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Approximation of fixed points of multifunctions in partial metric spaces

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Abstract

Recently, Reich and Zaslavski [S. Reich and A.J. Zaslavski, Convergence of Inexact Iterative Schemes for Nonexpansive Set-Valued Mappings, Fixed Point Theory Appl. 2010 (2010), Article ID 518243, 10 pages] have studied a new inexact iterative scheme for fixed points of contractive multifunctions. In this paper, using the partial Hausdorff metric introduced by Aydi et al., we prove an analogous to a result of Reich and Zaslavski for contractive multifunctions in the setting of partial metric spaces. An example is given to illustrate our result.

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1. Introduction

The study of iterative schemes for various classes of contractive and nonexpansive mappings is a central topic in metric fixed point theory. It started with the work of Banach [3] who proved a classical theorem, known as the Banach contraction principle, for the existence of a unique fixed point for a contraction. The importance of this result is also in the fact that it gives the convergence of an iterative scheme to a unique fixed point. Since Banach's result, there has been a lot of activity in this area and many developments have been taken place. In metric fixed point theory, there are many existence and approximation results for fixed points of those nonexpansive mappings which are not necessarily strictly contractive. Some authors have also provided results dealing with

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the existence and approximation of fixed points of certain classes of contractive multifunctions [4, 5, 7, 9, 12, 13]. In [12] Reich and Zaslavski introduced and studied new inexact iterative schemes for approximating fixed points of contractive and nonexpansive multifunctions. More recently, Aydi et al. [2] introduced a notion of partial Hausdoff metric type, that is a metric type associated to a partial metric. In [2] the authors using the partial Hausdorff metric proved an analogous to the well known Nadler's fixed point theorem [9]. In this paper, using the partial Hausdorff metric we prove an analogous to a result of [12] for contractive multifunctions in the setting of partial metric spaces. An example is given to illustrate our result.

2. Preliminaries

First, we recall some definitions of partial metric spaces that can be found in [6, 8, 10, 11, 14]. A partial metric on a nonempty set X is a function $p : X \times X \to [0, +\infty)$ such that for all $x, y, z \in X$:

(p1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;
(p2) $p(x, x) \leq p(x, y)$;
(p3) $p(x, y) = p(y, x)$;
(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x, y) = 0$, then from $(p1)$ and $(p2)$ it follows that $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic example of partial metric space is the pair $([0, +\infty), p)$, where $p(x, y) = \max\{x, y\}$.

Each partial metric p on X generates a T_0 topology τ_p on X, which has as a base the family of open p-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where

$$B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$$

for all $x \in X$, $\epsilon > 0$.

Let (X, p) be a partial metric space. A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to+\infty} p(x, x_n)$. A sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n, x_m)$. A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to+\infty} p(x_n, x_m)$.

A sequence $\{x_n\}$ in (X, p) is called $0-$Cauchy if $\lim_{n,m\to+\infty} p(x_n, x_m) = 0$. We say that (X, p) is $0-$complete if every $0-$Cauchy sequence in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = 0$.

Now, we recall the definition of partial Hausdorff metric and some property that can be found in [2]. Let $CB^p(X)$ be the family of all nonempty, closed and bounded subsets of the partial metric space (X, p), induced by the partial metric p. Note that closedness is taken from (X, τ_p) and boundedness is given as follows: A is a bounded subset in (X, p) if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0, M)$, that is, $p(x_0, a) < p(x_0, x_0) + M$.

For $A, B \in CB^p(X)$ and $x \in X$, define

$$\begin{aligned} p(x, A) &= \inf\{p(x, a),\ a \in A\},\ \delta_p(A, B) = \sup\{p(a, B) : a \in A\} \text{ and} \\ \delta_p(B, A) &= \sup\{p(b, A) : b \in B\}. \end{aligned}$$

2.1. Remark (see [1]). Let (X, p) be a partial metric space and A any nonempty set in (X, p), then

(2.1) $\quad a \in \bar{A}$ if and only if $p(a, A) = p(a, a)$, where $\bar{A}$ denotes the closure of A with respect to the partial metric p. Note that A is closed in (X,p) if and only if $A=\bar{A}$.

In the following proposition, we bring some properties of the mapping $\delta_p : CB^p(X)\times CB^p(X)\to[0,+\infty)$.

