On QF rings and artinian principal ideal rings

Alejandro Alvarado-García[1], César Cejudo-Castilla[2], Hugo Alberto
Rincón-Mejía[*1], Ivan Fernando Vilchis-Montalvo[2], Manuel Gerardo
Zorrilla-Noriega[1]

[1] *Facultad de Ciencias, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., C.P. 04510,
Ciudad de México , México.*

[2] *Facultad de Ciencias Físico Matemáticas, Benemérita Universidad Autónoma de Puebla, Av. San
Claudio y 18 Sur, Col. San Manuel, Ciudad Universitaria, C.P. 72570, Puebla, México*

Abstract

In this work we give sufficient conditions for a ring R to be quasi-Frobenius, such as R being
left artinian and the class of injective cogenerators of R-Mod being closed under projective
covers. We prove that R is a division ring if and only if R is a domain and the class of
left free R-modules is closed under injective hulls. We obtain some characterizations of
artinian principal ideal rings. We characterize the rings for which left cyclic modules
coincide with left cocyclic R-modules. Finally, we obtain characterizations of left artinian
and left coartinian rings.

1. Introduction

In [1] and [3] the authors obtained characterizations of artinian principal ideal rings
using big lattices of classes of modules closed under certain closure properties. Also,
in [4] the authors deal with rings over which all injective hulls of left simple modules are
noetherian. These rings are called left coartinian rings. In this work we further investigate
these notions, among others.

In the sequel, R denotes an associative ring with identity and R-Mod denotes the
category of left unitary R-modules, to which all "modules" and "R-modules" will belong,
unless otherwise specified. A *left uniserial* ring will be a ring whose left ideals are linearly
ordered. By "QF" we mean "quasi-Frobenius". Also, "$N \leq_e M$" will stand for "N is
essential in M".

2. Artinian principal ideal rings

Recall that an artinian principal ideal ring is a left and right artinian, left and right principal ideal ring.

Definition 2.1. We will say that $_RM$ is a *paraprojective* module if $_RM$ embeds in $_RN$ whenever $_RM$ is an epimorphic image of $_RN$. Dually, we say that $_RM$ is a *parainjective* module if $_RM$ is quotient of each module $_RN$ in which $_RM$ embeds.

Theorem 2.2. *The following statements are equivalent for a ring R.*

1) *R is an artinian principal ideal ring.*
2) *Every class of R-modules closed under submodules and direct sums is also closed under quotients.*
3) *Every class of R-modules closed under quotients and direct products is also closed under submodules.*
4) *There exists an epimorphism $P \twoheadrightarrow M$ precisely when there exists a monomorphism $M \rightarrowtail P$ for each R-module M and for each projective R-module P.*
5) *R is left noetherian, and every cyclic R-module C is parainjective.*

Proof. $1) \Rightarrow 2), 3), 4)$ and $5)$ They follow from [3, Theorem 38].

$4) \Rightarrow 1)$ For each module M there exists an epimorphism $R^{(X)} \twoheadrightarrow M$ for some set X. Hence, by hypothesis there exists a monomorphism $M \rightarrowtail R^{(X)}$. Therefore, by [6, Corollary 24.15], R is a QF ring. Now, let us take a left ideal I of R. By hypothesis, there exists an epimorphism $R \twoheadrightarrow I$. Thus, $I = Rx$ for some $x \in I$. Then, R is a left principal ideal ring. Therefore, by [5, Sec. 4, Theorem 1], R is an artinian principal ideal ring.

$2) \Rightarrow 1)$ Consider the class of modules

$$\mathscr{C} = \{M \mid \text{there exists a monomorphism } M \rightarrowtail R^{(X)} \text{ for some set } X\}.$$

It is clear that $\mathscr{C}$ is closed under submodules and direct sums. Then, by hypothesis, $\mathscr{C}$ is closed under quotients. Also, $R^{(X)} \in \mathscr{C}$ for each set X, so $\mathscr{C} = R$-Mod. Then for each module M, there exists a monomorphism $M \rightarrowtail R^{(X)}$ for some set X, so, by [6, Corollary 24.15], R is a QF ring. Let I be any two sided ideal of R. It is straightforward to verify that the ring R/I also satisfies 2). It follows that R/I is QF. By [6, P. 217], R is an artinian principal ideal ring.

