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Distribution of zeros of sublinear dynamic equations with a damping term on time scales

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Abstract

In this paper, for a second order sublinear dynamic equation with a damping term we will study the lower bounds of the distance between zeros of a solution and/or its derivatives and then establish some new criteria for disconjugacy and disfocality. Our results present a slight improvement to some results proved in the litrature. As a special case when $\mathbb{T} = \mathbb{R}$, for a second order linear differential equation, we get some results proved by Brown and Harris as a consequence of our results. The results will be proved by employing the time scales Hölder inequality, the time scales chain rule and some new dynamic Opial-type inequalities designed and proved for this purpose.

Keywords: Disconjugacy, Disfocality, Lyapunov inequality, Opial's inequality, Sublinear dynamic equations, Time Scales.

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1. Introduction

In this paper, we will study the distribution of zeros of solutions of the second-order sublinear dynamic equation with a damping term

$$\left(r\left(t\right)\left(y^{\Delta}\left(t\right)\right)^{\beta}\right)^{\Delta}+p(t)\left(y^{\Delta}\left(t\right)\right)^{\beta}+q(t)\left(y^{\sigma}\left(t\right)\right)^{\beta}=0, \quad \text{on } [a,b]_{\mathbb{T}}, \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$, where $0 < \beta \leq 1$ is a quotient of odd positive integers, r, p and q are real $rd-$continuous functions defined on $\mathbb{T}$ with $r(t) > 0$. In particular, we will find the lower bounds of the distance between zeros of a solution and/or its derivatives and prove several results related to the problems:

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(i) obtain lower bounds for the spacing $b-a$ where y is a solution of (1.1) and satisfies
$y(a) = y^{\Delta}(b) = 0$, or $y^{\Delta}(a) = y(b) = 0$,

(ii) obtain lower bounds for the spacing between consecutive zeros of solutions of (1.1).

By a solution of (1.1) on an interval $\mathbb{I}$, we mean a nontrivial real-valued function $y \in C_{rd}(\mathbb{I})$, which has the property that $r(t)\, y^{\Delta}(t) \in C^1_{rd}(\mathbb{I})$ and satisfies (1.1) on $\mathbb{I}$. We say that a solution y of (1.1) has a generalized zero at t if $y(t) = 0$ and has a generalized zero in $(t, \sigma(t))$ in case $y(t)\, y^{\sigma}(t) < 0$ and $\mu(t) > 0$. Equation (1.1) is disconjugate on the interval $[a,b]_{\mathbb{T}}$, if there is no nontrivial solution of (1.1) with two (or more) generalized zeros in $[a,b]_{\mathbb{T}}$. The solution $y(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is oscillatory. We say that (1.1) is right disfocal (left disfocal) on $[a,b]_{\mathbb{T}}$ if the solutions of (1.1) such that $y^{\Delta}(a) = 0$ ($y^{\Delta}(b) = 0$) have no generalized zeros in $[a,b]_{\mathbb{T}}$. We refer the reader to the book [28] for more details about oscillation and nonoscillation theory of dynamic equations on time scales.

We note that, equation (1.1) in its general form covers several different types of differential and difference equations depending on the choice of the time scale $\mathbb{T}$. For example, when $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, $\mu(t) = 0$, $x^{\Delta}(t) = x'(t)$ and (1.1) becomes the second-order sublinear differential equation

$$(1.2) \qquad (r(t)(x'(t))^{\beta})' + p(t)(x'(t))^{\beta} + q(t)x^{\beta}(t) = 0.$$

When $\mathbb{T} = \mathbb{Z}$, we have $\sigma(t) = t+1$, $\mu(t) = 1$, $x^{\Delta}(t) = \Delta x(t) = x(t+1) - x(t)$ and (1.1) becomes the second-order difference equation

$$(1.3) \qquad \Delta(r(t)\,(\Delta x(t))^{\beta}) + p(t)\,(\Delta x(t))^{\beta} + q(t)x^{\beta}(t+1)) = 0.$$

We present in the sequel some of the results that serve and motivate the contents on this paper. The well known existence results in the literature for disconjugacy is due to C. de la Vallée Poussin [22]. He considered the general n^{th} order linear differential equation

$$(1.4) \qquad x^{(n)} + p_0(t)x^{(n-1)} + \cdots + p_{n-1}(t)x = 0,$$

where the coefficients p_i are real continuous functions on an interval $\mathbb{I} = [a,b]$, and proved that if $|p_i(t)| \le q_i$ on $\mathbb{I}$ and the inequality

$$(1.5) \qquad \sum_{i=1}^{n} \frac{q_i\,(b-a)^i}{i!} \le 1,$$

holds, then (1.4) is disconjugate (that is every nontrivial solution of (1.4) has less than n zeros on $\mathbb{I}$, multiple zeros being counted according to their multiplicity).

Lyapunov [17] investigated the best known existence result in the literature for the second order differential equation

$$(1.6) \qquad x''(t) + q(t)x(t) = 0, \quad t \in (a,b),$$

and proved that if $x(t)$ is a solution of (1.6) with $x(a) = x(b) = 0$ and $q(t)$ is a continuous and nonnegative function on the closed interval $[a,b]$, then

$$(1.7) \qquad \int_a^b q(t)dt > \frac{4}{b-a}.$$

The constant 4 is the best possible and cannot be replaced by a larger number. The inverse of (1.7) gives a sufficient condition for disconjugacy of (1.6). The Lyapunov inequality is very important and has been extended extensively in the study of various properties of ordinary differential equations, for example bounds for eigenvalues, oscillation theory, stability criteria for periodic differential equations, and estimates for intervals of disconjugacy.

Since the appearance of Lyapunov's fundamental paper, there are many improvements and generalizations of (1.7) in several papers and different conditions for the disconjugacy, for the second order differential equation (1.2) and its special cases, have been investigated by many authors. We refer the reader to the papers [12, 16, 19, 21, 30, 32]. A literature review of continuous and discrete Lyapunov-type inequalities and their applications can be found in the survey articles by Brown and Hinton [6], Cheng [8] and Tiryaki [31] and the references cited therein. Hartman in [11, Chap. XI] generalized the classical Lyapunov inequality for the second order linear differential equation

$$\left(r(t)x^{'}(t)\right)^{'}+q(t)x(t)=0, \quad r(t)>0, \tag{1.8}$$

and proved that if $x(a)=x(b)=0$, then

$$\int_a^b q^{+}(s)ds \geq \frac{4}{\int_a^b r^{-1}(s)ds}, \tag{1.9}$$

where $q^{+}(t)=\max\{0,q(t)\}$ is the nonnegative part of $q(t)$.