2.2. Proposition ([2], Proposition 2.2)**.** *Let (X,p) be a partial metric space. For any $A,B,C\in CB^p(X)$, we have the following:*

$$\begin{aligned} (i) &\ :\ \delta_p(A,A)=\sup\{p(a,a): a\in A\};\\ (ii) &\ :\ \delta_p(A,A)\le\delta_p(A,B);\\ (iii) &\ :\ \delta_p(A,B)=0 \textit{ implies that } A\subseteq B;\\ (iv) &\ :\ \delta_p(A,B)\le\delta_p(A,C)+\delta_p(C,B)-\inf_{c\in C}p(c,c). \end{aligned}$$

Let (X,p) be a partial metric space. For $A,B\in CB^p(X)$, define

$$H_p(A,B)=\max\{\delta_p(A,B),\delta_p(B,A)\}.$$

In the following proposition, we bring some properties of the mapping H_p.

2.3. Proposition ([2], Proposition 2.3)**.** *Let (X,p) be a partial metric space. For all $A,B,C\in CB^p(X)$, we have*

$$\begin{aligned} (h1) &\ :\ H_p(A,A)\le H_p(A,B);\\ (h2) &\ :\ H_p(A,B)=H_p(B,A);\\ (h3) &\ :\ H_p(A,B)\le H_p(A,C)+H_p(C,B)-\inf_{c\in C}p(c,c). \end{aligned}$$

2.4. Corollary ([2], Corollary 2.4)**.** *Let (X,p) be a partial metric space. For $A,B\in CB^p(X)$ the following holds:*

$$H_p(A,B)=0 \textit{ implies that } A=B.$$

2.5. Remark. The converse of Corollary 2.4 is not true in general as it is clear from the following example.

2.6. Example ([2], Example 2.6)**.** Let $X=[0,1]$ be endowed with the partial metric $p: X\times X\to\mathbb{R}^+$ defined by

$$p(x,y)=\max\{x,y\}.$$

From (i) of Proposition 2.2, we have

$$H_p(X,X)=\delta_p(X,X)=\sup\{x: 0\le x\le 1\}=1\neq 0.$$

In view of Proposition 2.3 and Corollary 2.4, we call the mapping $H_p : CB^p(X)\times CB^p(X)\to[0,+\infty)$, a partial Hausdorff metric induced by p.

2.7. Remark. It is easy to show that any Hausdorff metric is a partial Hausdorff metric. The converse is not true (see Example 2.6).

3. Main result

The following theorem is the main result.

3.1. Theorem. *Let (X,p) be a $0-$complete partial metric space, $T: X\to CB^p(X)$ a multifunctions and $\{\varepsilon_i\}$ and $\{\delta_i\}$ two sequences in $(0,+\infty)$ such that*

$$(3.1)\qquad \sum_{i=0}^{+\infty}\varepsilon_i<+\infty \quad \textit{and} \quad \sum_{i=0}^{+\infty}\delta_i<+\infty.$$

Suppose that there exists $k \in [0,1)$ such that

$$H_p(Tx, Ty) \leq kp(x,y) \quad \text{for all } x, y \in X. \tag{3.2}$$

Let $T_i : X \to 2^X \backslash \{\emptyset\}$ satisfy, for each integer $i \geq 0$,

$$H_p(Tx, T_i x) \leq \epsilon_i, \quad \text{for all } x \in X. \tag{3.3}$$

Assume that $x_0 \in X$ and that for each integer $i \geq 0$,

$$x_{i+1} \in T_i x_i, \quad p(x_i, x_{i+1}) \leq p(x_i, T_i x_i) + \delta_i. \tag{3.4}$$

Then, the sequence $\{x_i\}_{i=0}^{+\infty}$ converges to a fixed point of T.

