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Existence and nonexistence results for a fourth-order discrete Dirichlet boundary value problem

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Abstract

In this paper, a fourth-order nonlinear difference equation is considered. By making use of the critical point theory, we establish various sets of sufficient conditions for the existence and nonexistence of solutions for Dirichlet boundary value problem and give some new results. Our results generalize and complement the results in the literature.

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1. Introduction

Difference equations have attracted the interest of many researchers in the past twenty years since they provided a natural description of several discrete models. Such discrete models are often investigated in various fields of science and technology such as computer science, economics, *neural networks*, ecology, cybernetics, biological systems, optimal control, and population dynamics. These studies cover many of the branches of difference equations, such as stability, attractivity, periodicity, oscillation, and boundary value *problems*, see [6,12-14,16,18,19,21,26,27] and the references therein.

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Below **N**, **Z** and **R** denote the sets of all natural numbers, integers and real numbers respectively. k is a positive integer. For any a, $b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \cdots\}$, $\mathbf{Z}(a, b) = \{a, a+1, \cdots, b\}$ when $a < b$. Besides, * denotes the transpose of a vector.

The present paper considers the fourth-order nonlinear difference equation

$$\Delta^2\left(p_{n-1}\Delta^2 u_{n-2}\right) - \Delta\left(q_n \Delta u_{n-1}\right) = f(n, u_{n+1}, u_n, u_{n-1}),\ n \in \mathbf{Z}(1, k), \tag{1.1}$$

with boundary value conditions

$$u_{-1} = u_0 = 0,\ u_{k+1} = u_{k+2} = 0, \tag{1.2}$$

where Δ is the forward difference operator $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_n = \Delta(\Delta u_n)$, p_n is nonzero and real valued for each $n \in \mathbf{Z}(0, k+1)$, q_n is real valued for each $n \in \mathbf{Z}(1, k+1)$, $f \in C(\mathbf{R}^4, \mathbf{R})$.

In recent years the study of boundary value problems for differential equations develops at relatively rapid rate. By using various methods and techniques, such as fixed point theory, topological degree theory, coincidence degree theory, a series of existence results of nontrivial solutions for differential equations have been obtained in literatures, we refer to [1-3,5,15,30]. And critical point theory is also an important tool to deal with problems on differential equations [9,11,20,25,35]. Only since 2003, critical point theory has been employed to establish sufficient conditions on the existence of periodic solutions of difference equations. By using the critical point theory, Guo and Yu [12-14] and Shi *et al.* [28] have successfully proved the existence of periodic solutions of second-order nonlinear difference equations. We also refer to [32,33] for the discrete boundary value problems. Compared to *first-order* or second-order difference equations, the study of higher-order equations, and in particular, fourth-order equations, has received considerably less attention (see, for example, [7,8,10,23,24,26,29,31] and the references contained therein). Yan, Liu [31] in 1997 and Thandapani, Arockiasamy [29] in 2001 studied the following fourth-order difference equation of form,

$$\Delta^2\left(p_n \Delta^2 u_n\right) + f(n, u_n) = 0,\ n \in \mathbf{Z}, \tag{1.3}$$

and obtained criteria for the oscillation and nonoscillation of solutions for equation (1.3). In 2005, Cai, Yu and Guo [4] have obtained some criteria for the existence of periodic solutions of the fourth-order difference equation

$$\Delta^2\left(p_{n-2}\Delta^2 u_{n-2}\right) + f(n, u_n) = 0,\ n \in \mathbf{Z}. \tag{1.4}$$

In 1995, Peterson and Ridenhour considered the disconjugacy of equation (1.7) when $p_n \equiv 1$ and $f(n, u_n) = q_n u_n$ (see [23]).

The boundary value problem (BVP) for determining the existence of solutions of difference equations has been a very active area of research in the last twenty years, and for surveys of recent results, we refer the reader to the monographs by Agarwal *et al.* [17,21,27]. As far as we know results obtained in the literature for the BVP (1.1) with (1.2) are very scarce. Since f in (1.1) depends on u_{n+1} and u_{n-1}, the traditional ways of establishing the functional in [12-14,32-34] are inapplicable to our case. As a result, the goal of this paper is to fill the gap in this area.

