_*)$, we have*

$$g_1(\phi U, \phi V) = \cos^2\theta g_1(U, V), \tag{3.5}$$

and

$$g_1(\omega U, \omega V) = \sin^2\theta g_1(U, V). \tag{3.6}$$

We now denote complementary distribution of $\omega(kerF_*)$ in $(kerF_*)^\perp$ by μ. The proof of the following result is exactly same with slant submersion (see [42]), therefore we omit its proof.

Proposition 3.9. *Let F be a proper conformal slant submersion from an almost Hermitian manifold (M_1, g_1, J_1) onto a Riemannian manifold (M_2, g_2). Then μ is invariant with respect to J_1.*

Corollary 3.10. *Let F be a proper conformal slant submersion from an almost Hermitian manifold (M_1^m, g_1, J_1) onto a Riemannian manifold (M_2^n, g_2). Let*

$$\{e_1, ..., e_{m-n}\}$$

be a local orthonormal basis of $(kerF_*)$, then $\{\csc\theta\omega e_1, ..., \csc\theta\omega e_{m-n}\}$ is a local orthonormal basis of $\omega(kerF_*)$.

Proof. It will be enough to show that $g_1(\csc\theta\omega e_i, \csc\theta\omega e_j) = \delta_{ij}$, for any $i, j \in \{1, ..., \frac{m-n}{2}\}$. By using (3.6), we have

$$g_1(\csc\theta\omega e_i, csc\theta\omega e_j) = \csc^2\theta\sin^2\theta g_1(e_i, e_j) = \delta_{ij},$$

which proves the assertion. □

We note that above Proposition 3.9 tells that the distributions μ and $(kerF_*)\oplus\omega(kerF_*)$ are even dimensional. In fact it implies that the distribution $(kerF_*)$ is even dimensional. More precisely, we have the following result whose proof is similar to the above corollary.

Lemma 3.11. *Let F be a proper conformal slant submersion from an almost Hermitian manifold (M_1^m, g_1, J_1) onto a Riemannian manifold (M_2^n, g_2). If $e_1, e_2, ..., e_{\frac{m-n}{2}}$ are orthogonal unit vector fields in $(kerF_*)$, then*

$$\{e_1, \sec\theta\phi e_1, e_2, \sec\theta\phi e_2, ..., e_{\frac{m-n}{2}}, \sec\theta\phi e_{\frac{m-n}{2}}\}$$

is a local orthonormal basis of $(kerF_*)$.

Let F be a proper conformal slant submersion from an almost Hermitian manifold (M_1^{2n}, g_1, J_1) onto a Riemannian manifold (M_2^n, g_2). As in the case of slant immersions, we call such an orthonormal frame

$$\{e_1, \sec\theta\phi e_1, e_2, \sec\theta\phi e_2, ..., e_n, \sec\theta\phi e_n, \csc\theta\omega e_1, \csc\theta\omega e_2, ..., \csc\theta\omega e_n\}$$

an adapted slant frame for conformal slant submersions. In the sequel, we show that the conformal slant submersion puts some restrictions on the dimensions of the distributions and the base manifold.

Proposition 3.12. *Let F be a proper conformal slant submersion from an almost Hermitian manifold (M_1^m, g_1, J_1) onto a Riemannian manifold (M_2^n, g_2). Then $dim(\mu) = 2n - m$. If $\mu = 0$, then $n = \frac{m}{2}$.*

Proof. First note that $dim(kerF_*) = m - n$. Thus using Corollary 3.10, we have $dim((kerF_*) \oplus \omega(kerF_*)) = 2(m-n)$. Since M_1 is $m-$ dimensional, we get $dim(\mu) = 2n - m$. Second assertion is clear. □

We now check the harmonicity of conformal slant submersions. But we first give a preparatory lemma.

Lemma 3.13. *Let F be a proper conformal slant submersion from a Kähler manifold onto a Riemannian manifold. If ω is parallel with respect to ∇ on $(kerF_*)$, then we have*

$$T_{\phi U}\phi U = -\cos^2\theta T_U U \tag{3.7}$$

for $U \in \Gamma(kerF_*)$.