Proof. We first show that $\{x_i\}_{i=0}^{+\infty}$ is a 0-Cauchy sequence. To this end, let $i \geq 0$ be an integer. Then, we have

$$\begin{aligned}
p(x_{i+1}, x_{i+2}) &\leq p(x_{i+1}, T_{i+1}x_{i+1}) + \delta_{i+1} \\
&\leq p(x_{i+1}, Tx_{i+1}) + \delta_p(Tx_{i+1}, T_{i+1}x_{i+1}) - \inf_{c \in Tx_{i+1}} p(c,c) + \delta_{i+1} \\
&\leq p(x_{i+1}, Tx_{i+1}) + \delta_p(Tx_{i+1}, T_{i+1}x_{i+1}) + \delta_{i+1} \\
&\leq p(x_{i+1}, Tx_{i+1}) + H_p(Tx_{i+1}, T_{i+1}x_{i+1}) + \delta_{i+1} \\
&\leq p(x_{i+1}, Tx_{i+1}) + \epsilon_{i+1} + \delta_{i+1} \\
&\leq H_p(T_i x_i, Tx_{i+1}) + \epsilon_{i+1} + \delta_{i+1} \\
&\leq H_p(T_i x_i, Tx_i) + H_p(Tx_i, Tx_{i+1}) - \inf_{c \in Tx_i} p(c,c) + \epsilon_{i+1} + \delta_{i+1} \\
&\leq H_p(Tx_i, Tx_{i+1}) + \epsilon_i + \epsilon_{i+1} + \delta_{i+1}.
\end{aligned}$$

Hence,

$$p(x_{i+1}, x_{i+2}) \leq kp(x_i, x_{i+1}) + \epsilon_i + \epsilon_{i+1} + \delta_{i+1}. \tag{3.5}$$

Now, we show by induction that for each $n \geq 1$, we have

$$p(x_n, x_{n+1}) \leq k^n p(x_0, x_1) + \sum_{i=0}^{n-1} k^i (\epsilon_{n-i} + \delta_{n-i} + \epsilon_{n-i-1}). \tag{3.6}$$

In view of (3.5), inequality (3.6) holds for $n = 1, 2$. Assume that $j \geq 1$ is an integer and that (3.6) holds for $n = j$. When combined with (3.5), this implies that

$$\begin{aligned}
p(x_{j+1}, x_{j+2}) &\leq kp(x_j, x_{j+1}) + \epsilon_{j+1} + \delta_{j+1} + \epsilon_j \\
&\leq k^{j+1} p(x_0, x_1) + \sum_{i=0}^{j-1} k^{i+1} (\epsilon_{j-i} + \delta_{j-i} + \epsilon_{j-i-1}) + \epsilon_{j+1} + \delta_{j+1} + \epsilon_j \\
&= k^{j+1} p(x_0, x_1) + \sum_{i=0}^{j} k^i (\epsilon_{j+1-i} + \delta_{j+1-i} + \epsilon_{j-i}).
\end{aligned}$$

This implies that (3.6) holds for all $n \geq 1$. From (3.6), by (3.1), we get

$$\begin{aligned}
\sum_{n=1}^{+\infty} p(x_n, x_{n+1}) &\leq \sum_{n=1}^{+\infty} \Big(k^n p(x_0, x_1) + \sum_{i=0}^{n-1} k^i (\epsilon_{n-i} + \delta_{n-i} + \epsilon_{n-i-1}) \Big) \\
&= \sum_{n=1}^{+\infty} \Big(k^n p(x_0, x_1) + \sum_{i=1}^{n} k^{n-i} (\epsilon_i + \delta_i + \epsilon_{i-1}) \Big) \\
&= \sum_{n=1}^{+\infty} k^n p(x_0, x_1) + (k^0 + k^1 + k^2 + \cdots)(\epsilon_1 + \delta_1 + \epsilon_0) \\
&+ (k^0 + k^1 + k^2 + \ldots)(\epsilon_2 + \delta_2 + \epsilon_1) \\
&+ (k^0 + k^1 + k^2 + \cdots)(\epsilon_3 + \delta_3 + \epsilon_2) + \cdots \\
&= \sum_{n=1}^{+\infty} k^n p(x_0, x_1) + \sum_{i=1}^{+\infty} \Big(\sum_{j=0}^{+\infty} k^j \Big) (\epsilon_i + \delta_i + \epsilon_{i-1}) \\
&\leq \Big(\sum_{n=0}^{+\infty} k^n \Big) \Big[p(x_0, x_1) + \sum_{n=1}^{+\infty} (\epsilon_n + \delta_n + \epsilon_{n-1}) \Big] < +\infty.
\end{aligned}$$