Cohn [9] and Kwong [14] proved that if $x(t)$ is a solution of (1.6) with $x(a)=x^{'}(c)=0$, then

$$\int_a^c (t-a)\,q(t)dt>1,$$

and similarly if $x(t)$ is a solution of (1.6) with $x^{'}(c)=x(b)=0$, then

$$\int_c^b (b-t)\,q(t)dt>1.$$

Harris and Kong [10] proved that if $x(t)$ is a solution of (1.6) with $x(a)=x^{'}(b)=0$, then

$$(b-a)\sup_{a\leq t\leq b}\left|\int_t^b q(s)\Delta s\right|>1, \tag{1.10}$$

and if instead $x^{'}(a)=x(b)=0$, then

$$(b-a)\sup_{a\leq t\leq b}\left|\int_a^t q(s)\Delta s\right|>1. \tag{1.11}$$

Brown and Hinton [7] proved that if $x(t)$ is a solution of (1.6) with $x(a)=x^{'}(b)=0$, then

$$2\int_a^b Q_1^2(t)(t-a)dt>1, \tag{1.12}$$

where $Q_1(t)=\int_t^b q(s)ds$. If instead $x^{'}(a)=x(b)=0$, *then*

$$2\int_a^b Q_2^2(t)(b-t)dt>1, \tag{1.13}$$

where $Q_2(t) = \int\limits_a^t q(s)ds$.

In [29] the author considered the equation (1.2) when $\beta = 1$ and established some criteria for disconjugacy and disfocality of solutions in an interval $\mathbb{I} = [a, b] \subset \mathbb{R}$. He also applied Hardy and Wirtinger type inequalities and established an explicit formula for the lower bound of the first eigenvalue of the eigenvalue problem

$$(1.14) \quad -\left(x^{'}(t)\right)^{'} - p(t)x^{'}(t) + q(t)x(t) = \lambda x(t), \;\; x(a) = x(b) = 0.$$

For the study of dynamic equations on time scales, Bohner et al. [5] considered the dynamic equation

$$(1.15) \quad x^{\Delta\Delta}(t) + q(t)x^{\sigma}(t) = 0,$$

and proved a new Lyapunov dynamic inequality on a time scale $\mathbb{T}$, where $q(t)$ is a positive rd-continuous function defined on $\mathbb{T}$. Saker [24], employed some new dynamic Opial type inequalities and established new Lyapunov type inequalities for the equation

$$(1.16) \quad \left(r(t)\, x^{\Delta}(t)\right)^{\Delta} + q(t)x^{\sigma}(t) = 0, \quad \text{on } [a, b]_{\mathbb{T}},$$

where r; q are rd-continuous functions satisfy the conditions

$$\int_a^b \frac{1}{r(t)}\Delta t < \infty, \quad \text{and} \quad \int_a^b q(t)\Delta t < \infty.$$

For more results related to these results, we refer the reader to the papers by Karpuz [13] and Saker [23, 27] and the references cited therein.

Following this trend and to develop the study of oscillation of second-order sublinear dynamic equations on time scales, we will prove several results related to the problems $(i) - (ii)$. The rest of the paper is divided into three sections: In Section 2, we present some basic concepts of the time scales calculus and present some dynamic Opial-type inequalities, which are also interesting results in their own right, that will be used in the proof of our main results. In Section 3, we first prove some new generalizations of Opial's inequality on an arbitrary time scale $\mathbb{T}$, then we will employ these inequalities to prove several results related to the problems $(i) - (ii)$ above. In Section 4, we will discuss some special cases of the results. The results yield some conditions for disfocality and disconjugacy for equation (1.1).

2. Preliminaries and Some Opial's Inequalities

In this section, we briefly give some essentials of time scales calculus which are necessary for our results, then we present some dynamic Opial-type inequalities on an arbitrary time scale $\mathbb{T}$.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by: $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, where $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$, is said to be left–dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, is right–dense if $\sigma(t) = t$, is left–scattered if $\rho(t) < t$ and right–scattered if $\sigma(t) > t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided g is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$. We denote by $C_{rd}^{(n)}(\mathbb{T})$ the space of all functions $f \in C_{rd}(\mathbb{T})$ such that $f^{\Delta_i} \in C_{rd}(\mathbb{T})$ for $i = 0, 1, 2, ..., n$ for $n \in \mathbb{N}$.

The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t \geq 0$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. We assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. For more details of time scale analysis we refer the reader to the two books by Bohner and Peterson [2], [3] which summarize and organize much of the time scale calculus. In this paper, we will refer to the (delta) integral which we can define as follows. If $G^{\Delta}(t) = g(t)$, then the Cauchy (delta) integral of g is defined by $\int_a^t g(s)\Delta s := G(t) - G(a)$. It can be shown (see [2]) that if $g \in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^{\Delta}(t) = g(t)$, $t \in \mathbb{T}$. A simple consequence of Keller's chain rule [2, Theorem 1.90] is given by

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_0^1 [hx^{\sigma}(t) + (1-h)x(t)]^{\gamma-1}\, dh\, x^{\Delta}(t), \tag{2.1}$$

and the integration by parts formula on time scales is given by

$$\int_a^b u(t)v^{\Delta}(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t. \tag{2.2}$$

The Hölder inequality, see [2, Theorem 6.13], on time scales is given by

$$\int_a^b |f(t)g(t)|\Delta t \le \left[\int_a^b |f(t)|^{\gamma}\Delta t\right]^{\frac{1}{\gamma}} \left[\int_a^b |g(t)|^{\nu}\Delta t\right]^{\frac{1}{\nu}}, \tag{2.3}$$

where a, $b \in \mathbb{T}$ and f, $g \in \mathrm{C}_{rd}(\mathbb{I},\mathbb{R})$, $\gamma > 1$ and $\frac{1}{\gamma} + \frac{1}{\nu} = 1$. Throughout the paper, we will assume that the functions in the statements of the theorems are nonnegative and rd-continuous functions and the integrals are assumed to exist.

For completeness, in the following, we recall some of the Opial-type inequalities that serve and motivate the contents of the paper.