$3) \Rightarrow 1)$ Let E be a minimal injective cogenerator. Consider the class of modules $\mathscr{C} = \{M \mid \text{ there exists an epimorphism } E^X \twoheadrightarrow M \text{ for some set } X\}$. It is clear that $\mathscr{C}$ is closed under quotients and direct products. By hypothesis, $\mathscr{C}$ is then closed under submodules. Of course, $E^X \in \mathscr{C}$ for each set X. But, for each module M, there exists a monomorphism $M \rightarrowtail E^X$ for some set X. Then, $\mathscr{C} = R$-Mod. So, for each projective module P, there exists an epimorphism $E^X \twoheadrightarrow P$, so that P is a direct summand of E^X. Therefore, P is an injective module. Thus, R is a QF ring. Moreover, as the ring R/I also satisfies 3), R/I is a QF ring for each two sided ideal I of R. Then, by [6, P. 217], R is an artinian principal ideal ring.

$5) \Rightarrow 1)$ As there exists a monomorphism $R \rightarrowtail E(R)$, by hypothesis there exists an epimorphism $E(R) \twoheadrightarrow R$, so R is left self-injective and left noetherian. Therefore, R is a QF ring. As the ring R/I also satisfies 5), R/I is a QF ring for each two sided ideal I of R. Then, R is an artinian principal ideal ring. $\square$

Theorem 2.3. *Let R be an artinian principal ideal ring. Then the following conditions are equivalent for an R-module M.*

1) *M is finitely generated.*
2) *M is finitely cogenerated.*
3) *M is artinian.*
4) *M is noetherian.*

Proof. $4) \Rightarrow 1)$ and $3) \Rightarrow 2)$ They are clear.

$1) \Rightarrow 4)$ This is true over every left noetherian ring.

$2) \Rightarrow 3)$ Consider the class of modules $\mathscr{C} = \{M \mid M \text{ is finitely cogenerated}\}$, which is closed under submodules. By hypothesis and [3, Theorem 38] the class $\mathscr{C}$ is closed under quotients, so M is artinian for each $M \in \mathscr{C}$.

$4) \Rightarrow 2)$ By hypothesis R is left semiartinian, so that $\text{soc}(M) \leq_e M$ for each non zero module M. If M is noetherian, then $\text{soc}(M)$ is finitely generated, so M is finitely cogenerated.

$3) \Rightarrow 1)$ For this part, we will use freely [3, Theorem 38]. If M is artinian, then M is finitely cogenerated. Thus $\text{soc}(M) \leq_e M$ and $\text{soc}(M)$ is finitely generated. Therefore $E(\text{soc}(M)) = E(M)$ and $\text{soc}(M) = \bigoplus_{i=1}^{n} S_i$ with S_i a simple module for each $i \in \{1, \ldots, n\}$. Thus, $E(M) = E(\text{soc}(M)) = E(\bigoplus_{i=1}^{n} S_i) = \bigoplus_{i=1}^{n} E(S_i)$. We claim that each $E(S_i)$ is cyclic. Indeed, take any simple S. By hypothesis, there exists a monomorphism $S \rightarrowtail R$. Moreover, since every artinian principal ideal ring is QF and thus left self-injective, there exists a monomorphism $E(S) \rightarrowtail R$. Then there exists an epimorphism $R \twoheadrightarrow E(S)$. Therefore, $E(S)$ is cyclic, as we claim. Thus $E(M)$ is finitely generated, and as the class of finitely generated modules is closed under submodules by hypothesis, M is finitely generated. $\square$

Recall that an R-module M is cocyclic if M contains an essential simple submodule.