Proof. If ω is parallel, then from (3.3) we have $\mathcal{C}T_U V = T_U\phi V$ for $U, V \in \Gamma(kerF_*)$. Interchanging the role of U and V, we get $\mathcal{C}T_V U = T_V\phi U$. Thus we have

$$\mathcal{C}T_U V - \mathcal{C}T_V U = T_U\phi V - T_V\phi U.$$

Since T is symmetric, we derive

$$T_U\phi V = T_V\phi U. \tag{3.8}$$

Then substituting V by ϕU we get $T_U\phi^2 U = T_{\phi U}\phi U$. Finally using Theorem 3.7 we obtain (3.7). □

Theorem 3.14. *Let $F : (M_1^{2(m+r)}, g_1, J_1) \longrightarrow (M_2^{m+2r}, g_2)$ be a conformal slant submersion, where (M_1, g_1, J_1) is a Kähler manifold and (M_2, g_2) is a Riemannian manifold. Then the tension field τ of F is*

$$\tau(F) = -\frac{1}{m}F_*\left(T_{e_i}e_i + \sec^2\theta T_{\phi e_i}\phi e_i\right) + \left(\frac{2}{\lambda^2} - (m+2r)\right)F_*(gradln\lambda). \tag{3.9}$$

Proof. Let $\{e_1, ..., e_m, \sec\theta\phi e_1, ..., \sec\theta\phi e_m, \csc\theta\omega e_1, ..., \csc\theta\omega e_m, \mu_1, ..., \mu_r, J_1\mu_1, ..., J_1\mu_r\}$ be orthonormal basis of $\Gamma(TM_1)$ such that $\{e_1, ..., e_m, \sec\theta\phi e_1, ..., \sec\theta\phi e_m\}$ be orthonormal basis of $\Gamma(kerF_*)$,
$\{\csc\theta\omega e_1, ..., \csc\theta\omega e_m\}$ be orthonormal basis of $\Gamma(\omega(kerF_*))$ and $\{\mu_1, ..., \mu_r, J_1\mu_1, ..., J_1\mu_r\}$ be orthonormal basis of $\Gamma(\mu)$. Then the trace of second fundamental form (restriction to $kerF_* \times kerF_*$) is given by

$$\begin{aligned} trace^{kerF_*}\nabla F_* &= \sum_{i=1}^{m}(\nabla F_*)(e_i, e_i) + (\nabla F_*)(\sec\theta\phi e_i, \sec\theta\phi e_i) \\ &= \sum_{i=1}^{m}(\nabla F_*)(e_i, e_i) + \sec^2\theta(\nabla F_*)(\phi e_i, \phi e_i). \end{aligned}$$

Then using (2.9) we obtain

$$\begin{aligned} trace^{kerF_*}\nabla F_* &= -\frac{1}{m}F_*(T_{e_i}e_i) - \frac{1}{m}F_*(\sec^2\theta T_{\phi e_i}\phi e_i) \\ &= -\frac{1}{m}F_*(T_{e_i}e_i + \sec^2\theta T_{\phi e_i}\phi e_i). \end{aligned} \tag{3.10}$$

In a similar way, we have

$$\begin{aligned} trace^{(kerF_*)^\perp}\nabla F_* &= \sum_{i=1}^{m}(\nabla F_*)(\csc\theta\omega e_i, \csc\theta\omega e_i) + \sum_{i=1}^{2r}(\nabla F_*)(\mu_i, \mu_i) \\ &= \csc^2\theta\sum_{i=1}^{m}(\nabla F_*)(\omega e_i, \omega e_i) + \sum_{i=1}^{2r}(\nabla F_*)(\mu_i, \mu_i). \end{aligned}$$

Using Lemma 2.3 we arrive at

$$\begin{aligned} trace^{(kerF_*)^\perp}\nabla F_* &= \csc^2\theta\sum_{i=1}^{m}\{\omega e_i(ln\lambda)F_*\omega e_i+\omega e_i(ln\lambda)F_*\omega e_i \\ &- g_1(\omega e_i,\omega e_i)F_*(gradln\lambda)\} \\ &+ \sum_{i=1}^{2r}\{\mu_i(ln\lambda)F_*\mu_i+\mu_i(ln\lambda)F_*\mu_i-g_1(\mu_i,\mu_i)F_*(gradln\lambda)\} \\ &= \csc^2\theta\sum_{i=1}^{m}2g_1(\mathcal{H}gradln\lambda,\omega e_i)F_*\omega e_i \\ &- \csc^2\theta g_1(\omega e_i,\omega e_i)F_*(gradln\lambda) \\ &+ \sum_{i=1}^{2r}2g_1(\mathcal{H}gradln\lambda,\mu_i)F_*\mu_i-2rF_*(gradln\lambda). \end{aligned}$$