In 1960 Opial [20] published an inequality involving integrals of a function and its derivative. Since the discovery of Opial's inequality much work has been done, and many papers which deal with new proofs, various generalizations, extensions and their discrete analogues have been also proved in the literature. The discrete analogy of Opial's inequality has been proved in [15]. In [4] the authors extended the Opial inequality to an arbitrary time scale $\mathbb{T}$ and proved that if $y : [0,h]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(0) = 0$, then

$$\int_0^h |y(x) + y^{\sigma}(x)| \left|y^{\Delta}(x)\right| \Delta x \le h \int_0^h \left|y^{\Delta}(x)\right|^2 \Delta x. \tag{2.4}$$

They also proved that if r and q are positive rd-continuous functions on $[0,h]_{\mathbb{T}}$, $\int_0^h \frac{\Delta x}{r(x)} < \infty$, q nonincreasing and $y : [0,h]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(0) = 0$, then

$$\int_0^h q^{\sigma}(x) \left|(y(x) + y^{\sigma}(x))\, y^{\Delta}(x)\right| \Delta x \le \int_0^h \frac{\Delta x}{r(x)} \int_0^h r(t)q(x) \left|y^{\Delta}(x)\right|^2 \Delta x. \tag{2.5}$$

In [24] the author proved that if $y : [a,\tau]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(a) = 0$, then

$$\int_a^{\tau} s(x)\, |y(x) + y^{\sigma}(x)| \left|y^{\Delta}(x)\right| \Delta x \le K_1(a,\tau) \int_a^{\tau} r(x) \left|y^{\Delta}(x)\right|^2 \Delta x, \tag{2.6}$$

where $s \in C_{rd}([a,\tau]_{\mathbb{T}},\mathbb{R})$ and r be a positive rd-continuous function on $(a,\tau)_{\mathbb{T}}$ such that $\int_a^{\tau} r^{-1}(t)\Delta t < \infty$, and

$$K_1(a,\tau) = \sqrt{2}\left(\int_a^{\tau} \frac{s^2(x)}{r(x)} \left(\int_a^x \frac{\Delta t}{r(t)}\right) \Delta x\right)^{\frac{1}{2}} + \sup_{a \le x \le \tau} \left(\mu(x)\frac{|s(x)|}{r(x)}\right).$$

In [26] the author generalized (2.6) and proved that if $y : [a,\tau]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(a) = 0$, then

$$\int_a^\tau s(x)|y(x)+y^\sigma(x)|^\lambda |y^\Delta(x)|^\delta \Delta x \le H_1(a,\tau)\int_a^\tau r(x)|y^\Delta(x)|^{\lambda+\delta}\Delta x, \tag{2.7}$$

where r, s be nonnegative rd-continuous functions on $[a,\tau]_{\mathbb{T}}$ such that $\int_a^\tau r^{\frac{-1}{\lambda+\delta-1}}(t)\Delta t < \infty$, λ, δ be positive real numbers such that $\lambda \ge 1$ and

$$\begin{aligned} H_1(a,\tau) \quad : \quad &= 2^{\lambda-1}\sup_{a\le x\le\tau}\left(\mu^\lambda(x)\frac{s(x)}{r(x)}\right) + 2^{2\lambda-1}\left(\frac{\delta}{\lambda+\delta}\right)^{\frac{\delta}{\lambda+\delta}} \\ &\times \left(\int_a^\tau \frac{(s(x))^{\frac{\lambda+\delta}{\lambda}}}{(r(x))^{\frac{\delta}{\lambda}}}\left(\int_a^x r^{\frac{-1}{\lambda+\delta-1}}(t)\Delta t\right)^{\lambda+\delta-1}\Delta x\right)^{\frac{\lambda}{\lambda+\delta}}. \end{aligned}$$

In [25] the author proved that if $y : [a,\tau]_{\mathbb{T}} \to \mathbb{R}^+$ is delta differentiable with $y(a) = 0$, then

$$\int_a^\tau s(x)|y(x)+y^\sigma(x)|^p |y^\Delta(x)|^q \Delta x \le K_2(a,\tau)\int_a^\tau r(x)|y^\Delta(x)|^{p+q}\Delta x, \tag{2.8}$$

where p, $q > 0$ such that $p \le 1$, $p+q > 1$, r, s be nonnegative rd-continuous functions such that $\int_a^\tau r^{\frac{-1}{p+q-1}}(t)\Delta t < \infty$ and

$$\begin{aligned} K_2(a,\tau) \quad : \quad &= \sup_{a\le x\le\tau}\left(\mu^p(x)\frac{s(x)}{r(x)}\right) + 2^{p}\left(\frac{q}{p+q}\right)^{\frac{q}{p+q}} \\ &\times \left(\int_a^\tau \frac{(s(x))^{\frac{p+q}{p}}}{(r(x))^{\frac{q}{p}}}\left(\int_a^x r^{\frac{-1}{p+q-1}}(t)\Delta t\right)^{p+q-1}\Delta x\right)^{\frac{p}{p+q}}. \end{aligned} \tag{2.9}$$

If $[a,\tau]_{\mathbb{T}}$ is replaced by $[\tau,b]_{\mathbb{T}}$, then we get the following result

$$\int_\tau^b s(x)|y(x)+y^\sigma(x)|^p |y^\Delta(x)|^q \Delta x \le K_3(\tau,b)\int_\tau^b r(x)|y^\Delta(x)|^{p+q}\Delta x, \tag{2.10}$$

where

$$\begin{aligned} K_3(\tau,b) \quad : \quad &= \sup_{\tau\le x\le b}\left(\mu^p(x)\frac{s(x)}{r(x)}\right) + 2^{p}\left(\frac{q}{p+q}\right)^{\frac{q}{p+q}} \\ &\times \left(\int_\tau^b \frac{(s(x))^{\frac{p+q}{p}}}{(r(x))^{\frac{q}{p}}}\left(\int_x^b r^{\frac{-1}{p+q-1}}(t)\Delta t\right)^{p+q-1}\Delta x\right)^{\frac{p}{p+q}}. \end{aligned} \tag{2.11}$$

We assume that there exists $\tau \in (a,b)$ which is the unique solution of the equation

$$K(p,q) = K_2(a,\tau) = K_3(\tau,b) < \infty, \tag{2.12}$$

where $K_2(a,\tau)$ and $K_3(\tau,b)$ are defined as in (2.9) and (2.11). Combining (2.8) and (2.10), we get

$$\int_a^b s(x)|y(x)+y^\sigma(x)|^p |y^\Delta(x)|^q \Delta x \le K(p,q)\int_a^b r(x)|y^\Delta(x)|^{p+q}\Delta x, \tag{2.13}$$

where $y : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(a) = 0 = y(b)$, $\int_a^b r^{\frac{-1}{p+q-1}}(t)\Delta t < \infty$ and $K(p,q)$ is defined as in (2.12).

3. Main results

In this section, we prove some new Opial-type inequalities on a time scale $\mathbb{T}$ and apply these new inequalities on the second-order sublinear dynamic equation (1.1) to obtain some new Lyapunov-type inequalities related to problems $(i)-(ii)$. Throughout the rest of the paper, we will assume that the functions in the statements of the theorems are nonnegative and rd-continuous functions and the integrals considered are assumed to exist.

3.1. New Opial-type inequalities. Now, we will prove some new Opial type inequalities that will be needed in the proofs of our main results.