Theorem 2.4. *The classes of non-zero cyclic R-modules and of cocyclic R-modules coincide if and only if R is a left uniserial artinian principal ideal ring.*

Proof. $\Rightarrow$] Let us first prove that R must be left artinian. This is equivalent to every quotient of $_R R$ being finitely cogenerated. By the hypothesis, all we need to prove is that every cocyclic module is finitely cogenerated. Take then any cocyclic M. There is some simple $S \leq_e M$. It follows that $\text{soc}(M) = S$. Thus, M has a finitely generated essential socle, a condition well-known to be equivalent to M being finitely cogenerated.

We now show that R is left self-injective. Suppose otherwise, that is, that $R \lneq E(R)$. The hypothesis gives that $_R R$ is cocyclic, so there is some simple $S \leq_e R$. Hence, $E(R) = E(S)$, which is obviously cocyclic. Using the hypothesis, we get that $E(R)$ is cyclic, so that there is an epimorphism $R \xrightarrow{f} E(R)$. Consider the following commutative diagram, where i and j are inclusion maps.

$$
\begin{array}{ccc}
f^{-1}(R) & \xhookrightarrow{\ i\ } & R \\
{\scriptstyle f_{\upharpoonright f^{-1}(R)}}\big\downarrow & = & \big\downarrow {\scriptstyle f} \\
R & \xhookrightarrow[\neq]{\ j\ } & E(R)
\end{array}
$$

Note that if i were surjective, j would also be so. Thus, $f^{-1}(R) \lneq R$. Now we may construct another level of the diagram. Let us write $f_{\upharpoonright}$ for appropriate restrictions of f.

$$
\begin{array}{ccc}
f^{-1}(f^{-1}(R)) & \xhookrightarrow{\quad} & f^{-1}(R) \\
{\scriptstyle f_{\upharpoonright}}\big\downarrow & = & \big\downarrow {\scriptstyle f_{\upharpoonright}} \\
f^{-1}(R) & \xhookrightarrow[\neq]{\ i\ } & R \\
{\scriptstyle f_{\upharpoonright}}\big\downarrow & = & \big\downarrow {\scriptstyle f} \\
R & \xhookrightarrow[\neq]{\ j\ } & E(R)
\end{array}
$$

As above, the newest inclusion must be proper. Continuing in this manner, we obtain an infinite descending chain $R \gneqq f^{-1}(R) \gneqq f^{-1}(f^{-1}(R)) \gneqq \ldots$, contradicting that $_RR$ is artinian.

Now, R being left artinian and left self-injective is equivalent to R being QF. Take any two-sided ideal I. It is straightforward to verify (using that cocyclic modules are precisely those modules having simple essential submodules) that the ring R/I satisfies the hypothesis, i.e. that an arbitrary R/I-module is cyclic if and only if it is cocyclic. It follows that for each two-sided ideal I of R, R/I is QF. But, according to [6, P. 217], this is equivalent to R being an artinian principal ideal ring.

(One can prove directly that R is a left principal ideal ring. Indeed, take some non-zero left ideal I. By the hypothesis, it suffices to show that I is cocyclic. Note that the hypothesis gives some simple $S \leq_e R$. Also, as $_RR$ is artinian, there is some simple $T \leq I$. But then $T \leq R$, so necessarily $S = T$. And of course, $S \leq I \leq R$ implies that $S \leq_e I$. This establishes that R is a left principal ideal ring. As we have already shown R to be QF, [5, Sec. 4, Theorem 1] grants that R is an artinian principal ideal ring.)

As every nonzero quotient of R is cocyclic, then every nonzero quotient of R is uniform. Therefore by [7, Proposition 2.7] R is left uniserial.

