

Optimizing the convergence rate of the Wallis sequence

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Abstract

The aim of this paper is to introduce a method for increasing the convergence rate of the Wallis sequence. Some sharp inequalities are stated.

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1. Introduction

One of the most known formula for estimating of the number π is the Wallis' formula

$$(1.1) \quad \frac{\pi}{2} = \lim_{n \rightarrow \infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \dots \cdot 2n \cdot 2n}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot \dots \cdot (2n-1)(2n+1)},$$

e.g., [1, Rel. 6.1.49, p. 258].

It was discovered in 1655 by the English mathematician John Wallis (1616-1703), while he was preoccupied to calculate the value of π by finding the area under the quadrant of a circle. The Wallis' formula is also related to the problem of estimation of the large factorials, which plays a central role in combinatorics, graph theory, special functions and other branches of science as physics or applied statistics.

The Wallis' sequence

$$W_n = \prod_{k=1}^n \frac{4k^2}{4k^2 - 1}$$

converges very slowly to its limit. For example,

$$W_{100} \approx \frac{\pi}{2} - 3.9026 \times 10^{-3}, \quad W_{10000} \approx \frac{\pi}{2} - 3.9267 \times 10^{-5},$$

and in consequence, many authors are concerned to improve the speed of convergence of the Wallis formula (1.1), usually by indicate the upper and lower bounds of the Wallis sequence $(W_n)_{n \geq 1}$.

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Here, to every approximation of the form $\pi/2 \approx f(n)$, we define the relative error sequence $(\omega_n)_{n \geq 1}$ by the formula

$$\frac{\pi}{2} = f(n) \cdot \exp \omega_n$$

and we say that the approximation $\pi/2 \approx f(n)$ is as better as $(\omega_n)_{n \geq 1}$ faster converges to zero.

The speed of convergence of the sequence $(\omega_n)_{n \geq 1}$ is computed throughout this paper, using the following basic

Lemma 1.1. *If $(\omega_n)_{n \geq 1}$ is convergent to zero and there exists the limit*

$$(1.2) \quad \lim_{n \rightarrow \infty} n^k (\omega_n - \omega_{n+1}) = l \in \mathbb{R},$$

with $k > 1$, then

$$\lim_{n \rightarrow \infty} n^{k-1} \omega_n = \frac{l}{k-1}.$$

We see from Lemma 1.1 that the speed of convergence of the sequence $(\omega_n)_{n \geq 1}$ increases together with the value k satisfying (1.2). This Lemma was used by Mortici [2]-[15] for constructing asymptotic expansions, or to accelerate some convergences. For proof and other details, see, *e.g.*, [2], or [4].

We use these ideas by considering new factors to accelerate the convergence of the Wallis sequence and to obtain better approximations for π .

It is well known that the speed of convergence of the sequence $\ln W_n$ toward $\ln \frac{\pi}{2}$ is n^{-1} (this also follows from Theorem 2.1, i) from the next section).

We will see that a simple change in $(W_n)_{n \geq 1}$ of the last factor $2n$ by $2n - \frac{1}{4}$ in the denominator and of the last factor $2n + 1$ by $2n + \frac{1}{4}$ in the denominator, we obtain the quicker sequence

$$t_n = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \dots \cdot 2n \cdot (2n - \frac{1}{4})}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot \dots \cdot (2n - 1) (2n + \frac{1}{4})}$$

such that $\ln t_n$ converges to $\ln \frac{\pi}{2}$ with the speed of convergence n^{-3} .