Since F is a conformal slant submersion, we derive

$$\begin{aligned} trace^{(kerF_*)^\perp}\nabla F_* &= \csc^2\theta\sum_{i=1}^{m}\frac{2}{\lambda^2}g_2(F_*(gradln\lambda),F_*\omega e_i)F_*\omega e_i \\ &+ \sum_{i=1}^{2r}\frac{2}{\lambda^2}g_2(F_*(gradln\lambda),F_*\mu_i)F_*\mu_i-(m+2r)F_*(gradln\lambda) \\ &= \frac{2}{\lambda^2}F_*(gradln\lambda)-(m+2r)F_*(gradln\lambda) \\ &= \left(\frac{2}{\lambda^2}-(m+2r)\right)F_*(gradln\lambda). \end{aligned} \tag{3.11}$$

Then proof follows from (3.10) and (3.11). □

We note that for any C^2 real valued function f defined on an open subset of a Riemannian manifold M, the equation $\triangle f=0$ is called Laplace's equation and solutions are called harmonic functions on U. Let $F:M\longrightarrow N$ be a smooth map between Riemannian manifolds. Then F is called harmonic morphism if, for every harmonic function $f:V\longrightarrow \mathbf{R}$ defined an open subset V of N with $F^{-1}(V)$ non-empty, the composition $f\circ F$ is harmonic on $F^{-1}(V)$. It is known that a smooth map $F:M\longrightarrow N$ between Riemannian manifolds is a harmonic morphism if and only if F is both harmonic and horizontally weakly conformal [23] and [33]. Thus from Theorem 3.14 we deduce the following result.

Theorem 3.15. *Let $F:(M_1^{2(m+r)},g_1,J_1)\longrightarrow(M_2^{m+2r},g_2)$ be a conformal slant submersion such that $\frac{2}{(m+2r)}\neq\lambda^2$ where (M_1,g_1,J_1) is a Kähler manifold and (M_2,g_2) is a Riemannian manifold. Then any two conditions below imply the third:*

(i) *F is a harmonic morphism*
(ii) *ω is parallel with respect to ∇ on $(kerF_*)$*
(iii) *F is a horizontally homotetic map.*

We also have the following result.

Corollary 3.16. *Let F be a conformal slant submersion from a Kähler manifold $(M_1^{2(m+r)},g_1,J_1)$ to a Riemannian manifold (M_2^{m+2r},g_2). If $\frac{2}{(m+2r)}=\lambda^2$ then F is harmonic morphism if and only if ω is parallel with respect to ∇ on $(kerF_*)$.*

Remark 3.17. By arguing as in [6, Proposition 3.5.1, Theorem 4.5.4], one can see that Theorem 3.15 and Corollary 3.16 are also valid for a horizontally weakly conformal map.

We note that if $\frac{2}{(m+2r)} = \lambda^2$ is satisfied, then F is certainly horizontally homothetic. We now study the integrability of the distribution $(kerF_*)^\perp$ and then we investigate the geometry of leaves of $(kerF_*)^\perp$ and $(kerF_*)$. We note that it is known that the distribution $kerF_*$ is integrable.

Theorem 3.18. *Let F be a proper conformal slant submersion from a Kähler manifold (M_1, g_1, J_1) onto a Riemannian manifold (M_2, g_2). Then the following assertions are equivalent to each other;*

(i) $(kerF_*)^\perp$ *is integrable,*

(ii) $$\frac{1}{\lambda^2}\{g_2(\nabla^F_Y F_*\mathcal{C}X - \nabla^F_X F_*\mathcal{C}Y, F_*\omega V) - g_2(\nabla^F_Y F_*X - \nabla^F_X F_*Y, F_*\omega\phi V)\}$$
$$= g_1(A_X\mathcal{B}Y - A_Y\mathcal{B}X - \mathcal{C}Y(\ln\lambda)X + \mathcal{C}X(\ln\lambda)Y - 2g_1(\mathcal{C}X, Y)\ln\lambda, \omega V)$$

for $X, Y \in \Gamma((kerF_*)^\perp)$, $V \in \Gamma(kerF_*)$.

Proof. For $X, Y \in \Gamma((kerF_*)^\perp)$ and $V \in \Gamma(kerF_*)$, using (2.1), (2.2) and (3.1) we have

$$g_1([X,Y],V) = -g_1(\nabla_X Y, J\phi V) + g_1(\nabla_X JY, \omega V) + g_1(\nabla_Y X, J\phi V) - g_1(\nabla_Y JX, \omega V).$$

Then by using (3.2), we get

$$\begin{aligned} g_1([X,Y],V) = & -g_1(\nabla_X Y, \phi^2 V) - g_1(\nabla_X Y, \omega\phi V) + g_1(\nabla_X \mathcal{B}Y, \omega V) \\ & + g_1(\nabla_X \mathcal{C}Y, \omega V) \\ & + g_1(\nabla_Y X, \phi^2 V) + g_1(\nabla_Y X, \omega\phi V) - g_1(\nabla_Y \mathcal{B}X, \omega V) \\ & - g_1(\nabla_Y \mathcal{C}X, \omega V). \end{aligned}$$