$\Leftarrow$] Take any non-zero cyclic module, say R/I for some left ideal $I \leq R$. The submodule lattice of R/I is isomorphic to $[I, R]$, which is a chain. As $_RR$ is artinian, then there exists I' minimal such that $I \lneqq I' \leq R$. Linearity ensures that I' is essential in $[I, R]$. Therefore, I'/I is an essential simple submodule of R/I, proving its cocyclicity.

Conversely, let M be a cocyclic module. There is some simple $S \leq_e M$. Also, since $_RR$ is artinian, there is some simple $T \leq R$. From the hypothesis on linearity it follows that R must be local and thus left local, so that $S \cong T$. Now, any artinian principal ideal ring is in particular QF and then in particular left self-injective, so we may extend $S \cong T \hookrightarrow R$ to a mapping $M \to R$, which is monic due to the fact that $S \leq_e M$. The situation is depicted below.
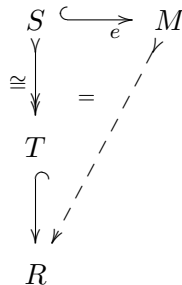
Thus, M is isomorphic to some left ideal of R, which by hypothesis is principal, i.e., cyclic. $\qquad\square$

Lemma 2.5. *If every semisimple R-module M is parainjective and paraprojective, then $R = R_1 \times R_2$, where R_1 is a semisimple ring and R_2 is a finite direct product of left local left artinian rings with all simple modules singular.*

Proof. By [2, Theorem 4.7] R is a finite direct product of left local, left and right perfect rings. Thus, R is a left semiartinian ring. Let Rx be a cyclic module. Then $\mathrm{soc}(Rx) \leq_e Rx$ and there exists an epimorphism $Rx \twoheadrightarrow soc(Rx)$ by hypothesis, so $soc(Rx)$ is finitely generated. Therefore, Rx is a finitely cogenerated module. Then R is a left artinian ring.

Now, if M is a projective semisimple R-module, there exists an epimorphism $E(M) \twoheadrightarrow M$ by hypothesis. Then M is injective. Analogously, if M is a semisimple injective R-module, M is projective. Thus, M is projective if and only if M is injective, for each semisimple R-module M.

Write $R = R_1 \times \cdots \times R_n$, where each R_i is a left local, left and right perfect ring. Let $1 \leq i \leq n$. Note that R_i is left artinian (either because R is left artinian or because

R_i satisfies the hypothesis on R), that $\text{soc}(R_i) \leq_e R_i$ (R_i being left semiartinian), that $\text{soc}(R_i)$ is a direct sum of copies of S_i (where $R_i\text{-}simp = \{S_i\}$) and is precisely the S_i-socle of R, and that for each $M \in R_i$-Mod, M is projective (respectively, injective) if and only if it is a projective (respectively, injective) R-module. We claim that S_i is projective if and only if R_i is a semisimple ring. Sufficiency is clear, because over any semisimple ring every module is projective. Conversely, suppose that S_i is projective. Then so is $\text{soc}(R_i)$, which is thus injective. but this makes $\text{soc}(R_i)$ an essential direct summand of R_i, whence $\text{soc}(R_i) = R_i$. This proves the claim.

Set $I_{SS} = \{i \in \{1, \ldots, n\} \mid R_i \text{ is semisimple}\}$, and put $R_I = \prod\limits_{i \in I_{SS}} R_i$ (which is the projective socle of R), and $R_{II} = \prod\limits_{i \in \{1, \ldots, n\} \backslash I_{SS}} R_i$. Then $R = R_I \times R_{II}$, R_I is a semisimple ring and R_{II} is a finite direct product of left local artinian rings over each of which all simple modules are singular. $\qquad\square$

Observe that the hypothesis of Lema 2.5 holds also for R/I for each two sided ideal I of R.

Recall that a ring R is called left quasi-duo if each maximal left ideal is two sided.

Remark 2.6. For a ring R the following conditions are equivalent.