3.1. Theorem. *Let $\mathbb{T}$ be a time scale with $a,\tau\in\mathbb{T}$ and λ, δ be positive real numbers such that $\lambda\leq 1$, $\lambda+\delta>1$, and let r, s be nonnegative rd-continuous functions on $(a,\tau)_{\mathbb{T}}$ such that $\int_a^\tau r^{\frac{-1}{\lambda+\delta-1}}(t)\Delta t<\infty$. If $y:[a,\tau]_{\mathbb{T}}\to\mathbb{R}$ is delta differentiable with $y(a)=0$, then*

$$\int_a^\tau s(x)|y^\sigma(x)|^\lambda|y^\Delta(x)|^\delta\Delta x\leq H_1(a,\tau,\lambda,\delta)\int_a^\tau r(x)|y^\Delta(x)|^{\lambda+\delta}\Delta x, \tag{3.1}$$

where

$$\begin{aligned} H_1(a,\tau,\lambda,\delta)\quad:\quad&=\left(\frac{\delta}{\lambda+\delta}\right)^{\frac{\delta}{\lambda+\delta}}\left(\int_a^\tau\frac{(s(x))^{\frac{\lambda+\delta}{\lambda}}}{(r(x))^{\frac{\delta}{\lambda}}}\left(\int_a^x r^{\frac{-1}{\lambda+\delta-1}}(t)\Delta t\right)^{\lambda+\delta-1}\Delta x\right)^{\frac{\lambda}{\lambda+\delta}}\\ &+\sup_{a\leq x\leq\tau}\left(\mu^\lambda(x)\frac{s(x)}{r(x)}\right), \end{aligned}\tag{3.2}$$

Proof. Since r is nonnegative on $(a,\tau)_{\mathbb{T}}$, it follows from the Hölder inequality with $f(t)=\frac{1}{(r(t))^{\frac{1}{\lambda+\delta}}}$, $g(t)=(r(t))^{\frac{1}{\lambda+\delta}}|y^\Delta(t)|$, $\gamma=\frac{\lambda+\delta}{\lambda+\delta-1}$ and $\beta=\lambda+\delta$, that

$$\begin{aligned} |y(x)|\quad&\leq\quad\int_a^x|y^\Delta(t)|\Delta t=\int_a^x\frac{1}{(r(t))^{\frac{1}{\lambda+\delta}}}(r(t))^{\frac{1}{\lambda+\delta}}|y^\Delta(t)|\Delta t\\ &\leq\quad\left(\int_a^x\frac{1}{(r(t))^{\frac{1}{\lambda+\delta-1}}}\Delta t\right)^{\frac{\lambda+\delta-1}{\lambda+\delta}}\left(\int_a^x r(t)|y^\Delta(t)|^{\lambda+\delta}\Delta t\right)^{\frac{1}{\lambda+\delta}}. \end{aligned}$$

Then, for $a\leq x\leq\tau$, we can write

$$|y(x)|^\lambda\leq\left(\int_a^x\frac{1}{(r(t))^{\frac{1}{\lambda+\delta-1}}}\Delta t\right)^{\lambda(\frac{\lambda+\delta-1}{\lambda+\delta})}\left(\int_a^x r(t)|y^\Delta(t)|^{\lambda+\delta}\Delta t\right)^{\frac{\lambda}{\lambda+\delta}}. \tag{3.3}$$

Now, since $y^\sigma=y+\mu y^\Delta$, by applying inequality (see [18], page 500)

$$2^{r-1}\left|a^r+b^r\right|\leq|a+b|^r\leq\left|a^r+b^r\right|,\quad\text{for}\quad 0\leq r\leq 1, \tag{3.4}$$

we have that

$$|y^\sigma|^\lambda=|y+\mu y^\Delta|^\lambda\leq|y|^\lambda+\mu^\lambda|y^\Delta|^\lambda. \tag{3.5}$$

Setting

$$z(x):=\int_a^x r(t)|y^\Delta(t)|^{\lambda+\delta}\Delta t, \tag{3.6}$$

we see that $z(a)=0$, and

$$z^\Delta(x)=r(x)|y^\Delta(x)|^{\lambda+\delta}>0. \tag{3.7}$$

This gives that

$$|y^{\Delta}(x)|^{\lambda+\delta}=\frac{z^{\Delta}(x)}{r(x)} \quad \text{and} \quad |y^{\Delta}(x)|^{\delta}=\left(\frac{z^{\Delta}(x)}{r(x)}\right)^{\frac{\delta}{\lambda+\delta}}. \tag{3.8}$$

Thus since s is nonnegative on $(a,\tau)_{\mathbb{T}}$, we get from (3.3), (3.5) and (3.8) that

$$\begin{aligned} s(x)|y^{\sigma}(x)|^{\lambda}|y^{\Delta}(x)|^{\delta} &\leq s(x)|y|^{\lambda}|y^{\Delta}(x)|^{\delta}+s(x)\mu^{\lambda}|y^{\Delta}|^{\lambda+\delta} \\ &\leq s(x)\left(\frac{1}{r(x)}\right)^{\frac{\delta}{\lambda+\delta}}\times\left(\int_a^x \frac{1}{(r(t))^{\frac{1}{\lambda+\delta-1}}}\Delta t\right)^{\lambda(\frac{\lambda+\delta-1}{\lambda+\delta})} \\ &\quad \times(z(x))^{\frac{\lambda}{\lambda+\delta}}(z^{\Delta}(x))^{\frac{\delta}{\lambda+\delta}}+s(x)\mu^{\lambda}(x)\left(\frac{z^{\Delta}(x)}{r(x)}\right). \end{aligned}$$

This implies that

$$\begin{aligned} &\int_a^{\tau} s(x)|y^{\sigma}(x)|^{\lambda}|y^{\Delta}(x)|^{\delta}\Delta x \\ \leq\ &\int_a^{\tau} s(x)\left(\frac{1}{r(x)}\right)^{\frac{\delta}{\lambda+\delta}}\times\left(\int_a^x \frac{1}{(r(t))^{\frac{1}{\lambda+\delta-1}}}\Delta t\right)^{\lambda(\frac{\lambda+\delta-1}{\lambda+\delta})} \\ &\times(z(x))^{\frac{\lambda}{\lambda+\delta}}(z^{\Delta}(x))^{\frac{\delta}{\lambda+\delta}}\Delta x+\int_a^{\tau}\left(\mu^{\lambda}\frac{s(x)}{r(x)}\right)z^{\Delta}(x)\Delta(x). \end{aligned}$$

$$\begin{aligned} \leq\ &\int_a^{\tau} s(x)\left(\frac{1}{r(x)}\right)^{\frac{\delta}{\lambda+\delta}}\times\left(\int_a^x \frac{1}{(r(t))^{\frac{1}{\lambda+\delta-1}}}\Delta t\right)^{\lambda(\frac{\lambda+\delta-1}{\lambda+\delta})} \\ &\times(z(x))^{\frac{\lambda}{\lambda+\delta}}(z^{\Delta}(x))^{\frac{\delta}{\lambda+\delta}}\Delta x+\sup_{a\leq x\leq\tau}\left(\mu^{\lambda}\frac{s(x)}{r(x)}\right)\int_a^{\tau} z^{\Delta}(x)\Delta(x). \end{aligned} \tag{3.9}$$