Motivated by the above results, *we use* the critical point theory to give some sufficient conditions *for* the existence and nonexistence of solutions for the BVP (1.1) with (1.2). We shall study the *superlinear* and sublinear cases. The main idea in this paper is to transfer the existence of the BVP (1.1) with (1.2) into the existence of the critical points of some functional. The proof is based on the notable Mountain Pass Lemma in combination with variational technique. The purpose of this paper is two-folded. On one hand, we shall further demonstrate the powerfulness of critical point theory in the study of solutions for boundary value problems of difference equations. On the other hand, we shall complement existing results. The motivation for the present work stems from the recent paper in [7].

Let

$$\bar{p} = \max\{p_n : n \in \mathbf{Z}(0, k+1)\},\ \underline{p} = \min\{p_n : n \in \mathbf{Z}(0, k+1)\},$$

$$\bar{q} = \max\{q_n : n \in \mathbf{Z}(1, k+1)\},\ \underline{q} = \min\{q_n : n \in \mathbf{Z}(1, k+1)\}.$$

Our main results are as follows.

Theorem 1.1. *Assume that the following hypotheses are satisfied:*
(p) *for any* $n \in \boldsymbol{Z}(0, k+1)$*,* $p_n < 0$*;*
(q) *for any* $n \in \boldsymbol{Z}(1, k+1)$*,* $q_n \leq 0$*;*
(F_1) *there exists a functional* $F(n, \cdot) \in C^1(\boldsymbol{Z} \times \boldsymbol{R}^2, \boldsymbol{R})$ *with* $F(0, \cdot) = 0$ *such that*

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = f(n, v_1, v_2, v_3),\ \forall n \in \boldsymbol{Z}(1, k);$$

(F_2) *there exists a constant* $M_0 > 0$ *such that for all* $(n, v_1, v_2) \in \boldsymbol{Z}(1, k) \times \boldsymbol{R}^2$

$$\left|\frac{\partial F(n, v_1, v_2)}{\partial v_1}\right| \leq M_0,\ \left|\frac{\partial F(n, v_1, v_2)}{\partial v_2}\right| \leq M_0.$$

Then the BVP (1.1) with (1.2) possesses at least one solution.

Remark 1.1. Assumption (F_2) implies that there exists a constant $M_1 > 0$ such that
(F_2') $|F(n, v_1, v_2)| \leq M_1 + M_0(|v_1| + |v_2|),\ \forall (n, v_1, v_2) \in \mathbf{Z}(1, k) \times \mathbf{R}^2$.

Theorem 1.2. *Suppose that* (F_1) *and the following hypotheses are satisfied:*
(p') *for any* $n \in \boldsymbol{Z}(0, k+1)$*,* $p_n > 0$*;*
(q') *for any* $n \in \boldsymbol{Z}(1, k+1)$*,* $q_n \geq 0$*;*
(F_3) *there exists a functional* $F(n, \cdot) \in C^1(\boldsymbol{Z} \times \boldsymbol{R}^2, \boldsymbol{R})$ *such that*

$$\lim_{r \to 0} \frac{F(n, v_1, v_2)}{r^2} = 0,\ r = \sqrt{v_1^2 + v_2^2},\ \forall n \in \boldsymbol{Z}(1, k);$$

(F_4) *there exists a constant* $\beta > 2$ *such that for any* $n \in \boldsymbol{Z}(1, k)$*,*

$$0 < \frac{\partial F(n, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(n, v_1, v_2)}{\partial v_2} v_2 < \beta F(n, v_1, v_2),\ \forall (v_1, v_2) \neq 0.$$

Then the BVP (1.1) with (1.2) possesses at least two nontrivial solutions.

Remark 1.2. Assumption (F_4) implies that there exist constants $a_1 > 0$ and $a_2 > 0$ such that
(F_4') $F(n, v_1, v_2) > a_1 \left(\sqrt{v_1^2 + v_2^2}\right)^{\beta} - a_2,\ \forall n \in \mathbf{Z}(1, k)$.