(1) R is a left quasi-duo ring.
(2) For each simple R-module S, and for all $x \in S$, $(0 : x)$ is a two-sided ideal of R.

Theorem 2.7. *For a left quasi-duo ring R, if every semisimple R-module is parainjective and paraprojective, then R is a finite direct product of left local left artinian rings and for each left ideal I of the factor ring R_i, $I = \text{rad}(R_i)^m$ for some $m \in \mathbb{N}$.*

Proof. Consider the decomposition supplied by [2, Theorem 4.7]. Let R be a factor ring. Note that R inherits the current hypotheses. Then, there exists an epimorphism $R \to \text{soc}(R)$, so $Rx = \text{soc}(R) = S_1 \oplus S_2 \oplus \cdots \oplus S_n$, where S_i is a simple module $\forall i \in \{1, \ldots, n\}$. Write $x = x_1 + x_2 + \cdots + x_n$, where each $x_i \in S_i \backslash \{0\}$. It is clear that $(0 : x) \subseteq (0 : x_i), \forall i \in \{1, \ldots, n\}$. Let $j \in \{2, \ldots, n\}$. Since R is left local, there is an isomorphism $f_j : S_1 \to S_j$. As $S_j = Rf_j(x_1)$, there is $r_j \in R$ such that $x_j = r_j f_j(x_1)$. Then, $x = x_1 + r_2 f_2(x_1) + \cdots + r_n f_n(x_1)$. Note that, for $2 \leq j \leq n$, $(0 : x_1)Rf_j(x_1) = 0$ because $(0 : x_1)$ is a two sided ideal by hypothesis and Remark 2.6. Thus, $(0 : x) = (0 : x_1)$. Then, $Rx \cong Rx_1$. Therefore, $Rx = \text{soc}(R)$ is a simple module.

As established in the proof of Lemma 2.5, R is a left artinian ring, so that $\text{rad}(R)$ is nilpotent. Let us prove by induction on the nilpotency index that for each left ideal I of R, $I = \text{rad}(R)^m$ for some $m \in \mathbb{N}$. If $n = 1$, then $\text{rad}(R) = 0$. Since R is left artinian, it is semilocal, so in this case it is semisimple and thus a division ring, so that the only two left ideals are $0 = \text{rad}(R)$ and $R = \text{rad}(R)^0$. Let us suppose that $n > 1$ is the nilpotency index. As $\text{rad}(R)^n = 0$, $\text{rad}(R)^{n-1} \neq 0$ is annhilated by $\text{rad}(R)$, so $\text{rad}(R)^{n-1}$ is a semisimple module (again by semilocality). Then, $\text{rad}(R)^{n-1} \subseteq \text{soc}(R)$, but $\text{soc}(R)$ is a simple module, so that $\text{rad}(R)^{n-1} = \text{soc}(R)$. Let I be a left ideal of R. Note that, R being left artinian and having a simple socle, $\text{soc}(R) \leq I$. Then $R/\text{soc}(R) = R/(\text{rad}(R)^{n-1})$ is a ring with the same hypothesis of R whose radical has nilpotency index $n-1$. Therefore, $I/soc(R) = \text{rad}(R)^m/soc(R)$ and by the Correspondence Theorem $I = \text{rad}(R)^m$, for some $m \in \mathbb{N}$. $\qquad\square$

Theorem 2.8. *For a commutative ring R the following statements are equivalent.*

(1) *Every semisimple R-module is parainjective and paraprojective.*
(2) *R is a finite direct product of uniserial artinian principal ideal rings.*