This implies that $\lim_{i,j \to +\infty} p(x_i, x_j) = 0$ and hence $\{x_i\}_{i=0}^{+\infty}$ is a 0-Cauchy sequence and so there exists $x^* \in X$ such that

$$\lim_{i \to +\infty} p(x_i, x^*) = p(x^*, x^*) = 0. \tag{3.7}$$

We claim that x^* is a fixed point of T, that is $x^* \in Tx^*$. From

$$H_p(Tx_i, Tx^*) \leq kp(x_i, x^*) \quad \text{for all } i \in \mathbb{N},$$

letting $i \to +\infty$, we obtain

$$\lim_{i \to +\infty} H_p(Tx_i, Tx^*) = 0. \tag{3.8}$$

As $x_{i+1} \in T_i x_i$ for all i, we have

$$\begin{aligned}
p(x_{i+1}, Tx^*) &\leq \delta_p(T_i x_i, Tx^*) \\
&\leq H_p(T_i x_i, Tx_i) + H_p(Tx_i, Tx^*) \\
&\leq \varepsilon_i + H_p(Tx_i, Tx^*).
\end{aligned}$$

Letting $i \to +\infty$, by (3.1) and (3.8), we obtain

$$\lim_{i \to +\infty} p(x_{i+1}, Tx^*) = 0. \tag{3.9}$$

Now, using (3.7) and (3.9), from

$$p(x^*, Tx^*) \leq p(x^*, x_{i+1}) + p(x_{i+1}, Tx^*) \quad \text{for all } i \in \mathbb{N},$$

as $i \to +\infty$ we deduce that $p(x^*, Tx^*) = 0$. Hence, $p(x^*, x^*) = p(x^*, Tx^*) = 0$ and so by Remark 2.1 we get that $x^* \in Tx^*$.

□

We also have the following result.

3.2. Theorem. *Let (X, p) be a $0-$complete partial metric space, $T : X \to CB^p(X)$ a multifunction and $\{\delta_i\}$ a sequence in $(0, +\infty)$ such that*

$$\sum_{i=0}^{+\infty} \delta_i < +\infty. \tag{3.10}$$

Suppose that there exists $k \in [0,1)$ such that

$$H_p(Tx,Ty) \leq kp(x,y) \quad \text{for all } x,y \in X. \tag{3.11}$$

Assume that $x_0 \in X$ and that for each integer $i \geq 0$,

$$x_{i+1} \in Tx_i, \quad p(x_i,x_{i+1}) \leq H_p(Tx_{i-1},Tx_i) + \delta_i. \tag{3.12}$$

Then, the sequence $\{x_i\}_{i=0}^{+\infty}$ converges to a fixed point of T.

Proof. We first show that $\{x_i\}_{i=0}^{+\infty}$ is a 0-Cauchy sequence. To this end, let $i \geq 0$ be an integer. Then, we have

$$p(x_{i+1},x_{i+2}) \leq H_p(Tx_i,Tx_{i+1}) + \delta_{i+1}.$$

Hence,

$$p(x_{i+1},x_{i+2}) \leq kp(x_i,x_{i+1}) + \delta_{i+1}. \tag{3.13}$$

Now, we show by induction that for each $n \geq 1$, we have

$$p(x_n,x_{n+1}) \leq k^n p(x_0,x_1) + \sum_{i=0}^{n-1} k^i \delta_{n-i}. \tag{3.14}$$

In view of (3.13), inequality (3.14) holds for $n = 1,2$. Assume that $j \geq 1$ is an integer and that (3.14) holds for $n = j$. When combined with (3.13), this implies that

$$\begin{aligned} p(x_{j+1},x_{j+2}) &\leq kp(x_j,x_{j+1}) + \delta_{j+1} \\ &\leq k^{j+1}p(x_0,x_1) + \sum_{i=0}^{j-1} k^{i+1}\delta_{j-i} + \delta_{j+1}. \end{aligned}$$