Since F is a conformal submersion, using (2.7), Theorem 3.7 and Lemma 2.3 we arrive at

$$\begin{aligned} & g_1([X,Y],V) = \cos^2\theta g_1([X,Y],V) + g_1(A_X\mathcal{B}Y - A_Y\mathcal{B}X, \omega V) \\ & + \frac{1}{\lambda^2} g_2((\nabla F_*)(X,Y), F_*\omega\phi V) \\ & - \frac{1}{\lambda^2} g_2(\nabla^F_X F_*Y, F_*\omega\phi V) - g_1(gradln\lambda, X) g_1(\mathcal{C}Y, \omega V) - g_1(gradln\lambda, \mathcal{C}Y) g_1(X, \omega V) \\ & + g_1(X, \mathcal{C}Y) g_1(gradln\lambda, \omega V) + \frac{1}{\lambda^2} g_2(\nabla^F_X F_*\mathcal{C}Y, F_*\omega V) - \frac{1}{\lambda^2} g_2((\nabla F_*)(Y,X), F_*\omega\phi V) \\ & + \frac{1}{\lambda^2} g_2(\nabla^F_Y F_*X, F_*\omega\phi V) + g_1(gradln\lambda, Y) g_1(\mathcal{C}X, \omega V) + g_1(gradln\lambda, \mathcal{C}X) g_1(Y, \omega V) \\ & - g_1(Y, \mathcal{C}X) g_1(gradln\lambda, \omega V) - \frac{1}{\lambda^2} g_2(\nabla^F_Y F_*\mathcal{C}X, F_*\omega V). \end{aligned}$$

Since ∇F_* is symmetric, we have

$$\begin{aligned} \sin^2\theta g_1([X,Y],V) &= g_1(A_X\mathcal{B}Y - A_Y\mathcal{B}X - \mathcal{C}Y(\ln\lambda)X + \mathcal{C}X(\ln\lambda)Y \\ &- 2g_1(\mathcal{C}X, Y)\ln\lambda, \omega V) \\ &+ \frac{1}{\lambda^2}\{g_2(\nabla^F_Y F_*X - \nabla^F_X F_*Y, F_*\omega\phi V) \\ &- g_2(\nabla^F_Y F_*\mathcal{C}X - \nabla^F_X F_*\mathcal{C}Y, F_*\omega V)\} \end{aligned}$$

which proves assertion. □

From Theorem 3.18, we deduce the following which shows that a conformal slant submersion with integrable $(kerF_*)^\perp$ turns out to be a horizontally homothetic submersion.

Theorem 3.19. *Let F be a proper conformal slant submersion from a Kähler manifold (M_1, g_1, J_1) onto a Riemannian manifold (M_2, g_2). Then any two conditions below imply the three;*

(i) $(kerF_*)^\perp$ *is integrable*

(ii) F *is horizontally homotetic.*

(iii) $$\frac{1}{\lambda^2}\{g_2(\nabla^F_Y F_*\mathcal{C}X - \nabla^F_X F_*\mathcal{C}Y, F_*\omega V) - g_2(\nabla^F_Y F_*X - \nabla^F_X F_*Y, F_*\omega\phi V)\} = g_1(A_X\mathcal{B}Y - A_Y\mathcal{B}X, \omega V)$$

for $X, Y \in \Gamma((kerF_*)^\perp)$, $V \in \Gamma(kerF_*)$.

Proof. For $X, Y \in \Gamma((kerF_*)^\perp)$, $V \in \Gamma(kerF_*)$, from Theorem 3.18, we have

$$\begin{aligned}\sin^2\theta g_1([X,Y],V) &= g_1(A_X\mathcal{B}Y - A_Y\mathcal{B}X - \mathcal{C}Y(\ln\lambda)X + \mathcal{C}X(\ln\lambda)Y\\ &- 2g_1(\mathcal{C}X,Y)\ln\lambda, \omega V)\\ &+ \frac{1}{\lambda^2}\{g_2(\nabla^F_Y F_*X - \nabla^F_X F_*Y, F_*\omega\phi V)\\ &- g_2(\nabla^F_Y F_*\mathcal{C}X - \nabla^F_X F_*\mathcal{C}Y, F_*\omega V)\}.\end{aligned}$$