Proof. 1) $\Rightarrow$ 2) Let R be any of the factor rings in the decomposition supplied by Theorem 2.7. We know that, for each ideal I of R, there exists $m \in \mathbb{N}$ such that $I = \text{rad}(R)^m$. In

the proof of Theorem 2.7, we showed that if $\mathrm{rad}(R) = 0$, then R must be a division ring, and thus in particular an uniserial artinian principal ideal ring. Suppose then that there is $x \in \mathrm{rad}(R)$ but $x \notin \mathrm{rad}(R)^2$. Then $\mathrm{rad}(R)^2 < Rx \leq \mathrm{rad}(R)$. Thus, $\mathrm{rad}(R) = Rx$. Then by an induction argument $\mathrm{rad}(R)^m = Rx^m, \forall m \in \mathbb{N}$. Now we prove that R is a self-injective ring. Consider the following diagram:

$$
\begin{array}{ccc}
Rx^n & \overset{i}{\lhook\joinrel\longrightarrow} & R \\
\downarrow{\scriptstyle f} & & \\
R & &
\end{array}
$$

where $f : Rx^n \to R$ is any homomorphism and $i : Rx^n \hookrightarrow R$ is the inclusion. Then $f(Rx^n) = Rx^m$ with $m \geq n$. Put $f(x^n) = rx^m$. Consider the homomorphism $h : R \to R$ such that $h(s) = s(rx^{m-n}), \forall s \in R$. Thus, for $t \in R$, $(hi)(tx^n) = h(i(tx^n)) = h(tx^n) = tx^n(rx^{m-n}) = trx^m = f(tx^n)$. Therefore, R is self-injective. As was established in the proof of Lemma 2.5, R is artinian. Then R is a QF-ring, Thus, by [5, Sec. 4, Theorem 1], R is an artinian principal ideal ring.

2) $\Rightarrow$ 1) Follows by [3, Theorem 38]. $\qquad\square$

3. Coartinian, conoetherian and quasi-Frobenius rings

The ring R is said to be left *coartinian* if for every $S \in R\text{-}simp$, $E(S)$ is noetherian.

Proposition 3.1. *Let R be a ring.*

1) *R is left artinian if and only if every finitely generated R-module is finitely cogenerated.*
2) *R is left coartinian if and only if every finitely cogenerated R-module is finitely generated.*

Proof. 1) Suppose that R is left artinian and take some finitely generated $M \in R$-Mod. As R is left noetherian, $\mathrm{soc}(M)$, being a submodule of M, is also finitely generated. Also, R being left semiartinian implies that $\mathrm{soc}(M) \leq_e M$. Therefore, M is finitely cogenerated.

Conversely, suppose that every finitely generated module is finitely cogenerated. Any quotient of $_RR$, being cyclic, is by hypothesis finitely cogenerated. Thus, $_RR$ is artinian.

2) Suppose that R is left coartinian and take some finitely cogenerated $M \in R$-Mod. Then, there are some simple $S_1, \ldots, S_n$ such that $\bigoplus\limits_{i=1}^{n} S_i = \mathrm{soc}(M) \leq_e M$. But then $E(M) = E(\mathrm{soc}(M)) = \bigoplus\limits_{i=1}^{n} E(S_i)$ is, by hypothesis, noetherian, so that its submodule M is finitely generated.

Conversely, suppose that every finitely cogenerated module is finitely generated. For every $S \in R\text{-}simp$, any submodule of $E(S)$, being finitely cogenerated, is by hypothesis finitely generated. Thus, $E(S)$ is noetherian. $\qquad\square$

A ring R is called left *conoetherian* if for every $S \in R\text{-}simp$, $E(S)$ is artinian. Accordingly, let us call R left *strongly conoetherian* if every indecomposable[†] injective R-module is artinian.

Theorem 3.2. *Let R be a ring. The following statements are equivalent.*

1) *R is left artinian and left coartinian.*
2) *The classes of finitely generated and of finitely cogenerated R-modules coincide.*

[†]By "indecomposable" we mean "directly indecomposable".

3) *R is left noetherian and left strongly conoetherian.*

Proof. 1) $\Leftrightarrow$ 2) Direct from Proposition 3.1.