Applying the Hölder inequality (2.3) with indices $(\lambda+\delta)/\lambda$ and $(\lambda+\delta)/\delta$, we have

$$\begin{aligned} &\int_a^{\tau} s(x)|y^{\sigma}(x)|^{\lambda}|y^{\Delta}(x)|^{\delta}\Delta x \\ \leq\ &\left(\int_a^{\tau} s(x)^{\frac{\lambda+\delta}{\lambda}}\left(\frac{1}{r(x)}\right)^{\frac{\delta}{\lambda}}\left(\int_a^x \frac{1}{(r(t))^{\frac{1}{\lambda+\delta-1}}}\Delta t\right)^{(\lambda+\delta-1)}\Delta x\right)^{\frac{\lambda}{\lambda+\delta}} \\ &\times\left(\int_a^{\tau} z^{\frac{\lambda}{\delta}}(x)z^{\Delta}(x)\Delta x\right)^{\frac{\delta}{\lambda+\delta}}+\sup_{a\leq x\leq\tau}\left(\mu^{\lambda}\frac{s(x)}{r(x)}\right)\int_a^{\tau} z^{\Delta}(x)\Delta(x). \end{aligned} \tag{3.10}$$

From (3.7), and the chain rule (2.1), we get that

$$z^{\frac{\lambda}{\delta}}(x)z^{\Delta}(x)\leq\frac{\delta}{\lambda+\delta}\left(z^{\frac{\lambda+\delta}{\delta}}(x)\right)^{\Delta}. \tag{3.11}$$

Substituting (3.11) into (3.10) and using the fact that $z(a) = 0$, we obtain

$$\begin{aligned}
&\int_a^\tau s(x)|y^\sigma(x)|^\lambda |y^\Delta(x)|^\delta \Delta x \\
\leq\; & \left(\int_a^\tau s(x)^{\frac{\lambda+\delta}{\lambda}} \left(\frac{1}{r(x)}\right)^{\frac{\delta}{\lambda}} \left(\int_a^x \frac{1}{(r(t))^{\frac{1}{\lambda+\delta-1}}}\Delta t\right)^{(\lambda+\delta-1)} \Delta x\right)^{\frac{\lambda}{\lambda+\delta}} \\
& \times \left(\frac{\delta}{\lambda+\delta}\right)^{\frac{\delta}{\lambda+\delta}} \left(\int_a^\tau \left(z^{\frac{\lambda+\delta}{\delta}}(x)\right)^\Delta \Delta t\right)^{\frac{\delta}{\lambda+\delta}} + \sup_{a\leq x\leq \tau}\left(\mu^\lambda \frac{s(x)}{r(x)}\right)\int_a^\tau z^\Delta(x)\Delta(x) \\
=\; & \left(\int_a^\tau s(x)^{\frac{\lambda+\delta}{\lambda}} \left(\frac{1}{r(x)}\right)^{\frac{\delta}{\lambda}} \left(\int_a^x \frac{1}{(r(t))^{\frac{1}{\lambda+\delta-1}}}\Delta t\right)^{(\lambda+\delta-1)} \Delta x\right)^{\frac{\lambda}{\lambda+\delta}} \\
& \times \left(\frac{\delta}{\lambda+\delta}\right)^{\frac{\delta}{\lambda+\delta}} z(\tau) + \sup_{a\leq x\leq \tau}\left(\mu^\lambda \frac{s(x)}{r(x)}\right) z(\tau).
\end{aligned}$$

Using (3.6), we have from the last inequality that

$$\int_a^\tau s(x)|y^\sigma(x)|^\lambda |y^\Delta(x)|^\delta \Delta x \leq H_1(a,\tau,\lambda,\delta)\int_a^\tau r(x)|y^\Delta(x)|^{\lambda+\delta}\Delta x,$$

which is the required inequality (3.1) with (3.2). This completes the proof. □

Next, we will just state the following theorem, since its proof is the same as that of Theorem 3.1, with $[a,\tau]_{\mathbb{T}}$ replaced by $[\tau,b]_{\mathbb{T}}$.

3.2. Theorem. *Let $\mathbb{T}$ be a time scale with $\tau, b \in \mathbb{T}$ and λ, δ be positive real numbers such that $\lambda \leq 1$, $\lambda+\delta > 1$, and let r, s be nonnegative rd-continuous functions on $(\tau, b)_{\mathbb{T}}$ such that $\int_\tau^b r^{\frac{-1}{\lambda+\delta-1}}(t)\Delta t < \infty$. If $y : [\tau, b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(b) = 0$, then*

$$\int_\tau^b s(x)|y^\sigma(x)|^\lambda |y^\Delta(x)|^\delta \Delta x \leq H_2(\tau,b,\lambda,\delta)\int_\tau^b r(x)|y^\Delta(x)|^{\lambda+\delta}\Delta x, \tag{3.12}$$

where

$$\begin{aligned}
H_2(\tau,b,\lambda,\delta) \;=\;& \left(\frac{\delta}{\lambda+\delta}\right)^{\frac{\delta}{\lambda+\delta}} \left(\int_\tau^b \frac{(s(x))^{\frac{\lambda+\delta}{\lambda}}}{(r(x))^{\frac{\delta}{\lambda}}} \left(\int_x^b r^{\frac{-1}{\lambda+\delta-1}}(t)\Delta t\right)^{\lambda+\delta-1} \Delta x\right)^{\frac{\lambda}{\lambda+\delta}} \\
& + \sup_{\tau\leq x\leq b}\left(\mu^\lambda(x)\frac{s(x)}{r(x)}\right).
\end{aligned} \tag{3.13}$$

In the following, we assume that there exists $\tau \in (a,b)_{\mathbb{T}}$ which is the unique solution of the equation

$$H(a,b) = H_1(a,\tau,\lambda,\delta) = H_2(\tau,b,\lambda,\delta) < \infty, \tag{3.14}$$

where $H_1(a,\tau,\lambda,\delta)$ and $H_2(\tau,b,\lambda,\delta)$ are defined as in Theorems 3.1 and 3.2. Note that since

$$\begin{aligned}
\int_a^b s(x)|y^\sigma(x)|^\lambda |y^\Delta(x)|^\delta \Delta x \;=\;& \int_a^\tau s(x)|y^\sigma(x)|^\lambda |y^\Delta(x)|^\delta \Delta x \\
& + \int_\tau^b s(x)|y^\sigma(x)|^\lambda |y^\Delta(x)|^\delta \Delta x,
\end{aligned}$$

then the proof of the following theorem is just a combination of Theorems 3.1 and 3.2 and so, we remove it.