Now, if we have (i) and (iii), then we arrive at

$$\begin{aligned}&-g_1(\mathcal{H}grad\ln\lambda, \mathcal{C}Y)g_1(X,\omega V) + g_1(\mathcal{H}grad\ln\lambda, \mathcal{C}X)g_1(Y,\omega V)\\ &-2g_1(\mathcal{C}X,Y)g_1(\mathcal{H}grad\ln\lambda, \omega V) = 0.\end{aligned} \tag{3.12}$$

Now, taking $Y = JV$ in (3.12) for $V \in \Gamma(kerF_*)$, we get

$$g_1(\mathcal{H}grad\ln\lambda, \mathcal{C}X)g_1(\omega V, \omega V) = 0.$$

Hence λ is a constant on $\Gamma(\mu)$. On the other hand, taking $Y = \mathcal{C}X$ in (3.12) for $X \in \Gamma(\mu)$, we derive

$$\begin{aligned}&-g_1(\mathcal{H}grad\ln\lambda, \mathcal{C}^2X)g_1(X,\omega V) + g_1(\mathcal{H}grad\ln\lambda, \mathcal{C}X)g_1(\mathcal{C}X,\omega V)\\ &-2g_1(\mathcal{C}X,\mathcal{C}X)g_1(\mathcal{H}grad\ln\lambda, \omega V) = 0,\end{aligned}$$

hence, we arrive at

$$g_1(\mathcal{C}X,\mathcal{C}X)g_1(\mathcal{H}grad\ln\lambda, \omega V) = 0.$$

From above equation, λ is a constant on $\Gamma(\omega(kerF_*))$. Similarly, one can obtain the other assertions. □

Theorem 3.20. *Let F be a proper conformal slant submersion from a Kähler manifold (M_1, g_1, J_1) onto a Riemannian manifold (M_2, g_2). Then the distribution $(kerF_*)^\perp$ defines a totally geodesic foliation on M_1 if and only if*

$$\begin{aligned}\frac{1}{\lambda^2}\{g_2(\nabla^F_X F_*Y, F_*\omega\phi V) - g_2(\nabla^F_X F_*\mathcal{C}Y, F_*\omega V)\} &= g_1(A_X\mathcal{B}Y, \omega V)\\ &+ g_1(gradln\lambda, X)g_1(Y, \omega\phi V)\\ &+ g_1(gradln\lambda, Y)g_1(X, \omega\phi V)\\ &- g_1(X,Y)g_1(gradln\lambda, \omega\phi V)\\ &- g_1(gradln\lambda, X)g_1(\mathcal{C}Y, \omega V)\\ &- g_1(gradln\lambda, \mathcal{C}Y)g_1(X, \omega V)\\ &+ g_1(X, \mathcal{C}Y)g_1(gradln\lambda, \omega V)\end{aligned}$$

for $X, Y \in \Gamma((kerF_*)^\perp)$, $V \in \Gamma(kerF_*)$.

Proof. For $X, Y \in \Gamma((kerF_*)^\perp)$ and $V \in \Gamma(kerF_*)$, using (2.1), (2.2), (3.1) and (3.2) we have

$$g_1(\nabla_X Y, V) = -g_1(\nabla_X Y, \phi^2 V) - g_1(\nabla_X Y, \omega\phi V) + g_1(\nabla_X\mathcal{B}Y, \omega V) + g_1(\nabla_X\mathcal{C}Y, \omega V).$$

Since F is a conformal submersion, using (2.7), Theorem 3.7 and Lemma 2.3 we arrive at

$$\begin{aligned} g_1(\nabla_X Y, V) = \cos^2\theta g_1(\nabla_X Y, V) &+ g_1(A_X \mathcal{B}Y, \omega V) + g_1(gradln\lambda, X) g_1(Y, \omega\phi V) \\ &+ g_1(gradln\lambda, Y) g_1(X, \omega\phi V) - g_1(X, Y) g_1(gradln\lambda, \omega\phi V) \\ &- \frac{1}{\lambda^2} g_2(\nabla_X^F F_* Y, F_*\omega\phi V) \\ &- g_1(gradln\lambda, X) g_1(\mathcal{C}Y, \omega V) - g_1(gradln\lambda, \mathcal{C}Y) g_1(X, \omega V) \\ &+ g_1(X, \mathcal{C}Y) g_1(gradln\lambda, \omega V) + \frac{1}{\lambda^2} g_2(\nabla_X^F F_* \mathcal{C}Y, F_*\omega V). \end{aligned}$$