Other idea to improve the speed of convergence is to consider new factors of the form

$$T_n = W_n \cdot \frac{P(n)}{Q(n)},$$

where P, Q are polynomials of equal degrees. More precisely, we define in this paper the sequences

$$\begin{aligned} \rho_n &= W_n \cdot \frac{n + \frac{5}{8}}{n + \frac{3}{8}} \\ \sigma_n &= W_n \cdot \frac{n^2 + \frac{9}{8}n + \frac{23}{64}}{n^2 + \frac{7}{8}n + \frac{15}{64}} \\ \tau_n &= W_n \cdot \frac{n^3 + \frac{13}{8}n^2 + \frac{37}{32}n + \frac{167}{512}}{n^3 + \frac{11}{8}n^2 + \frac{29}{32}n + \frac{105}{512}} \\ \chi_n &= W_n \cdot \frac{n^4 + \frac{17}{8}n^3 + \frac{161}{64}n^2 + \frac{389}{256}n + \frac{1473}{4096}}{n^4 + \frac{15}{8}n^3 + \frac{137}{64}n^2 + \frac{291}{256}n + \frac{945}{4096}} \end{aligned}$$

having increasingly speed of convergence. In fact, the sequences $\ln \rho_n, \ln \sigma_n, \ln \tau_n, \ln \chi_n$ converge to $\ln \frac{\pi}{2}$ with the speed of convergence n^{-3}, n^{-5}, n^{-7} and n^{-9} respectively.

2. Modifying the last fraction

First we modify the last fraction from (1.1) to define the sequence

$$(2.1) \quad t_n = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \dots \cdot 2n \cdot (2n + a)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot \dots \cdot (2n - 1)(2n + b)} = W_n \frac{(2n + a)(2n + 1)}{2n(2n + b)},$$

where a, b are real parameters. For $a = 0$ and $b = 1$, the Wallis sequence is obtained and we prove that for $a = -1/4$ and $b = 1/4$, the resulting sequence has a superior speed of convergence.

In this sense, let us define the sequence $(\omega_n)_{n \geq 1}$ by

$$(2.2) \quad \frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \dots \cdot 2n \cdot (2n + a)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot \dots \cdot (2n - 1)(2n + b)} \cdot \exp \omega_n = t_n \cdot \exp \omega_n.$$

Now we are in position to state the following

Theorem 2.1. *Let $(\omega_n)_{n \geq 1}$ be the sequence defined by (2.2), where a, b are real parameters. Then:*

i) *If $b - a \neq \frac{1}{2}$, then the speed of convergence of $(\omega_n)_{n \geq 1}$ is n^{-1} , since*

$$\lim_{n \rightarrow \infty} n \omega_n = \frac{1}{2} \left(b - a - \frac{1}{2} \right) \neq 0.$$

ii) *If $b - a = \frac{1}{2}$ and $(a, b) \neq \left(-\frac{1}{4}, \frac{1}{4}\right)$, then the speed of convergence of $(\omega_n)_{n \geq 1}$ is n^{-2} , since*

$$\lim_{n \rightarrow \infty} n^2 \omega_n = -\frac{4a + 1}{32} \neq 0.$$

iii) *If $(a, b) = \left(-\frac{1}{4}, \frac{1}{4}\right)$, then the speed of convergence of $(\omega_n)_{n \geq 1}$ is n^{-3} , since*

$$\lim_{n \rightarrow \infty} n^3 \omega_n = \frac{3}{256} \neq 0.$$

Proof. As we are interested to compute limits of the form (1.2), we develop in power series of n^{-1} the sequence

$$\omega_n - \omega_{n+1} = \ln \frac{2n(2n+2)(2n+2+a)(2n+b)}{(2n+1)^2(2n+2+b)(2n+a)},$$

namely

$$\begin{aligned} \omega_n - \omega_{n+1} &= \left(\frac{1}{2}b - \frac{1}{2}a - \frac{1}{4} \right) \frac{1}{n^2} + \left(\frac{1}{4}a^2 + \frac{1}{2}a - \frac{1}{4}b^2 - \frac{1}{2}b + \frac{1}{4} \right) \frac{1}{n^3} - \\ &- \left(\frac{1}{8}a^3 + \frac{3}{8}a^2 + \frac{1}{2}a - \frac{1}{8}b^3 - \frac{3}{8}b^2 - \frac{1}{2}b + \frac{7}{32} \right) \frac{1}{n^4} + O\left(\frac{1}{n^5}\right) \end{aligned}$$

and now the conclusion easily follows using Lemma 1.1. \square

3. Adding new rational factors

Other interesting method to improve the speed of convergence of the Wallis sequence is to add new factors of the form

$$T_n = W_n \cdot \frac{P(n)}{Q(n)},$$

where P, Q are polynomials of equal degrees, having the leading coefficient equal to one. In this sense, we give the following results:

Theorem 3.1. *Let us define the sequence $(\mu_n)_{n \geq 1}$ by*

$$\frac{\pi}{2} = W_n \cdot \frac{n+a}{n+b} \cdot \exp \mu_n.$$

i) If $a - b \neq \frac{1}{4}$, then the speed of convergence of $(\mu_n)_{n \geq 1}$ is n^{-1} , since

$$\lim_{n \rightarrow \infty} n\mu_n = b - a + \frac{1}{4} \neq 0.$$

ii) If $a - b = \frac{1}{4}$ and $(a, b) \neq (\frac{5}{8}, \frac{3}{8})$, then the speed of convergence of $(\mu_n)_{n \geq 1}$ is n^{-2} , since

$$\lim_{n \rightarrow \infty} n^2\mu_n = \frac{8a - 5}{32} \neq 0.$$

iii) If $(a, b) = (\frac{5}{8}, \frac{3}{8})$, then the speed of convergence of $(\mu_n)_{n \geq 1}$ is n^{-3} , since

$$\lim_{n \rightarrow \infty} n^3\mu_n = \frac{-3}{256} \neq 0.$$

Proof. We have

$$\mu_n - \mu_{n+1} = \ln \left(\frac{(2n+2)^2}{(2n+1)(2n+3)} \cdot \frac{n+1+a}{n+1+b} \cdot \frac{n+b}{n+a} \right)$$

or

$$\begin{aligned} \mu_n - \mu_{n+1} &= \left(b - a + \frac{1}{4} \right) \frac{1}{n^2} + \left(a^2 + a - b^2 - b - \frac{1}{2} \right) \frac{1}{n^3} - \\ &- \left(a^3 + \frac{3}{2}a^2 + a - b^3 - \frac{3}{2}b^2 - b - \frac{25}{32} \right) \frac{1}{n^4} + O\left(\frac{1}{n^5}\right), \end{aligned}$$

and the conclusion follows using Lemma 1.1. \square

Theorem 3.2. Let us define the sequence $(\nu_n)_{n \geq 1}$ by

$$\frac{\pi}{2} = W_n \cdot \frac{n^2 + an + b}{n^2 + cn + d} \cdot \exp \nu_n$$

where a, b, c, d are real parameters and denote:

$$\alpha = c - a + \frac{1}{4}$$

$$\beta = a^2 + a - c^2 - c - 2b + 2d - \frac{1}{2}$$

$$\gamma = 3b - a + c - 3d + 3ab - 3cd - \frac{3}{2}a^2 - a^3 + \frac{3}{2}c^2 + c^3 + \frac{25}{32}$$

$$\begin{aligned} \delta &= a^4 + 2a^3 - 4a^2b + 2a^2 - 6ab + a + 2b^2 - 4b - c^4 - \\ &- 2c^3 + 4c^2d - 2c^2 + 6cd - c - 2d^2 + 4d - \frac{9}{8} \end{aligned}$$

i) If $\alpha \neq 0$, then the speed of convergence of the sequence $(\nu_n)_{n \geq 1}$ is n^{-1} , since

$$\lim_{n \rightarrow \infty} n\nu_n = \alpha \neq 0.$$

ii) If $\alpha = 0$ and $\beta \neq 0$, then the speed of convergence of $(\nu_n)_{n \geq 1}$ is n^{-2} , since

$$\lim_{n \rightarrow \infty} n^2\nu_n = \frac{\beta}{2} \neq 0.$$

iii) If $\alpha = \beta = 0$, and $\gamma \neq 0$, then the speed of convergence of $(\mu_n)_{n \geq 1}$ is n^{-3} , since

$$\lim_{n \rightarrow \infty} n^3\mu_n = \frac{\gamma}{3} \neq 0.$$

iv) If $\alpha = \beta = \gamma = 0$ and $\delta \neq 0$, then the speed of convergence of $(\mu_n)_{n \geq 1}$ is n^{-4} , since