3.3. Theorem. *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$ and λ, δ be positive real numbers such that $\lambda \leq 1$, $\lambda + \delta > 1$, and let r, s be nonnegative rd-continuous functions on $(a,b)_{\mathbb{T}}$ such that $\int_a^b r^{\frac{-1}{\lambda+\delta-1}}(t)\Delta t < \infty$. If $y : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(a) = 0 = y(b)$, then*

$$\int_a^b s(x)|y^\sigma(x)|^\lambda |y^\Delta(x)|^\delta \Delta x \leq H(a,b) \int_a^b r(x)|y^\Delta(x)|^{\lambda+\delta}\Delta x, \tag{3.15}$$

where $H(a,b)$ is defined as in (3.14).

3.2. New Lyapunov Inequalities. Now, we are ready to prove the results related to problems $(i) - (ii)$. For simplicity, we set the following notations:

$$K_1(a,b,\beta) := \sup_{a\leq t\leq b}\left(\mu^\beta(t)\frac{Q_1(t)}{r(t)}\right) + 2^\beta\left(\frac{1}{\beta+1}\right)^{\frac{1}{\beta+1}}\left(\int_a^b \frac{|Q_1(t)|^{\frac{\beta+1}{\beta}}}{r^{\frac{1}{\beta}}(t)} R_1(t)\Delta t\right)^{\frac{\beta}{\beta+1}},$$

$$H_1(a,b,\beta) := \left(\frac{\beta}{\beta+1}\right)^{\frac{\beta}{\beta+1}}\left(\int_a^b \frac{(p(t))^{\beta+1}}{(r(t))^\beta} R_1(t)\Delta t\right)^{\frac{1}{1+\beta}} + \sup_{a\leq t\leq b}\left(\mu(t)\frac{p(t)}{r(t)}\right),$$

where $Q_1(t) = \int\limits_t^b q(s)\Delta s$ and $R_1(t) = \left(\int_a^t r^{-\frac{1}{\beta}}(\theta)\Delta\theta\right)^\beta$,

$$K_2(a,b,\beta) := \sup_{a\leq t\leq b}\left(\mu^\beta(t)\frac{Q_2(t)}{r(t)}\right) + 2^\beta\left(\frac{1}{\beta+1}\right)^{\frac{1}{\beta+1}}\left(\int_a^b \frac{|Q_2(t)|^{\frac{\beta+1}{\beta}}}{r^{\frac{1}{\beta}}(t)} R_2(t)\Delta t\right)^{\frac{\beta}{\beta+1}},$$

and

$$H_2(a,b,\beta) := \left(\frac{\beta}{\beta+1}\right)^{\frac{\beta}{\beta+1}}\left(\int_a^b \frac{(p(t))^{\beta+1}}{(r(t))^\beta} R_2(t)\Delta t\right)^{\frac{1}{1+\beta}} + \sup_{a\leq t\leq b}\left(\mu(t)\frac{p(t)}{r(t)}\right),$$

where $Q_2(t) = \int\limits_a^t q(s)\Delta s$ and $R_2(t) = \left(\int_t^b r^{-\frac{1}{\beta}}(\theta)\Delta\theta\right)^\beta$.

3.4. Theorem. *Assume that y is a nontrivial solution of (1.1). If $y(a) = y^\Delta(b) = 0$, then*

$$2^{1-\beta}K_1(a,b,\beta) + H_1(a,b,\beta) \geq 1. \tag{3.16}$$

Proof. Without loss of generality, we may assume that $y(t) > 0$ in $[a,b]_{\mathbb{T}}$. Multiplying (1.1) by y^σ and integrating by parts (see 2.2), we get

$$\begin{aligned}
&\int\limits_a^b \left(r(t)\left(y^\Delta(t)\right)^\beta\right)^\Delta y^\sigma(t)\,\Delta t + \int\limits_a^b p(t)\left(y^\Delta(t)\right)^\beta y^\sigma(t)\,\Delta t \\
= \quad & r(t)\left(y^\Delta(t)\right)^\beta y(t)\Big|_a^b - \int\limits_a^b r(t)\left(y^\Delta(t)\right)^{\beta+1}\Delta t + \int\limits_a^b p(t)\left(y^\Delta(t)\right)^\beta y^\sigma(t)\,\Delta t \\
= \quad & -\int\limits_a^b q(t)\left(y^\sigma(t)\right)^{\beta+1}\Delta t.
\end{aligned}$$

Using the assumptions that $y(a) = y^{\Delta}(b) = 0$ and $Q(t) = \int_t^b q(s)\Delta s$, we have

$$\text{(3.17)} \quad \int_a^b r(t)\left(y^{\Delta}(t)\right)^{\beta+1}\Delta t = \int_a^b p(t)\left(y^{\Delta}(t)\right)^{\beta} y^{\sigma}(t)\,\Delta t - \int_a^b Q^{\Delta}(t)\,(y^{\sigma}(t))^{\beta+1}\,\Delta t.$$

Integrating by parts the term $\int_a^b Q^{\Delta}(t)\,(y^{\sigma}(t))^{\beta+1}\,\Delta t$, and using the facts that $y(a) = 0 = Q(b)$, we obtain

$$\text{(3.18)} \quad \int_a^b r(t)\left(y^{\Delta}(t)\right)^{\beta+1}\Delta t = \int_a^b p(t)\left(y^{\Delta}(t)\right)^{\beta} y^{\sigma}(t)\,\Delta t + \int_a^b Q(t)\left(y^{\beta+1}(t)\right)^{\Delta}\Delta t.$$

Applying the chain rule formula (2.1) and the inequality (3.4), we see that

$$\begin{aligned} \left|\left(y^{\beta+1}(t)\right)^{\Delta}\right| &\leq (\beta+1)\int_0^1 \left|hy^{\sigma}(t) + (1-h)\,y(t)\right|^{\beta} dh\left|y^{\Delta}(t)\right| \\ &\leq 2^{1-\beta}\left|y^{\sigma}(t) + y(t)\right|^{\beta}\left|y^{\Delta}(t)\right|. \qquad \text{(3.19)} \end{aligned}$$

This and (3.18) imply that

$$\begin{aligned} \int_a^b r(t)\left|y^{\Delta}(t)\right|^{\beta+1}\Delta t &\leq \int_a^b |p(t)|\left|y^{\sigma}(t)\right|\left|y^{\Delta}(t)\right|^{\beta}\Delta t \\ &\quad + 2^{1-\beta}\int_a^b |Q(t)|\left|y^{\sigma}(t) + y(t)\right|^{\beta}\left|y^{\Delta}(t)\right|\Delta t. \qquad \text{(3.20)} \end{aligned}$$