Hence we have

$$\begin{aligned} \sin^2\theta g_1(\nabla_X Y, V) = g_1(A_X \mathcal{B}Y, \omega V) &+ g_1(gradln\lambda, X) g_1(Y, \omega\phi V) \\ &+ g_1(gradln\lambda, Y) g_1(X, \omega\phi V) \\ &- g_1(X, Y) g_1(gradln\lambda, \omega\phi V) - g_1(gradln\lambda, X) g_1(\mathcal{C}Y, \omega V) \\ &- g_1(gradln\lambda, \mathcal{C}Y) g_1(X, \omega V) + g_1(X, \mathcal{C}Y) g_1(gradln\lambda, \omega V) \\ &- \frac{1}{\lambda^2}\{g_2(\nabla_X^F F_* Y, F_*\omega\phi V) - g_2(\nabla_X^F F_* \mathcal{C}Y, F_*\omega V)\} \end{aligned}$$

which proves assertion. □

In a similar way we have the following.

Theorem 3.21. *Let F be a proper conformal slant submersion from a Kähler manifold (M_1, g_1, J_1) onto a Riemannian manifold (M_2, g_2). Then the distribution $(kerF_*)$ defines a totally geodesic foliation on M_1 if and only if*

$$\begin{aligned} \frac{1}{\lambda^2}\{g_2((\nabla F_*)(U, \omega\phi V), F_* Z) - g_2(\nabla_{\omega V}^F F_*\omega U, F_* JCZ)\} = g_1(A_{\omega V}\phi U &\\ + g_1(\omega U, \omega V) grad \ln\lambda, JCZ) &\\ + g_1(T_U \mathcal{B}Z, \omega V) & \end{aligned}$$

for $U, V \in \Gamma(kerF_*)$ and $Z \in \Gamma((kerF_*)^\perp)$.

From Theorem 3.21, we deduce that:

Theorem 3.22. *Let F be a proper conformal slant submersion from a Kähler manifold (M_1, g_1, J_1) onto a Riemannian manifold (M_2, g_2). Then any two conditions below imply the three:*

(i) *$kerF_*$ defines a totally geodesic foliation on M_1.*
(ii) *λ is a constant on $\Gamma(\mu)$.*
(iii) $\frac{1}{\lambda^2}\{g_2((\nabla F_*)(U, \omega\phi V), F_* Z) - g_2(\nabla_{\omega V}^F F_*\omega U, F_* JCZ)\} = g_1(A_{\omega V}\phi U, JCZ) + g_1(T_U \mathcal{B}Z, \omega V)$

for $U, V \in \Gamma(kerF_*)$ and $Z \in \Gamma((kerF_*)^\perp)$.

Proof. For $U, V \in \Gamma(kerF_*)$ and $Z \in \Gamma((kerF_*)^\perp)$, from Theorem 3.21, we have

$$\begin{aligned} \sin^2\theta g_1(\nabla_U V, Z) = g_1(T_U \omega V, \mathcal{B}Z) &- g_1(A_{\omega V}\phi U, J\mathcal{C}Z) \\ &- g_1(\omega V, \omega U) g_1(\mathcal{H} gradln\lambda, J\mathcal{C}Z) \\ &+ \frac{1}{\lambda^2}\{g_2((\nabla F_*)(U, \omega\phi V), F_* Z) - g_2(\nabla_{\omega V}^F F_*\omega U, F_* J\mathcal{C}Z)\}. \end{aligned}$$

Now, if we have (i) and (iii), then we get

$$g_1(\omega V, \omega U) g_1(\mathcal{H} gradln\lambda, J\mathcal{C}Z) = 0.$$

From above equation, λ is a constant on $\Gamma(\mu)$. Similarly, one can obtain the other assertions. □

From Theorem 3.20 and Theorem 3.21 we have the following result.

Corollary 3.23. *Let F be a proper conformal slant submersion from a Kähler manifold (M_1, g_1, J_1) onto a Riemannian manifold (M_2, g_2). Then M_1 is a locally product Riemannian manifold if and only if*

$$\begin{aligned}\frac{1}{\lambda^2}\{g_2(\nabla^F_X F_*Y, F_*\omega\phi V) - g_2(\nabla^F_X F_*\mathcal{C}Y, F_*\omega V)\} = {} & g_1(A_X\mathcal{B}Y, \omega V)\\ & + g_1(gradln\lambda, X)g_1(Y, \omega\phi V)\\ & + g_1(gradln\lambda, Y)g_1(X, \omega\phi V)\\ & - g_1(X, Y)g_1(gradln\lambda, \omega\phi V)\\ & - g_1(gradln\lambda, X)g_1(\mathcal{C}Y, \omega V)\\ & - g_1(gradln\lambda, \mathcal{C}Y)g_1(X, \omega V)\\ & + g_1(X, \mathcal{C}Y)g_1(gradln\lambda, \omega V)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\lambda^2}\{g_2((\nabla F_*)(U, \omega\phi V), F_*Z) - g_2(\nabla^F_{\omega V} F_*\omega U, F_*JCZ)\} & = g_1(A_{\omega V}\phi U\\ & + g_1(\omega U, \omega V)grad \ln\lambda, JCZ)\\ & + g_1(T_U\mathcal{B}Z, \omega V)\end{aligned}$$

for $X, Y, Z \in \Gamma((kerF_*)^\perp)$ and $U, V \in \Gamma(kerF_*)$.