Theorem 1.3. *Suppose that* (p')*,* (q')*,* (F_1) *and the following assumption are satisfied:*
(F_5) *there exist constants* $R > 0$ *and* $1 < \alpha < 2$ *such that for* $n \in \boldsymbol{Z}(1, k)$ *and* $\sqrt{v_1^2 + v_2^2} \geq R$*,*

$$0 < \frac{\partial F(n, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(n, v_1, v_2)}{\partial v_2} v_2 \leq \alpha F(n, v_1, v_2).$$

Then the BVP (1.1) with (1.2) possesses at least one solution.

Remark 1.3. Assumption (F_5) implies that for each $n \in \mathbf{Z}(1, k)$ there exist constants

$a_3 > 0$ and $a_4 > 0$ such that

(F_5') $F(n, v_1, v_2) \leq a_3 \left(\sqrt{v_1^2 + v_2^2}\right)^\alpha + a_4, \ \forall (n, v_1, v_2) \in \mathbf{Z}(1, k) \times \mathbf{R}^2.$

Theorem 1.4. *Suppose that* (p), (q), (F_1) *and the following assumption are satisfied:*

(F_6) $v_2 f(n, v_1, v_2, v_3) > 0$, *for* $v_2 \neq 0$, $\forall n \in \boldsymbol{Z}(1, k)$.

Then the BVP (1.1) with (1.2) has no nontrivial solutions.

Remark 1.4. In the existing literature, results on the nonexistence of solutions of discrete boundary value problems are scarce. Hence, Theorem 1.4 complements existing ones.

The *remainder* of this paper is organized as follows. *First*, in Section 2, we shall establish the variational framework for the BVP (1.1) with (1.2) and transfer the problem of the existence of the BVP (1.1) with (1.2) into that of the existence of critical points of the corresponding functional. Some related fundamental results will also be recalled. Then, in Section 3, we shall complete the proof of the results by using the critical point method. Finally, in Section 4, we shall give three examples to illustrate the main results.

For the basic knowledge *of* variational methods, *the reader is referred to* [20,22,25,35].

2. Variational structure and some lemmas

In order to apply the critical point theory, we shall establish the corresponding variational framework for the BVP (1.1) with (1.2) and give some lemmas which will be of fundamental importance in proving our main results. Firstly, we state some basic notations.

Let $\mathbf{R}^k$ be the real Euclidean space with dimension k. Define the inner product on $\mathbf{R}^k$ as follows:

$$\langle u, v \rangle = \sum_{j=1}^{k} u_j v_j, \ \forall u, v \in \mathbf{R}^k, \tag{2.1}$$

by which the norm $\| \cdot \|$ can be induced by

$$\|u\| = \left(\sum_{j=1}^{k} u_j^2 \right)^{\frac{1}{2}}, \ \forall u \in \mathbf{R}^k. \tag{2.2}$$

On the other hand, we define the norm $\| \cdot \|_r$ on $\mathbf{R}^k$ as follows:

$$\|u\|_r = \left(\sum_{j=1}^{k} |u_j|^r \right)^{\frac{1}{r}}, \tag{2.3}$$

for all $u \in \mathbf{R}^k$ and $r > 1$.

Since $\|u\|_r$ and $\|u\|_2$ are equivalent, there exist constants c_1, c_2 such that $c_2 \geq c_1 > 0$, and

$$c_1 \|u\|_2 \leq \|u\|_r \leq c_2 \|u\|_2, \ \forall u \in \mathbf{R}^k. \tag{2.4}$$

Clearly, $\|u\| = \|u\|_2$. For any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$, for the BVP (1.1) with (1.2), consider the functional J defined on $\mathbf{R}^k$ as follows:

$$J(u) = \frac{1}{2} \sum_{n=-1}^{k} p_{n+1} \left(\Delta^2 u_n\right)^2 + \frac{1}{2} \sum_{n=0}^{k} q_{n+1} \left(\Delta u_n\right)^2 - \sum_{n=1}^{k} F(n, u_{n+1}, u_n), \tag{2.5}$$

where

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = f(n, v_1, v_2, v_3),$$

$$u_{-1} = u_0 = 0, \ u_{k+1} = u_{k+2} = 0.$$

Clearly, $J \in C^1(\mathbf{R}^k, \mathbf{R})$ and for any $u = \{u_n\}_{n=1}^k = (u_1, u_2, \ldots, u_k)^*$, by using $u_{-1} = u_0 = 0, \ u_{k+1} = u_{k+2} = 0$, we can compute the partial derivative as

$$\frac{\partial J}{\partial u_n} = \Delta^2 \left(p_{n-1}\Delta^2 u_{n-2}\right) - \Delta\left(q_n \Delta u_{n-1}\right) - f(n, u_{n+1}, u_n, u_{n-1}), \ \forall n \in \mathbf{Z}(1,k).$$