1) $\Rightarrow$ 3) Suppose that 1) holds. Of course, every left artinian ring is left noetherian. Let $E \in R$-Mod be injective and indecomposable. Since R is artinian, there is some simple S such that $E = E(S)$. Then, as R is left coartinian, E is noetherian, and in particular finitely generated. But over a left artinian ring, every finitely generated module is artinian. Therefore, 3) holds.

3) $\Rightarrow$ 1) Suppose now that 3) holds. Since R is left noetherian, in order to prove that it is left artinian it suffices to show that it is left semiartinian. Take then some non-zero $M \in R$-Mod. As is well-known, left noetherian rings are characterized by the fact that over them, every injective module is a direct sum of indecomposable modules. We can apply this to $E(M)$ and then use the fact that R is left strongly conoetherian to obtain some simple $S \leq E(M)$. Then, by simplicity, $S \leq M$.

Let now $S \in R\text{-}simp$. Let us write $J = \mathrm{rad}(R)$. We have already established that R is left artinian, so J is nilpotent. Then there is a least $n \in \mathbb{N}$ such that $J^n E(S) = 0$. (Of course, $n > 0$.) Observe that both of $J^{n-1}E(S)$ and $J^{n-2}E(S)/J^{n-1}E(S)$ are artinian and semisimple. Indeed, they are subquotients of $E(S)$, an artinian module, and they are annihilated by J, i.e. they are R/J-modules (since R is left artinian, it is semilocal). Thus, $J^{n-1}E(S)$ and $J^{n-2}E(S)/J^{n-1}E(S)$ are noetherian, so that the short exact sequence

$$0 \to J^{n-1}E(S) \to J^{n-2}E(S) \to J^{n-2}E(S)/J^{n-1}E(S) \to 0$$

shows that $J^{n-2}E(S)$ is noetherian. Next, we use

$$0 \to J^{n-2}E(S) \to J^{n-3}E(S) \to J^{n-3}E(S)/J^{n-2}E(S) \to 0$$

to show that $J^{n-3}E(S)$ is noetherian, and so on. At the n-th step, we obtain that $E(S)$ is noetherian. $\qquad\square$

Theorem 3.3. *Let R be a ring. The following conditions are equivalent.*

1) *R is a domain[‡] and the class of free R-modules is closed under taking injective hulls.*

2) *R is a division ring.*

Proof. 1) $\Rightarrow$ 2) We claim that every free module is injective. Consider $R^{(X)}$ for some set X. Suppose first that $|X| > |R|$. By hypothesis, $E(R^{(X)}) = R^{(Y)}$ for some set Y, so that $|R^{(X)}| \leq |R^{(Y)}|$. Let us verify that $|X| \leq |Y|$.

In case both of $R^{(X)}$ and $R^{(Y)}$ are infinite, we have that

$$\max\{|R|, |Y|\} = |R^{(Y)}| \geq |R^{(X)}| = \max\{|R|, |X|\} = |X| > |R|,$$

so that it must happen that $\max\{|R|, |Y|\} = |Y|$. Thus, $|X| \leq |Y|$.

In case both of $R^{(X)}$ and $R^{(Y)}$ are finite, we have that

$$|R|^{|X|} = |R^{(X)}| \leq |R^{(Y)}| = |R|^{|Y|},$$

so that necessarily $|X| \leq |Y|$.

Lastly, in case $R^{(X)}$ is finite and $R^{(Y)}$ is infinite, we must have that $|R|$ and $|X|$ are finite, and thus that $|Y|$ is infinite (seeing as, for any set A, $R^{(A)}$ is finite if and only if R and A are finite).

Thus, we always have that $|R| < |X| \leq |Y|$.

Let us write $\{\delta_y\}_{y \in Y}$ for the canonical basis of $R^{(Y)}$ [§]. Since $R^{(X)} \hookrightarrow_e R^{(Y)}$, for each $y \in Y$ there is an $r_y \in R$ such that $0 \neq r_y \delta_y \in R^{(X)}$. By the hypothesis on R, the set

---

[‡]That is, every product of non-zero elements of R is non-zero.