Finally we obtain necessary and sufficient condition for a conformal slant submersion to be totally geodesic. We recall that a differentiable map F between two Riemannian manifolds is called totally geodesic if

$$(\nabla F_*)(X, Y) = 0 \quad \forall X, Y \in \Gamma(TM).$$

A geometric interpretation of a totally geodesic map is that it maps every geodesic in the total manifold into a geodesic in the base manifold in proportion to arc lengths.

Theorem 3.24. *Let F be a proper conformal slant submersion from a Kähler manifold (M_1, g_1, J_1) onto a Riemannian manifold (M_2, g_2). Then F is a totally geodesic map if and only if*

(i) $\frac{1}{\lambda^2}\{g_2((\nabla F_*)(U, \omega\phi V), F_*Z) - g_2(\nabla^F_{\omega V} F_*\omega U, F_*JCZ)\} = g_1(A_{\omega V}\phi U, JCZ) + g_1(T_U\omega V, \mathcal{B}Z)$,

(ii) $\frac{1}{\lambda^2}\{g_2((\nabla F_*)(U, \omega\mathcal{B}X), F_*Y) + g_2((\nabla F_*)(U, \mathcal{C}X), F_*CY)\} = g_1(T_U\phi\mathcal{B}X, Y) - g_1(T_U\mathcal{C}X, \mathcal{B}Y)$,

(iii) *F is a horizontally homothetic map*

for $U, V \in \Gamma(kerF_*)$ and $X, Y, Z \in \Gamma((kerF_*)^\perp)$.

Proof. (i) For $U, V \in \Gamma(kerF_*)$ and $Z \in \Gamma((kerF_*)^\perp)$, using (2.1), (2.2), (3.1), (3.2 and Lemma 2.3 we have

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla F_*)(U, V), F_*Z) & = g_1(\nabla_U\phi^2 V, Z) + g_1(\nabla_U\omega\phi V, Z)\\ & - g_1(\nabla_U\omega V, \mathcal{B}Z) - g_1(\nabla_U\omega V, \mathcal{C}Z)\\ & = g_1(\nabla_U\phi^2 V, Z) + g_1(\nabla_U\omega\phi V, Z)\\ & - g_1(\nabla_U\omega V, \mathcal{B}Z) + g_1(\nabla_{\omega V} JU, J\mathcal{C}Z).\end{aligned}$$

Using (2.5), Theorem 3.7 and Lemma 2.3 we arrive at

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla F_*)(U,V),F_*Z) &= -\cos^2\theta g_1(\nabla_U V,Z) - g_1(T_U\omega V,\mathcal{B}Z) + g_1(A_{\omega V}\phi U, J\mathcal{C}Z)\\ &- \frac{1}{\lambda^2}g_2((\nabla F_*)(U,\omega\phi V),F_*Z) - g_1(grad\ln\lambda,\omega V)g_1(\omega U,J\mathcal{C}Z)\\ &- g_1(grad\ln\lambda,\omega U)g_1(\omega V,J\mathcal{C}Z) + g_1(\omega V,\omega U)g_1(gradln\lambda,J\mathcal{C}Z)\\ &+ \frac{1}{\lambda^2}g_2(\nabla^F_{\omega V}F_*\omega U,F_*J\mathcal{C}Z).\end{aligned}$$

Hence we have

$$\begin{aligned}\sin^2\theta\frac{1}{\lambda^2}g_2((\nabla F_*)(U,V),F_*Z) &= g_1(A_{\omega V}\phi U,J\mathcal{C}Z) - g_1(\omega V,\omega U)g_1(gradln\lambda,J\mathcal{C}Z)\\ &- g_1(T_U\omega V,\mathcal{B}Z)\\ &+ \frac{1}{\lambda^2}\{g_2((\nabla F_*)(U,\omega\phi V),F_*Z) + g_2(\nabla^F_{\omega V}F_*\omega U,F_*J\mathcal{C}Z)\}.\end{aligned}$$

(ii) For $X,Y\in\Gamma((kerF_*)^\perp)$ and $U\in\Gamma(kerF_*)$, in a similar way