Thus, u is a critical point of J on $\mathbf{R}^k$ if and only if

$$\Delta^2 \left(p_{n-1}\Delta^2 u_{n-2}\right) - \Delta\left(q_n \Delta u_{n-1}\right) = f(n, u_{n+1}, u_n, u_{n-1}), \ \forall n \in \mathbf{Z}(1,k).$$

We reduce the existence of the BVP (1.1) with (1.2) to the existence of critical points of J on $\mathbf{R}^k$. That is, the functional J is just the variational framework of the BVP (1.1) with (1.2).

Let P and Q be the $k \times k$ matrices defined by

$$P = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & 6 & -4 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -4 & 6 \end{pmatrix},$$

$$Q = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

Clearly, P and Q are positive definite. Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be the eigenvalues of P, $\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots, \tilde{\lambda}_k$ be the eigenvalues of Q. Applying matrix theory, we know $\lambda_j > 0, \ \tilde{\lambda}_j > 0, \ j = 1, 2, \cdots, k$. Without loss of generality, we may assume that

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k, \tag{2.6}$$

$$0 < \tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \cdots \le \tilde{\lambda}_k. \tag{2.7}$$

Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\left\{u^{(l)}\right\} \subset E$ for which $\left\{J\left(u^{(l)}\right)\right\}$ is bounded and $J'\left(u^{(l)}\right) \to 0 (l \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1 *(Mountain Pass Lemma [25]). Let E be a real Banach space and $J \in C^1(E, \boldsymbol{R})$ satisfy the P.S. condition. If $J(0) = 0$ and*
(J_1) there exist constants $\rho, \ a > 0$ such that $J|_{\partial B_\rho} \ge a$, and
(J_2) there exists $e \in E \setminus B_\rho$ such that $J(e) \le 0$.
Then J possesses a critical value $c \ge a$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} J(g(s)), \tag{2.8}$$

where

(2.9) $\Gamma = \{g \in C([0,1], E) | g(0) = 0,\ g(1) = e\}.$

Lemma 2.2. *Suppose that* (p'), (q'), (F_1), (F_3) *and* (F_4) *are satisfied. Then the functional* J *satisfies the P.S. condition.*

Proof. Let $u^{(l)} \in \mathbf{R}^k,\ l \in \mathbf{Z}(1)$ be such that $\left\{J\left(u^{(l)}\right)\right\}$ is bounded. Then there exists a positive constant M_2 such that

$$-M_2 \leq J\left(u^{(l)}\right) \leq M_2,\ \forall l \in \mathbf{N}.$$

By (F_4'), we have

$$\begin{aligned}
-M_2 \leq J\left(u^{(l)}\right) &= \frac{1}{2}\sum_{n=-1}^{k} p_{n+1}\left(\Delta^2 u_n^{(l)}\right)^2 + \frac{1}{2}\sum_{n=0}^{k} q_{n+1}\left(\Delta u_n^{(l)}\right)^2 - \sum_{n=1}^{k} F\left(n, u_{n+1}^{(l)}, u_n^{(l)}\right) \\
&\leq \frac{1}{2}\bar{p}\sum_{n=-1}^{k}\left(u_{n+2}^{(l)} - 2u_{n+1}^{(l)} + u_n^{(l)}\right)^2 + \frac{1}{2}\bar{q}\sum_{n=0}^{k}\left(u_{n+1}^{(l)} - u_n^{(l)}\right)^2 \\
&\quad - a_1\sum_{n=1}^{k}\left[\sqrt{\left(u_{n+1}^{(l)}\right)^2 + \left(u_n^{(l)}\right)^2}\right]^\beta + a_2 k \\
&\leq \frac{1}{2}\bar{p}\left(u^{(l)}\right)^* P u^{(l)} + \frac{1}{2}\bar{q}\left(u^{(l)}\right)^* Q u^{(l)} - a_1 c_1^\beta \left\|u^{(l)}\right\|^\beta + a_2 k \\
&\leq \frac{1}{2}\bar{p}\lambda_k\left\|u^{(l)}\right\|^2 + \frac{1}{2}\bar{q}\tilde{\lambda}_k\left\|u^{(l)}\right\|^2 - a_1 c_1^\beta\left\|u^{(l)}\right\|^\beta + a_2 k,
\end{aligned}$$