This implies that (3.14) holds for all $n \geq 1$. From (3.14), by (3.10), proceeding as in the proof of Theorem 3.1, we get

$$\begin{aligned} \sum_{n=1}^{+\infty} p(x_n,x_{n+1}) &\leq \sum_{n=1}^{+\infty} \Big(k^n p(x_0,x_1) + \sum_{i=0}^{n-1} k^i \delta_{n-i}\Big) \\ &= \sum_{n=1}^{+\infty} k^n p(x_0,x_1) + \sum_{i=1}^{+\infty} \Big(\sum_{j=0}^{\infty} k^j\Big)\delta_i \\ &\leq \Big(\sum_{n=0}^{+\infty} k^n\Big)\Big[p(x_0,x_1) + \sum_{n=1}^{+\infty} \delta_n\Big] < +\infty. \end{aligned}$$

This implies that $\lim_{n,m\to+\infty} p(x_i,x_j) = 0$ and hence $\{x_i\}_{i=0}^{+\infty}$ is a 0-Cauchy sequence and so there exists $x^* \in X$ such that

$$\lim_{i\to+\infty} p(x_i,x^*) = p(x^*,x^*) = 0. \tag{3.15}$$

We claim that x^* is a fixed point of T, that is $x^* \in Tx^*$. From

$$H_p(Tx_i,Tx^*) \leq kp(x_i,x^*) \qquad \text{for all } i \in \mathbb{N},$$

letting $i \to +\infty$, we obtain

$$\lim_{i\to\infty} H_p(Tx_i,Tx^*) = 0.$$

As $x_{i+1} \in Tx_i$ for all i, we have

$$p(x_{i+1},Tx^*) \leq \delta_p(Tx_i,Tx^*) \leq H_p(Tx_i,Tx^*).$$

Letting $i \to +\infty$, we get that

$$\lim_{i\to+\infty} p(x_{i+1},Tx^*) = 0. \tag{3.16}$$

Now, using (3.15) and (3.16), from

$$p(x^*, Tx^*) \leq p(x^*, x_{i+1}) + p(x_{i+1}, Tx^*) \qquad \text{for all } \; i \in \mathbb{N},$$

as $i \to +\infty$ we deduce that $p(x^*, Tx^*) = 0$. Hence, $p(x^*, x^*) = p(x^*, Tx^*) = 0$ and so $x^* \in Tx^*$, that is x^* is a fixed point of T. □

3.3. Lemma. *Let (X, p) be a partial metric space, $A, B \in CB^p(X)$ and $\alpha > 0$. For any $a \in A$, there exists $b = b(a) \in B$ such that*

(3.17) $\quad p(a, b) \leq H_p(A, B) + \alpha.$

Proof. Without loss of generality, we can assume that $H_p(A, B) > 0$. If we choose $h > 1$ such that $h\, H_p(A, B) = H_p(A, B) + \alpha$, the existence of $b \in B$ satisfying (3.17) follows from Lemma 3.1 of [2]. □

3.4. Lemma. *Let (X, p) be a partial metric space, $T : X \to CB^p(X)$ a multifunction. Suppose that there exists $k \in [0, 1)$ such that*

(3.18) $\quad H_p(Tx, Ty) \leq kp(x, y) \qquad$ *for all* $\; x, y \in X.$

Then for all $x_0 \in X$ there exists a sequence $\{x_i\}_{i=0}^{+\infty}$ such that

(3.19) $\quad x_{i+1} \in Tx_i, \quad p(x_i, x_{i+1}) \leq H_p(Tx_{i-1}, Tx_i) + k^i.$

Proof. We may assume $k > 0$. Choose $x_1 \in Tx_0$. As $Tx_0, Tx_1 \in CB^p(X)$ and $x_1 \in Tx_0$, there is a point $x_2 \in Tx_1$ such that

$$p(x_1, x_2) \leq H_p(Tx_0, Tx_1) + k.$$

Now, since $Tx_1, Tx_2 \in CB^p(X)$ and $x_2 \in Tx_1$ there is a point $x_3 \in Tx_2$ such that $p(x_2, x_3) \leq H_p(Tx_1, Tx_2) + k^2$. Continuing in this way we produce a sequence $\{x_i\}_{i=0}^{+\infty}$ of points of X such that $x_{i+1} \in Tx_i$ and $p(x_i, x_{i+1}) \leq H_p(Tx_{i-1}, Tx_i) + k^i$ for all $i \geq 1$. □

From Theorem 3.2 and Lemma 3.4, we deduce the following result, which generalizes Theorem 3.2 of [2].