$$\lim_{n \rightarrow \infty} n^4\mu_n = \frac{\delta}{4} \neq 0.$$

v) If $\alpha = \beta = \gamma = \delta = 0$, (equivalent with $a = \frac{9}{8}$, $b = \frac{23}{64}$, $c = \frac{7}{8}$, $d = \frac{15}{64}$), then the speed of convergence of $(\mu_n)_{n \geq 1}$ is n^{-5} , since

$$\lim_{n \rightarrow \infty} n^5 \nu_n = \frac{45}{16384}.$$

Proof. We have

$$\nu_n - \nu_{n+1} = \ln \left(\frac{(2n+2)^2}{(2n+1)(2n+3)} \cdot \frac{(n+1)^2 + a(n+1) + b}{(n+1)^2 + c(n+1) + d} \cdot \frac{n^2 + cn + d}{n^2 + an + b} \right),$$

or

$$\begin{aligned} \nu_n - \nu_{n+1} &= \left(c - a + \frac{1}{4} \right) \frac{1}{n^2} + \left(a^2 + a - c^2 - c - 2b + 2d - \frac{1}{2} \right) \frac{1}{n^3} + \\ &+ \left(3b - a + c - 3d + 3ab - 3cd - \frac{3}{2}a^2 - a^3 + \frac{3}{2}c^2 + c^3 + \frac{25}{32} \right) \frac{1}{n^4} + \\ &+ \left(a^4 + 2a^3 - 4a^2b + 2a^2 - 6ab + a + 2b^2 - 4b - c^4 - \right. \\ &\left. - 2c^3 + 4c^2d - 2c^2 + 6cd - c - 2d^2 + 4d - \frac{9}{8} \right) \frac{1}{n^5} + O\left(\frac{1}{n^6}\right) \end{aligned}$$

and we recognize the coefficients $\alpha, \beta, \gamma, \delta$ in this power series. The conclusion follows using Lemma 1.1. \square

More accurate results can be established in case of the family of approximations of the form

$$\frac{\pi}{2} \approx W_n \cdot \frac{n^3 + an^2 + bn + c}{n^3 + dn^2 + fn + g},$$

where a, b, c, d, f, g are real parameters. As above, we introduce the sequence $(\psi_n)_{n \geq 1}$ by

$$\frac{\pi}{2} = W_n \cdot \frac{n^3 + an^2 + bn + c}{n^3 + dn^2 + fn + g} \cdot \exp \psi_n$$

and we can state the following

Theorem 3.3. *The fastest sequence of the form $(\psi_n)_{n \geq 1}$ is obtained for*

$$(3.1) \quad a = \frac{13}{8}, \quad b = \frac{37}{32}, \quad c = \frac{167}{512}, \quad d = \frac{11}{8}, \quad f = \frac{29}{32}, \quad g = \frac{105}{512}.$$

In this case, we have

$$\lim_{n \rightarrow \infty} n^7 \psi_n = -\frac{1575}{1048576}.$$

For every other real parameters a, b, c, d, f, g , different from the values (3.1), the speed of convergence of the sequence $(\psi_n)_{n \geq 1}$ is at most n^{-6} .

Proof. We have

$$\begin{aligned} \psi_n - \psi_{n+1} &= \ln \frac{(2n+2)^2}{(2n+1)(2n+3)} + \\ &+ \ln \left(\frac{(n+1)^3 + a(n+1)^2 + b(n+1) + c}{(n+1)^3 + d(n+1)^2 + f(n+1) + g} \cdot \frac{n^3 + dn^2 + fn + g}{n^3 + an^2 + bn + c} \right), \end{aligned}$$

or

$$\begin{aligned} \psi_n - \psi_{n+1} &= -\left(a - d - \frac{1}{4} \right) \frac{1}{n^2} + \left(a^2 + a - d^2 - d - 2b + 2f - \frac{1}{2} \right) \frac{1}{n^3} + \\ &+ \left(3b - a - 3c + d - 3f + 3g + 3ab - 3df - \frac{3}{2}a^2 - a^3 + \frac{3}{2}d^2 + d^3 + \frac{25}{32} \right) \frac{1}{n^4} - \end{aligned}$$