where $u^{(l)} = \left(u_1^{(l)}, u_2^{(l)}, \cdots, u_k^{(l)}\right)^*,\ u^{(l)} \in \mathbf{R}^k$. That is,

$$a_1 c_1^\beta\left\|u^{(l)}\right\|^\beta - \frac{1}{2}\left(\bar{p}\lambda_k + \bar{q}\tilde{\lambda}_k\right)\left\|u^{(l)}\right\|^2 \leq M_2 + a_2 k.$$

Since $\beta > 2$, there exists a constant $M_3 > 0$ such that

$$\left\|u^{(l)}\right\| \leq M_3,\ \forall l \in \mathbf{N}.$$

Therefore, $\left\{u^{(l)}\right\}$ is bounded on $\mathbf{R}^k$. As a consequence, $\left\{u^{(l)}\right\}$ possesses a convergence subsequence in $\mathbf{R}^k$. Thus the P.S. condition is verified. □

3. Proof of the main results

In this Section, we shall prove our main results by using the critical point theory.

3.1. Proof of Theorem 1.1

Proof. By (F_2'), for any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$, we have

$$\begin{aligned}
J(u) &= \frac{1}{2}\sum_{n=-1}^{k} p_{n+1}\left(\Delta^2 u_n\right)^2 + \frac{1}{2}\sum_{n=0}^{k} q_{n+1}\left(\Delta u_n\right)^2 - \sum_{n=1}^{k} F(n, u_{n+1}, u_n) \\
&\leq \frac{1}{2}\bar{p}\sum_{n=-1}^{k}\left(u_{n+2} - 2u_{n+1} + u_n\right)^2 + \frac{1}{2}\bar{q}\sum_{n=0}^{k}\left(u_{n+1} - u_n\right)^2 + M_0\sum_{n=1}^{k}\left(|u_{n+1}| + |u_n|\right) + M_1 k \\
&\leq \frac{1}{2}\bar{p}u^* P u + \frac{1}{2}\bar{q}u^* Q u + 2M_0\sum_{n=1}^{k}|u_n| + M_1 k
\end{aligned}$$

$$\begin{aligned}&\le \frac{1}{2}\bar{p}\lambda_1\|u\|^2+\frac{1}{2}\bar{q}\tilde{\lambda}_1\|u\|^2+2M_0\sqrt{k}\|u\|+M_1k\\ &\quad\to -\infty \text{ as } \|u\|\to+\infty.\end{aligned}$$

The above inequality means that $-J(u)$ is coercive. By the continuity of $J(u)$, J attains its maximum at some point, and we denote it $\breve{u}$, that is,

$$J(\breve{u})=\max\left\{J(u)|u\in\mathbf{R}^k\right\}.$$

Clearly, $\breve{u}$ is a critical point of the functional J. This completes the proof of Theorem 1.1. □

3.2. Proof of Theorem 1.2

Proof. By (F_3), for any $\epsilon=\frac{1}{8}\left(\underline{p}\lambda_1+\underline{q}\tilde{\lambda}_1\right)$($\lambda_1$ and $\tilde{\lambda}_1$ can be referred to (2.6) and (2.7)), there exists $\rho>0$, such that

$$|F(n,v_1,v_2)|\le\frac{1}{8}\left(\underline{p}\lambda_1+\underline{q}\tilde{\lambda}_1\right)\left(v_1^2+v_2^2\right),\forall n\in\mathbf{Z}(1,k),$$

for $\sqrt{v_1^2+v_2^2}\le\sqrt{2}\rho$.

For any $u=(u_1,u_2,\cdots,u_k)^*\in\mathbf{R}^k$ and $\|u\|\le\rho$, we have $|u_n|\le\rho,\ n\in\mathbf{Z}(1,k)$.