Applying the inequality (2.7) on the integral $\int_a^b |Q(t)|\left|y^{\sigma}(t) + y(t)\right|^{\beta}\left|y^{\Delta}(t)\right|\Delta t$, with $s = Q,\ p = \beta$, and $q = 1$, we have

$$\text{(3.21)} \quad \int_a^b |Q(t)|\left|y^{\sigma}(t) + y(t)\right|^{\beta}\left|y^{\Delta}(t)\right|\Delta t \leq K_1(a,b,\beta)\int_a^b r(t)\left|y^{\Delta}(t)\right|^{\beta+1}\Delta t.$$

Applying the inequality (3.1) on the integral $\int_a^b |p(t)|\left|y^{\sigma}(t)\right|\left|y^{\Delta}(t)\right|^{\beta}\Delta t$ with $s = p,\ \lambda = 1$, and $\delta = \beta$, we obtain

$$\text{(3.22)} \quad \int_a^b p(t)|y^{\sigma}(t)||y^{\Delta}(t)|^{\beta}\Delta t \leq H_1(a,b,\beta)\int_a^b r(t)|y^{\Delta}(t)|^{\beta+1}\Delta t.$$

Substituting (3.21) and (3.22) into (3.20), we get

$$\begin{aligned} \int_a^b r(t)\left|y^{\Delta}(t)\right|^{\beta+1}\Delta t &\leq 2^{1-\beta}K_1(a,b,\beta)\int_a^b r(t)\left|y^{\Delta}(t)\right|^{\beta+1}\Delta t \\ &\quad + H_1(a,b,\beta)\int_a^b r(t)|y^{\Delta}(t)|^{\beta+1}\Delta t. \qquad \text{(3.23)} \end{aligned}$$

Then, we have from (3.23) after cancelling the term $\int_a^b r(t)\left|y^{\Delta}(t)\right|^{\beta+1}\Delta t$, the desired inequality (3.16). The proof is complete. □

3.5. Remark. Theorem 3.4 gives us a condition for right disfocality of (1.1). In particular, if

$$2^{1-\beta}K_1(a,b,\beta)+H_1(a,b,\beta)<1,$$

then (1.1) is right disfocal in $[a,b]_{\mathbb{T}}$. This means that there is no nontrivial solution of (1.1) in $[a,b]_{\mathbb{T}}$ satisfies $y(a)=y^{\Delta}(b)=0$.

3.6. Theorem. *Assume that y is a nontrivial solution of (1.1). If $y^{\Delta}(a)=y(b)=0$, then*

$$2^{1-\beta}K_2(a,b,\beta)+H_2(a,b,\beta)\geq 1. \tag{3.24}$$

Proof. The proof of (3.24) is similar to (3.16) by employing Opial-type inequalities (2.8) and (3.12) instead of (2.7) and (3.1). The proof is complete. □

3.7. Remark. Theorem 3.6 gives us a condition for left disfocality of (1.1). In particular, if

$$2^{1-\beta}K_2(a,b,\beta)+H_2(a,b,\beta)<1.$$

then (1.1) is left disfocal in $[a,b]_{\mathbb{T}}$. This means that there is no nontrivial solution of (1.1) in $[a,b]_{\mathbb{T}}$ satisfies $y^{\Delta}(a)=y(b)=0$.

In the following, we employ inequalities (2.13) and (3.15) to determine the lower bound for the distance between consecutive zeros of a solution of (1.1).

3.8. Theorem. *Assume that $Q^{\Delta}(t)=q(t)$ and y is a nontrivial solution of (1.1). If $y(a)=y(b)=0$, then*

$$2^{1-\beta}K(a,b)+H(a,b)\geq 1, \tag{3.25}$$

where $K(a,b)$ and $H(a,b)$ are defined as in (2.12) and (3.14), respectively.

Proof. Multiplying (1.1) by y^{σ} and integrating by parts, we get that

$$\int_a^b r(t)\left(y^{\Delta}(t)\right)^{\beta+1}\Delta t=\int_a^b p(t)\left(y^{\Delta}(t)\right)^{\beta}y^{\sigma}(t)\,\Delta t-\int_a^b Q^{\Delta}(t)\left(y^{\sigma}(t)\right)^{\beta+1}\Delta t. \tag{3.26}$$

Using the facts that $y(a)=0=y(b)$, we obtain

$$\begin{aligned}\int_a^b r(t)\left(y^{\Delta}(t)\right)^{\beta+1}\Delta t &\leq \int_a^b p(t)\left(y^{\Delta}(t)\right)^{\beta}y^{\sigma}(t)\,\Delta t\\ &\quad+2^{1-\beta}\int_a^b |Q(t)|\left|y^{\sigma}(t)+y(t)\right|^{\beta}\left|y^{\Delta}(t)\right|\Delta t.\end{aligned} \tag{3.27}$$

Applying the inequality (2.13) on the integral $\int_a^b |Q(t)|\left|y^{\sigma}(t)+y(t)\right|^{\beta}\left|y^{\Delta}(t)\right|\Delta t$, with $s=|Q|,\ \lambda=\beta,\ \delta=1$, we have that

$$\int_a^b |Q(t)|\left|y^{\sigma}(t)+y(t)\right|^{\beta}\left|y^{\Delta}(t)\right|\Delta t\leq K(a,b)\int_a^b r(t)\left|y^{\Delta}(t)\right|^{\beta+1}\Delta t, \tag{3.28}$$

where $K(a,b)$ is defined as in (2.12). Applying the inequality (3.15) on the integral $\int\limits_a^b |p(t)| \, |y^\sigma(t)| \, \left|y^\Delta(t)\right|^\beta \Delta t$ with $s = p, \ \lambda = 1, \ \delta = \beta$, we have that

$$\int_a^b p(t)|y^\sigma(t)||y^\Delta(t)|^\beta \Delta t \leq H(a,b) \int_a^b r(t)|y^\Delta(t)|^{\beta+1}\Delta t, \tag{3.29}$$

where $H(a,b)$ is defined as in (3.14). Substituting (3.28) and (3.29) into (3.27), we get that

(3.30)

$$\int\limits_a^b r(t) \left|y^\Delta(t)\right|^{\beta+1} \Delta t \leq 2^{1-\beta} K(a,b) \int\limits_a^b r(t) \left|y^\Delta(t)\right|^{\beta+1} \Delta t + H(a,b) \int_a^b r(t)|y^\Delta(t)|^{\beta+1}\Delta t.$$

Then, we have from (3.30) after cancelling the term $\int\limits_a^b r(t) \left|y^\Delta(t)\right|^{\beta+1} \Delta t$, that

$$2^{1-\beta} K(a,b) + H(a,b) \geq 1,$$

which is the desired inequality (3.25). The proof is complete. □

3.9. Remark. Theorem 3.8 gives us a condition for disconjugacy of (1.1). In particular, if

$$K(a,b) + H(a,b) < 1,$$

then (1.1) is disconjugate in $[a,b]_\mathbb{T}$. This means that there is no nontrivial solution of (1.1) in $[a,b]_\mathbb{T}$ satisfies $y(a) = y(b) = 0$.