3.5. Theorem. *Let (X, p) be a 0-complete partial metric space. If $T : X \to CB^p(X)$ is a multifunction such that for all $x, y \in X$, we have*

(3.20) $\quad H_p(Tx, Ty) \leq k\, p(x, y)$

where $k \in [0, 1)$. Then T has a fixed point.

To illustrate the usefulness of our result, we give the following example.

3.6. Example. Let $X = [0, 2]$ be endowed with the usual metric. Define the multifunctions $T, T_i : X \to CB^p(X)$ by

$$Tx = \begin{cases} [\frac{x}{8}, \frac{x}{4}] & \text{if } x \in [0, 1], \\ \{0\} & \text{otherwise.} \end{cases}$$

$$T_i x = \begin{cases} [\frac{x}{8} - \frac{x}{8^{i+2}}, \frac{x}{4}] & \text{if } x \in [0, 1], \\ \{0\} & \text{otherwise.} \end{cases}$$

It is easy to see that Theorem 2.1 of [12] is not applicable in this case. Indeed, for $x = \dfrac{19}{18}$ and $y = \dfrac{8}{9}$, we have

$$\begin{aligned} H(T(\frac{19}{18}), T(\frac{8}{9})) &= H(\{0\}, [\frac{1}{9}, \frac{2}{9}]) \\ &= \frac{2}{9} \not\leq \frac{k}{6} = k\, d(\frac{19}{18}, \frac{8}{9}), \end{aligned}$$

for any $k \in [0,1)$.

On the other hand, if we endow X with the partial metric defined by

$$p(x,y) = \begin{cases} |x-y| & \text{if } x,y \in [0,1], \\ \frac{|x-y|}{4} + \frac{\max\{x,y\}}{2} & \text{otherwise.} \end{cases}$$

Then (X,p) is a complete partial metric space and Tx is closed for all $x \in X$.

We shall show that for all $x,y \in X$, (3.20) is satisfied with $k = \frac{2}{3}$.

Consider the following cases:

- If $x \in [0,1]$ and $y \in (1,2]$, then $p(x,y) = \frac{3}{4}y - \frac{x}{4} > \frac{1}{2}$ and

$$\begin{aligned} H_p(Tx,Ty) &= H_p([\frac{x}{8},\frac{x}{4}],\{0\}) \\ &= \max\{\frac{x}{8},\frac{x}{4}\} = \frac{x}{4} \le \frac{1}{4} < \frac{1}{3} = \frac{k}{2} < kp(x,y). \end{aligned}$$

- If $x,y \in (1,2]$, then $H_p(Tx,Ty) = H_p(\{0\},\{0\}) = 0$ and (3.20) is satisfied obviously.
- If $x,y \in [0,1]$, with $x \le y$, then

$$\begin{aligned} H_p(Tx,Ty) &= H_p([\frac{x}{8},\frac{x}{4}],[\frac{y}{8},\frac{y}{4}]) \\ &= \max\{\frac{y-x}{8},\frac{y-x}{4}\} \\ &= \frac{y-x}{4} < \frac{2}{3}(y-x) = kp(x,y). \end{aligned}$$

It is easy to see that $H_p(T_ix,Tx) \le 1/8^{i+2}$ for all $x \in X$. Moreover, for all $x_0 \in [0,1]$ the sequence $\{x_i\}_{i=0}^{+\infty}$ defined by $x_{i+1} = x_i/4 - x_i/4^{i+2} \in T_ix_i$ for all $i \ge 0$ is such that

$$p(x_i,x_{i+1}) = \frac{3}{4}x_i + \frac{x_i}{4^{i+2}} = p(x_i,T_ix_i) + \frac{x_i}{4^{i+2}} \le p(x_i,T_ix_i) + \frac{1}{4^{i+2}}.$$

If $x_0 \in (1,2]$, then we choose $x_i = 0$ for all $i > 0$.

Thus, all the conditions of Theorem 3.1 are satisfied with $\varepsilon_i = 1/8^{i+2}$ and $\delta_i = 1/4^{i+2}$. Moreover, $x_i \to 0$ and $x = 0$ is a fixed point of T in X.

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