For any $n\in\mathbf{Z}(1,k)$,

$$\begin{aligned}J(u)&=\frac{1}{2}\sum_{n=-1}^{k}p_{n+1}\left(\Delta^2u_n\right)^2+\frac{1}{2}\sum_{n=0}^{k}q_{n+1}\left(\Delta u_n\right)^2-\sum_{n=1}^{k}F(n,u_{n+1},u_n)\\ &\ge\frac{1}{2}\underline{p}\sum_{n=-1}^{k}\left(u_{n+2}-2u_{n+1}+u_n\right)^2+\frac{1}{2}\underline{q}\sum_{n=0}^{k}\left(u_{n+1}-u_n\right)^2-\frac{1}{8}\left(\underline{p}\lambda_1+\underline{q}\tilde{\lambda}_1\right)\sum_{n=1}^{k}\left(u_{n+1}^2+u_n^2\right)\\ &\ge\frac{1}{2}\underline{p}u^*Pu+\frac{1}{2}\underline{q}u^*Qu-\frac{1}{4}\left(\underline{p}\lambda_1+\underline{q}\tilde{\lambda}_1\right)\|u\|^2\\ &\ge\frac{1}{2}\underline{p}\lambda_1\|u\|^2+\frac{1}{2}\underline{q}\tilde{\lambda}_1\|u\|^2-\frac{1}{4}\left(\underline{p}\lambda_1+\underline{q}\tilde{\lambda}_1\right)\|u\|^2\\ &=\frac{1}{4}\left(\underline{p}\lambda_1+\underline{q}\tilde{\lambda}_1\right)\|u\|^2,\end{aligned}$$

where $u=(u_1,u_2,\cdots,u_k)^*,\ u\in\mathbf{R}^k$.

Take $a=\frac{1}{4}\left(\underline{p}\lambda_1+\underline{q}\tilde{\lambda}_1\right)\rho^2>0$. Therefore,

$$J(u)\ge a>0,\ \forall u\in\partial B_\rho.$$

At the same time, we have also proved that there exist constants $a>0$ and $\rho>0$ such that $J|_{\partial B_\rho}\ge a$. That is to say, J satisfies the condition (J_1) of the Mountain Pass Lemma.

For our setting, clearly $J(0)=0$. In order to exploit the Mountain Pass Lemma in critical point theory, we need to verify *all* other conditions of the Mountain Pass Lemma. By Lemma 2.2, J satisfies the P.S. condition. So it suffices to verify the condition (J_2).

From the proof of the P.S. condition *in Lemma 2.2*, we know

$$J(u)\le\frac{1}{2}\left(\bar{p}\lambda_k+\bar{q}\tilde{\lambda}_k\right)\|u\|^2-a_1c_1^\beta\|u\|^\beta+a_2k.$$

Since $\beta>2$, we can choose $\bar{u}$ large enough to ensure that $J(\bar{u})<0$.

By the Mountain Pass Lemma, J possesses a critical value $c\ge a>0$, where

$$c=\inf_{h\in\Gamma}\sup_{s\in[0,1]}J(h(s)),$$

and

$$\Gamma=\{h\in C([0,1],\mathbf{R}^k)\mid h(0)=0,\ h(1)=\bar{u}\}.$$

Let $\tilde{u} \in \mathbf{R}^k$ be a critical point associated to the critical value c of J, i.e., $J(\tilde{u}) = c$. Similar to the proof of the P.S. condition, we know that there exists $\hat{u} \in \mathbf{R}^k$ such that

$$J(\hat{u}) = c_{\max} = \max_{s\in[0,1]} J(h(s)).$$

Clearly, $\hat{u} \neq 0$. If $\tilde{u} \neq \hat{u}$, then the conclusion of Theorem 1.2 holds. Otherwise, $\tilde{u} = \hat{u}$. Then $c = J(\tilde{u}) = c_{\max} = \max\limits_{s\in[0,1]} J(h(s))$. That is,

$$\sup_{u\in\mathbf{R}^k} J(u) = \inf_{h\in\Gamma} \sup_{s\in[0,1]} J(h(s)).$$

Therefore,

$$c_{\max} = \max_{s\in[0,1]} J(h(s)), \ \forall h \in \Gamma.$$

By the continuity of $J(h(s))$ with respect to s, $J(0) = 0$ and $J(\bar{u}) < 0$ imply that there exists $s_0 \in (0,1)$ such that

$$J\left(h\left(s_0\right)\right) = c_{\max}.$$

Choose $h_1,\ h_2 \in \Gamma$ such that $\{h_1(s) \mid s \in (0,1)\} \cap \{h_2(s) \mid s \in (0,1)\}$ is empty, then there exists $s_1,\ s_2 \in (0,1)$ such that

$$J\left(h_1\left(s_1\right)\right) = J\left(h_2\left(s_2\right)\right) = c_{\max}.$$