$$\begin{aligned}
& - (4b - a - 6c + d - 4f + 6g + 4a^2b - 4d^2f + 6ab - 4ac - 6df + 4dg - \\
& - 2a^2 - 2a^3 - a^4 - 2b^2 + 2d^2 + 2d^3 + d^4 + 2f^2 + \frac{9}{8}) \frac{1}{n^5} - \\
& - (a - 5b + 10c - d + 5f - 10g + 5ab^2 - 10a^2b - 5a^3b + 5a^2c - 5df^2 + \\
& + 10d^2f + 5d^3f - 5d^2g - 10ab + 10ac - 5bc + 10df - 10dg + 5fg + \frac{5}{2}a^2 + \\
& + \frac{10}{3}a^3 + \frac{5}{2}a^4 + 5b^2 + a^5 - \frac{5}{2}d^2 - \frac{10}{3}d^3 - \frac{5}{2}d^4 - d^5 - 5f^2 - \frac{301}{192}) \frac{1}{n^6} + \\
& + (-a^6 - 3a^5 + 6a^4b - 5a^4 + 15a^3b - 6a^3c - 5a^3 - 9a^2b^2 + 20a^2b - \\
& - 15a^2c - 3a^2 - 15ab^2 + 12abc + 15ab - 20ac - a + 2b^3 - 10b^2 + \\
& + 15bc + 6b - 3c^2 - 15c + d^6 + 3d^5 - 6d^4f + 5d^4 - 15d^3f + 6d^3g + \\
& + 5d^3 + 9d^2f^2 - 20d^2f + 15d^2g + 3d^2 + 15df^2 - 12dfg - 15df + \\
& + 20dg + d - 2f^3 + 10f^2 - 15fg - 6f + 3g^2 + 15g + \frac{69}{32}) \frac{1}{n^7} + O\left(\frac{1}{n^8}\right).
\end{aligned}$$

If we impose that the first coefficients of n^{-k} , for $k = 2, 3, 4, 5, 6, 7$, to vanish, then the obtained system with unknowns a, b, c, d, f, g has the unique solution (3.1).

Otherwise, if p denotes the smallest element of $\{2, 3, 4, 5, 6, 7\}$ such that the coefficient of n^{-p} is not zero, then the corresponding limit is non-zero:

$$\lim_{n \rightarrow \infty} n^p (\psi_n - \psi_{n+1}) \neq 0.$$

According with Lemma 1.1, the speed of convergence of the sequence $(\psi_n)_{n \geq 1}$ is n^{p-1} , which is less than n^{-7} . \square

4. Concluding Remarks

Similar results, which are increasingly accurate can be obtained if we consider approximations of the form

$$\frac{\pi}{2} \approx W_n \cdot \frac{P(n)}{Q(n)},$$

where $\deg P = \deg Q \geq 4$. In case of the polynomials of fourth degree, it can be proved that the best approximation is

$$\frac{\pi}{2} \approx W_n \cdot \frac{n^4 + \frac{17}{8}n^3 + \frac{161}{64}n^2 + \frac{389}{256}n + \frac{1473}{4096}}{n^4 + \frac{15}{8}n^3 + \frac{137}{64}n^2 + \frac{291}{256}n + \frac{945}{4096}}.$$

We omit the proof for sake of simplicity.

As above, the coefficients of these polynomials are the solution of the system defined by the first eight coefficients from the associated power series.