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla F_*)(U,X),F_*Y) &= g_1(\nabla_U\phi\mathcal{B}X,Y) + g_1(\nabla_U\omega\mathcal{B}X,Y)\\ &- g_1(\nabla_U\mathcal{C}X,\mathcal{B}Y) - g_1(\nabla_U\mathcal{C}X,\mathcal{C}Y).\end{aligned}$$

Using also (2.5), Theorem 3.7 and Lemma 2.3 we arrive at

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla F_*)(U,X),F_*Y) &= g_1(T_U\phi\mathcal{B}X,Y) + \frac{1}{\lambda^2}g_2(F_*(\nabla_U\omega\mathcal{B}X),F_*Y)\\ &- g_1(T_U\mathcal{C}X,\mathcal{B}Y) - \frac{1}{\lambda^2}g_2(F_*(\nabla_U\mathcal{C}X),F_*\mathcal{C}Y)\\ &= g_1(T_U\phi\mathcal{B}X,Y) + \frac{1}{\lambda^2}g_2(-(\nabla F_*)(U,\omega\mathcal{B}X) + \nabla^F_U\omega\mathcal{B}X,F_*Y)\\ &- g_1(T_U\mathcal{C}X,\mathcal{B}Y) - \frac{1}{\lambda^2}g_2(-(\nabla F_*)(U,\mathcal{C}X) + \nabla^F_U\mathcal{C}X,F_*\mathcal{C}Y)\\ &= g_1(T_U\phi\mathcal{B}X,Y) - g_1(T_U\mathcal{C}X,\mathcal{B}Y)\\ &+ \frac{1}{\lambda^2}\{g_2((\nabla F_*)(U,\mathcal{C}X),F_*\mathcal{C}Y) - g_2((\nabla F_*)(U,\omega\mathcal{B}X),F_*Y).\end{aligned}$$

(iii) For $X,Y\in\Gamma(\mu)$, from Lemma 2.3, we have

$$(\nabla F_*)(X,Y) = X(ln\lambda)F_*Y + Y(ln\lambda)F_*X - g_1(X,Y)F_*(gradln\lambda).$$

From above equation, taking $Y=JX$ for $X\in\Gamma(\mu)$ we obtain

$$\begin{aligned}(\nabla F_*)(X,JX) &= X(ln\lambda)F_*JX + JX(ln\lambda)F_*X - g_1(X,JX)F_*(gradln\lambda)\\ &= X(ln\lambda)F_*JX + JX(ln\lambda)F_*X.\end{aligned}$$

If $(\nabla F_*)(X,JX)=0$, we obtain

$$X(ln\lambda)F_*JX + JX(ln\lambda)F_*X = 0. \tag{3.13}$$

Taking inner product in (3.13) with F_*JX we have

$$g_1(gradln\lambda,X)g_2(F_*JX,F_*JX) + g_1(gradln\lambda,X)g_2(F_*X,F_*JX) = 0.$$

From above equation, it follows that λ is a constant on $\Gamma(\mu)$. In a similar way, for $U,V\in\Gamma(kerF_*)$, using Lemma 2.3 we have

$$(\nabla F_*)(\omega U,\omega V) = \omega U(ln\lambda)F_*\omega V + \omega V(ln\lambda)F_*\omega U - g_1(\omega U,\omega V)F_*(gradln\lambda).$$

From above equation, taking $V = U$ we obtain

$$\begin{aligned}(\nabla F_*)(\omega U, \omega U) &= \omega U(ln\lambda)F_*\omega U + \omega U(ln\lambda)F_*\omega U - g_1(\omega U, \omega U)F_*(gradln\lambda) \\ &= 2\omega U(ln\lambda)F_*\omega U - g_1(\omega U, \omega U)F_*(gradln\lambda). \end{aligned} \quad (3.14)$$

Taking inner product in (3.14) with $F_*\omega U$ and since F is a conformal submersion, we derive

$$2g_1(gradln\lambda, \omega U)g_2(F_*\omega U, F_*\omega U) - g_1(\omega U, \omega U)g_2(F_*(gradln\lambda), F_*\omega U) = 0.$$

From above equation, it follows that λ is a constant on $\Gamma(\omega(kerF_*))$. Thus λ is a constant on $\Gamma((kerF_*)^\perp)$. On the other hand, if F is a horizontally homothetic map, it is obvious that $(\nabla F_*)(X, Y) = 0$. Thus proof is complete. □

Acknowledgment. This paper is supported by The Scientific and Technological Council of Turkey (TUBITAK) with number (114F339).

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