Thus, we get two different critical points of J on $\mathbf{R}^k$ denoted by

$$u^1 = h_1\left(s_1\right),\ u^2 = h_2\left(s_2\right).$$

The above argument implies that the BVP (1.1) with (1.2) possesses at least two nontrivial solutions. The proof of Theorem 1.2 is finished. □

3.3. Proof of Theorem 1.3

Proof. We only need to find at least one critical point of the functional J defined as in (2.5).

By (F_5'), for any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$, we have

$$\begin{aligned}
J(u) &= \frac{1}{2}\sum_{n=-1}^{k} p_{n+1}\left(\Delta^2 u_n\right)^2 + \frac{1}{2}\sum_{n=0}^{k} q_{n+1}\left(\Delta u_n\right)^2 - \sum_{n=1}^{k} F(n, u_{n+1}, u_n) \\
&\geq \frac{1}{2}\underline{p}\sum_{n=-1}^{k}\left(u_{n+2} - 2u_{n+1} + u_n\right)^2 + \frac{1}{2}\underline{q}\sum_{n=0}^{k}\left(u_{n+1} - u_n\right)^2 - a_3\sum_{n=1}^{k}\left(\sqrt{u_{n+1}^2 + u_n^2}\right)^{\alpha} - a_4 k \\
&= \frac{1}{2}\underline{p}u^* P u + \frac{1}{2}\underline{q}u^* Q u - a_3\left\{\left[\sum_{n=1}^{k}\left(\sqrt{u_{n+1}^2 + u_n^2}\right)^{\alpha}\right]^{\frac{1}{\alpha}}\right\}^{\alpha} - a_4 k \\
&\geq \frac{1}{2}\underline{p}\lambda_1\|u\|^2 + \frac{1}{2}\underline{q}\tilde{\lambda}_1\|u\|^2 - a_3 c_2^{\alpha}\left\{\left[\sum_{n=1}^{k}\left(u_{n+1}^2 + u_n^2\right)\right]^{\frac{1}{2}}\right\}^{\alpha} - a_4 k \\
&\geq \frac{1}{2}\left(\underline{p}\lambda_1 + \underline{q}\tilde{\lambda}_1\right)\|u\|^2 - 2^{\alpha} a_3 c_2^{\alpha}\|u\|^{\alpha} - a_4 k \\
&\quad \to +\infty \text{ as } \|u\| \to +\infty.
\end{aligned}$$

By the continuity of J, we know from the above inequality that there exist lower bounds of values of the functional. And this means that J attains its minimal value at some point which is just the critical point of J with the finite norm. □

3.4. Proof of Theorem 1.4

Proof. Assume, for the sake of contradiction, that the BVP (1.1) with (1.2) has a nontrivial solution. Then J has a nonzero critical point $u^\star$. Since

$$\frac{\partial J}{\partial u_n} = \Delta^2\left(p_{n-1}\Delta^2 u_{n-2}\right) - \Delta\left(q_n \Delta u_{n-1}\right) - f(n, u_{n+1}, u_n, u_{n-1}),$$

we get

$$\sum_{n=1}^{k} f(n, u^\star_{n+1}, u^\star_n, u^\star_{n-1})u^\star_n = \sum_{n=1}^{k}\left[\Delta^2\left(p_{n-1}\Delta^2 u^\star_{n-2}\right) - \Delta\left(q_n\Delta u^\star_{n-1}\right)\right]u^\star_n$$

$$= \sum_{n=-1}^{k} p_{n+1}\left(\Delta^2 u^\star_n\right)^2 + \sum_{n=0}^{k} q_{n+1}\left(\Delta u^\star_n\right)^2 \le 0. \tag{3.1}$$

On the other hand, it follows from (F_6) that

$$\sum_{n=1}^{k} f(n, u^\star_{n+1}, u^\star_n, u^\star_{n-1})u^\star_n > 0. \tag{3.2}$$

This contradicts (3.1) and hence the proof is complete. □

4. Examples

As an application of Theorems 1.2, 1.3 and 1.4, we give three examples to illustrate our main results.