Moreover, the sequence $(\zeta_n)_{n \geq 1}$ defined by

$$\frac{\pi}{2} = W_n \cdot \frac{n^4 + \frac{17}{8}n^3 + \frac{161}{64}n^2 + \frac{389}{256}n + \frac{1473}{4096}}{n^4 + \frac{15}{8}n^3 + \frac{137}{64}n^2 + \frac{291}{256}n + \frac{945}{4096}} \cdot \exp \zeta_n$$

satisfies

$$\zeta_n - \zeta_{n+1} = \frac{893025}{67108864} \cdot \frac{1}{n^{10}} + O\left(\frac{1}{n^{11}}\right).$$

According to Lemma 1.1, the speed of convergence of $(\zeta_n)_{n \geq 1}$ is n^{-9} , since

$$\lim_{n \rightarrow \infty} n^9 \zeta_n = \frac{99225}{67108864}.$$

Our new defined sequences have great superiority over the Wallis sequence. Precisely, we tabulate the following numerical results:

n	$\frac{\pi}{2} - W_n$	$\rho_n - \frac{\pi}{2}$	$\frac{\pi}{2} - \sigma_n$	$\tau_n - \frac{\pi}{2}$	$\frac{\pi}{2} - \chi_n$
1	0.23746	5.0×10^{-3}	4.3×10^{-4}	7.2×10^{-5}	1.8×10^{-5}
10	3.7×10^{-2}	1.6×10^{-5}	3.4×10^{-8}	1.6×10^{-10}	1.4×10^{-12}
20	1.9×10^{-2}	2.1×10^{-6}	1.2×10^{-9}	1.5×10^{-12}	3.6×10^{-15}
50	7.8×10^{-3}	1.4×10^{-7}	1.3×10^{-11}	2.8×10^{-15}	1.1×10^{-18}
100	3.9×10^{-3}	1.8×10^{-8}	4.2×10^{-13}	2.3×10^{-17}	2.2×10^{-21}
300	1.3×10^{-3}	6.8×10^{-10}	1.8×10^{-15}	1.1×10^{-20}	1.2×10^{-25}
500	7.8×10^{-4}	1.5×10^{-10}	1.4×10^{-16}	3.0×10^{-22}	1.1×10^{-27}
1000	3.9×10^{-4}	1.8×10^{-11}	4.3×10^{-18}	2.4×10^{-24}	7.6×10^{-29}

4. Sharp bounds

Whenever an approximation formula of the form $f(n) \approx g(n)$ is given, there is a tendency to improve it by using a series of the form

$$f(n) \sim g(n) \exp\left(\sum_{k=1}^{\infty} \frac{a_k}{n^k}\right),$$

also called an asymptotic series. Although in asymptotic analysis, the problem of constructing asymptotic expansions is considered to be technically difficult, an elementary method for establishing the asymptotic series associated to the Wallis sequence was given by Mortici [2]

$$W_n \sim \frac{\pi}{2} \exp\left(-\frac{1}{4n} + \frac{1}{8n^2} - \frac{5}{96n^3} + \frac{1}{64n^4} - \dots\right).$$

Even if such asymptotic series may not converge, in a truncated form, it provides approximations of any desired accuracy. We prove the following sharp bounds for the sequence $(\rho_n)_{n \geq 1}$, arising from its asymptotic expansion, which can be constructed for example, using the method from [2].

Theorem 3.1. *For every integer $n \geq 1$, it holds*

$$\frac{\pi}{2} \exp\left(\frac{3}{256n^3} - \frac{9}{512n^4}\right) < \rho_n < \frac{\pi}{2} \exp\left(\frac{3}{256n^3} - \frac{9}{512n^4} + \frac{1441}{81920n^5}\right).$$

Proof. The sequences

$$x_n = \frac{\pi}{2\rho_n} \exp\left(\frac{3}{256n^3} - \frac{9}{512n^4}\right), \quad y_n = \frac{\pi}{2\rho_n} \exp\left(\frac{3}{256n^3} - \frac{9}{512n^4} + \frac{1441}{81920n^5}\right)$$

converge to 1, and we prove that $(x_n)_{n \geq 1}$ is strictly increasing and $(y_n)_{n \geq 1}$ is strictly decreasing. In consequence, $x_n < 1$ and $y_n > 1$ and the theorem is proved.