[§]That is, for $y \in Y$, $\delta_y : Y \to R$ is such that $\delta_y : z \mapsto \begin{cases} 1 & \text{if } z = y \\ 0 & \text{if } z \neq y \end{cases}$, although any basis will do.

$\{r_y\delta_y\}_{y\in Y}$ is linearly independent. Thus, the submodule of $R^{(X)}$ spanned by $\{r_y\delta_y\}_{y\in Y}$ is a free module. Therefore,

$$R^{(Y)} \cong R^{(\{r_y\delta_y\}_{y\in Y})} \cong \langle\{r_y\delta_y\}_{y\in Y}\rangle \le R^{(X)},$$

so that $|R^{(Y)}| \le |R^{(X)}|$. As we already had the reverse inequality, we obtain that $|R^{(X)}| = |R^{(Y)}|$. This cardinality may or may not be finite, but it is now easy to show that, in any case, $|X| = |Y|$. This implies that $R^{(X)} \cong R^{(Y)}$, whence $R^{(X)}$ is injective.

Now, in case $|X| \le |R|$, simply take some set Z such that $|Z| > |R|$ and note that $R^{(X)}$ embeds as a direct summand in $R^{(Z)}$, which, by the above argument, is injective. Therefore, the claim is proved.

Since every projective module is a direct summand of a free module, every projective module is injective. This condition is well-known to be equivalent to R being QF. Therefore R is a left artinian domain; thus, there exists a minimal left ideal Rx, which is isomorphic to R. Therefore R is a division ring.

2) $\Rightarrow$ 1) It is clear.   □

Theorem 3.4. *Suppose that R is left artinian and that the class of injective cogenerators of R-Mod is closed under taking projective covers. Then R is a QF ring.*

Proof. Take some injective cogenerator of R-Mod, and let P stand for its projective cover, which exists because R is left perfect. By hypothesis, P is a projective and injective cogenerator. Let $S \in R\text{-}simp$. As S is cogenerated by P, by simplicity S embeds in P, so that $E(S)$ embeds in P as a direct summand, which makes it projective. Also observe that, over any artinian ring, any injective indecomposable module E is the injective hull of some simple $S \le E$.

Let $M \in R$-Mod be injective. As R is left noetherian, M is a direct sum of injective indecomposable modules. By the above remark, M is a direct sum of injective hulls of simple modules, which we know are projective. Therefore, every injective module is projective. This condition is well-known to be equivalent to R being QF.   □

References

[1] A. Alvarado-García, C. Cejudo-Castilla, H.A. Rincón Mejía, I.F. Vilchis-Montalvo, *On classes of modules closed under injective hulls and artinian principal ideal rings*, Int. Electron. J. Algebra **15**, 1-12, 2014.

[2] A. Alvarado-García, C. Cejudo-Castilla, H.A. Rincón-Mejía, I.F. Vilchis-Montalvo, *Pseudocomplements and strong pseudocomplements in lattices of module classes*, J. Algebra Appl. **17** (1), 1850016 (14 pages), 2018.

[3] A. Alvarado-García, H.A. Rincón-Mejía, J. Ríos-Montes, *On some lattices of module classes*, J. Algebra Appl. **5** (1), 105-117, 2006.

[4] A. El Mejdani, H. Essannouni, A. Kaidi, *Rings with nice artinian modules*, Int. J. Algebra. **2** (18), 895-904, 2008.

[5] C. Faith, *On Köthe rings*, Math. Annalen **164**, 207-212, 1966.

[6] C. Faith, *Algebra II, Ring Theory*, Grundlehren der Matematischen Wissenschaften **191**, Springer-Verlag, 1976.

[7] L. Shen and J.L. Chen, *On strong Goldie dimension*, Comm. Algebra **35**, 3018-3025, 2007.