Example 4.1. For $n \in \mathbf{Z}(1,k)$, assume that

$$\Delta^4 u_{n-2} - \Delta\left(9^n\Delta u_{n-1}\right) = \beta u_n\left[\varphi(n)\left(u_{n+1}^2 + u_n^2\right)^{\frac{\beta}{2}-1} + \varphi(n-1)\left(u_n^2 + u_{n-1}^2\right)^{\frac{\beta}{2}-1}\right], \tag{4.1}$$

with boundary value conditions (1.2), where $\beta > 2$, φ is continuously differentiable and $\varphi(n) > 0,\ n \in \mathbf{Z}(1,k)$ with $\varphi(0) = 0$.

We have

$$p_n \equiv 1,\ q_n = 9^n,\ f(n, v_1, v_2, v_3) = \beta v_2\left[\varphi(n)\left(v_1^2 + v_2^2\right)^{\frac{\beta}{2}-1} + \varphi(n-1)\left(v_2^2 + v_3^2\right)^{\frac{\beta}{2}-1}\right]$$

and

$$F(n, v_1, v_2) = \varphi(n)\left(v_1^2 + v_2^2\right)^{\frac{\beta}{2}}.$$

It is easy to verify all the assumptions of Theorem 1.2 are satisfied and then the BVP (4.1) with (1.2) possesses at least two nontrivial solutions.

Example 4.2. For $n \in \mathbf{Z}(1,k)$, assume that

$$\Delta^2\left(8^{n-1}\Delta^2 u_{n-2}\right) - \Delta\left(6^n\Delta u_{n-1}\right) = \alpha u_n\left[\psi(n)\left(u_{n+1}^2 + u_n^2\right)^{\frac{\alpha}{2}-1} + \psi(n-1)\left(u_n^2 + u_{n-1}^2\right)^{\frac{\alpha}{2}-1}\right], \tag{4.2}$$

with boundary value conditions (1.2), where $1 < \alpha < 2$, ψ is continuously differentiable and $\psi(n) > 0,\ n \in \mathbf{Z}(1,k)$ with $\psi(0) = 0$.

We have

$$p_n = 8^n,\ q_n = 6^n,\ f(n, v_1, v_2, v_3) = \alpha v_2\left[\psi(n)\left(v_1^2 + v_2^2\right)^{\frac{\alpha}{2}-1} + \psi(n-1)\left(v_2^2 + v_3^2\right)^{\frac{\alpha}{2}-1}\right]$$

and

$$F(n, v_1, v_2) = \psi(n) \left(v_1^2 + v_2^2\right)^{\frac{\alpha}{2}}.$$

It is easy to verify all the assumptions of Theorem 1.3 are satisfied and then the BVP (4.2) with (1.2) possesses at least one solution.

Example 4.3. For $n \in \mathbf{Z}(1, k)$, assume that

$$-\Delta^4 u_{n-2} + \Delta\left(7^n \Delta u_{n-1}\right) = \frac{8}{5} u_n \left[\left(u_{n+1}^2 + u_n^2\right)^{-\frac{1}{5}} + \left(u_n^2 + u_{n-1}^2\right)^{-\frac{1}{5}}\right], \tag{4.3}$$

with boundary value conditions (1.2).

We have

$$p_n \equiv -1,\ q_n = -7^n,\ f(n, v_1, v_2, v_3) = \frac{8}{5} v_2 \left[\left(v_1^2 + v_2^2\right)^{-\frac{1}{5}} + \left(v_2^2 + v_3^2\right)^{-\frac{1}{5}}\right]$$

and

$$F(n, v_1, v_2) = \left(v_1^2 + v_2^2\right)^{\frac{4}{5}}.$$

It is easy to verify all the assumptions of Theorem 1.4 are satisfied and then the BVP (4.3) with (1.2) has no nontrivial solutions.

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