In this sense, we denote $x_{n+1}/x_n = \exp u(n)$ and $y_{n+1}/y_n = \exp v(n)$, where

$$u(t) = \ln(r(t)) + \frac{3}{256(t+1)^3} - \frac{9}{512(t+1)^4} - \frac{3}{256t^3} + \frac{9}{512t^4}$$

$$v(t) = \ln(r(t)) + \frac{3}{256(t+1)^3} - \frac{9}{512(t+1)^4} + \frac{1441}{81920(t+1)^5} - \frac{3}{256t^3} + \frac{9}{512t^4} - \frac{1441}{81920t^5}$$

with

$$r(t) = \frac{(2t+1)(2t+3)(8t+5)(8t+11)}{4(8t+3)(8t+13)(t+1)^2}.$$

We have $u' < 0$ and $v' > 0$, since

$$u'(t) = -\frac{9P(t)}{256t^5(t+1)^5(2t+1)(2t+3)(8t+3)(8t+5)(8t+11)(8t+13)}$$

$$v'(t) = \frac{Q(t)}{16384t^6(t+1)^6(2t+1)(2t+3)(8t+3)(8t+5)(8t+11)(8t+13)},$$

where

$$P(t) = 164427t + 919901t^2 + 2964474t^3 + 6067340t^4 + 8187804t^5 + 7305696t^6 + 4169712t^7 + 1382400t^8 + 202240t^9 + 12870$$

and

$$Q(t) = 124965786t + 745324705t^2 + 2597843148t^3 + 5881230025t^4 + 9094461530t^5 + 9845176300t^6 + 7519686680t^7 + 4003302912t^8 + 1413183488t^9 + 291323904t^{10} + 25165824t^{11} + 9272835.$$

Finally, u is strictly decreasing, v is strictly increasing, with $u(\infty) = v(\infty) = 0$, so $u > 0$ and $v < 0$ and the conclusion follows.

Now it is clear that our new method is suitable for establishing similar better results for all other sequences discussed here.

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References

- [1] M. Abramowitz and I. A. Stegun (Eds.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards, Applied Mathematics Series, 55, 9th printing, Dover, New York, 1972.
- [2] C. Mortici, Product approximations via asymptotic integration, Amer. Math. Monthly, 117 (2010), no. 5, 434-441.
- [3] C. Mortici, An ultimate extremely accurate formula for approximation of the factorial function, Arch. Math. (Basel), 93 (2009), no. 1, 37-45.
- [4] C. Mortici, New approximations of the gamma function in terms of the digamma function, Appl. Math. Lett., 23 (2010), no. 1, 97-100.
- [5] C. Mortici, New sharp bounds for gamma and digamma functions, An. Ştiinţ. Univ. A. I. Cuza Iaşi Ser. N. Matem., 57 (2011), no. 1, 57-60.
- [6] C. Mortici, Completely monotonic functions associated with gamma function and applications, Carpathian J. Math., 25 (2009), no. 2, 186-191.
- [7] C. Mortici, The proof of Muqattash-Yahdi conjecture, Math. Comput. Modelling, (2010), 51 (2010), no. 9-10, 1154-1159.
- [8] C. Mortici, Monotonicity properties of the volume of the unit ball in \mathbb{R}^n , Optimization Lett., 4 (2010), no. 3, 457-464.
- [9] C. Mortici, Sharp inequalities related to Gosper's formula, C. R. Math. Acad. Sci. Paris, 348 (2010), no. 3-4, 137-140.
- [10] C. Mortici, A class of integral approximations for the factorial function, Comp. Math. Appl., 59 (2010), no. 6, 2053-2058.
- [11] C. Mortici, Best estimates of the generalized Stirling formula, Appl. Math. Comp., 215 (2010), no. 11, 4044-4048.

- [12] C. Mortici, Very accurate estimates of the polygamma functions, *Asympt. Anal.*, 68 (2010), no. 3, 125-134.
- [13] C. Mortici, Improved convergence towards generalized Euler-Mascheroni constant, *Appl. Math. Comp.*, 215 (2010), no. 9, 3443-3448.
- [14] C. Mortici, A quicker convergence toward the γ constant with the logarithm term involving the constant e , *Carpathian J. Math.*, 26 (2010), no. 1, 86-91.
- [15] C. Mortici, Optimizing the rate of convergence in some new classes of sequences convergent to Euler's constant, *Anal. Appl. (Singap.)*, 8 (2010), no. 1, 99-107.