4. Applications

In Theorem 3.4 if $\beta = 1$, then we have the following result, which improves the obtained result in [23, Corollary 2.2] by removing the additional constant c in the conditions.

4.1. Corollary. *Assume that y is a nontrivial solution of (1.1). If $y(a) = y^\Delta(b) = 0$, then*

$$\sup_{a\leq t\leq b} \frac{1}{r(t)} \left[p(t)\mu(t) + Q_1(t)\mu(t)\right] + \sqrt{2} \left(\int\limits_a^b \frac{|Q_1(t)|^2}{r(t)} R_1(t)\Delta t \right)^{\frac{1}{2}} + \frac{1}{\sqrt{2}} \left(\int_a^b \frac{p^2(t)}{r(t)} R_1(t)\Delta t \right)^{\frac{1}{2}} \geq 1, \tag{4.1}$$

where $Q_1(t) = \int\limits_t^b q(s)\Delta s$ and $R_1(t) = \int_a^t \frac{\Delta\tau}{r(\tau)}$. If $y^\Delta(a) = y(b) = 0$, then

$$\sup_{a\leq t\leq b} \frac{1}{r(t)} \left[p(t)\mu(t) + Q_2(t)\mu(t)\right] + \sqrt{2} \left(\int\limits_a^b \frac{|Q_2(t)|^2}{r(t)} R_2(t)\Delta t \right)^{\frac{1}{2}} + \frac{1}{\sqrt{2}} \left(\int_a^b \frac{p^2(t)}{r(t)} R_2(t)\Delta t \right)^{\frac{1}{2}} \geq 1, \tag{4.2}$$

where $Q_2(t) = \int\limits_a^t q(s)\Delta s$ and $R_2(t) = \int_t^b \frac{\Delta\tau}{r(\tau)}$.

As a special case of Theorem 3.4, when $r(t) = 1$, we obtain the following result, which improves the result that is obtained in [23, Corollary 2.1] by removing the additional constant c in the obtained results.

4.2. Corollary. *Assume that y is a nontrivial solution of (1.1). If $y(a) = y^{\Delta}(b) = 0$, then*

$$\sup_{a\le t\le b}\left[\mu(t)p(t) + 2^{1-\beta}\mu^{\beta}(t)Q(t)\right] + \frac{2}{(\beta+1)^{\frac{1}{\beta+1}}}\left(\int_a^b |Q(t)|^{\frac{\beta+1}{\beta}}(\tau-a)^{\beta}\Delta t\right)^{\frac{\beta}{\beta+1}}$$
$$+\left(\frac{\beta}{\beta+1}\right)^{\frac{\beta}{\beta+1}}\left(\int_a^b |p(t)|^{\beta+1}(\tau-a)^{\beta}\Delta t\right)^{\frac{1}{1+\beta}} \ge 1,$$

where $Q(t) = \int_t^b q(s)\Delta s$. If $y^{\Delta}(a) = y(b) = 0$, then

$$\sup_{a\le t\le b}\left[\mu(t)p(t) + 2^{1-\beta}\mu^{\beta}(t)Q(t)\right] + \frac{2}{(\beta+1)^{\frac{1}{\beta+1}}}\left(\int_a^b |Q(t)|^{\frac{\beta+1}{\beta}}(b-\tau)^{\beta}\Delta t\right)^{\frac{\beta}{\beta+1}}$$
$$+\left(\frac{\beta}{\beta+1}\right)^{\frac{\beta}{\beta+1}}\left(\int_a^b |p(t)|^{\beta+1}(b-\tau)^{\beta}\Delta t\right)^{\frac{1}{1+\beta}} \ge 1,$$

where $Q(t) = \int_a^t q(s)\Delta s$.

As a special case of Corollary 4.1, when $p(t) = 0$, we have the following results.

4.3. Corollary. *Assume that y is a nontrivial solution of (1.16). If $y(a) = y^{\Delta}(b) = 0$, then*

$$\sup_{a\le t\le b}\frac{1}{r(t)}Q_1(t)\mu(t) + \sqrt{2}\left(\int_a^b \frac{|Q_1(t)|^2}{r(t)}R_1(t)\Delta t\right)^{\frac{1}{2}} \ge 1, \tag{4.3}$$

where $Q_1(t) = \int_t^b q(s)\Delta s$ and $R_1(t) = \int_a^t \frac{\Delta\tau}{r(\tau)}$. If instead $y^{\Delta}(a) = y(b) = 0$, then

$$\sup_{a\le t\le b}\frac{1}{r(t)}Q_1(t)\mu(t) + \sqrt{2}\left(\int_a^b \frac{|Q_1(t)|^2}{r(t)}R_2(t)\Delta t\right)^{\frac{1}{2}} \ge 1, \tag{4.4}$$

where $Q_2(t) = \int_a^t q(s)\Delta s$ and $R_2(t) = \int_t^b \frac{\Delta\tau}{r(\tau)}$.

Using the maximum of $|Q_1(t)|$ on $[a,b]_{\mathbb{T}}$ in Corollary 4.3, we get the following results.

4.4. Corollary. *Assume that y is a nontrivial solution of (1.16). If $y(a) = y^{\Delta}(b) = 0$, then*

$$\sup_{a\le t\le b}\frac{1}{r(t)}\left|\int_t^b q(s)\Delta s\right|\mu(t) + \sqrt{2}\max_{a\le t\le b}\left|\int_t^b q(s)\Delta s\right|\left(\int_a^b \frac{R_1(t)}{r(t)}\Delta t\right)^{\frac{1}{2}} \ge 1, \tag{4.5}$$

where $R_1(t) = \int_a^t \frac{\Delta\tau}{r(\tau)}$. If instead $y^{\Delta}(a) = y(b) = 0$, then

$$\sup_{a\le t\le b}\frac{1}{r(t)}\left|\int_a^t q(s)\Delta s\right|\mu(t) + \sqrt{2}\max_{a\le t\le b}\left|\int_a^t q(s)\Delta s\right|\left(\int_a^b \frac{R_2(t)}{r(t)}\Delta t\right)^{\frac{1}{2}} \ge 1, \tag{4.6}$$

where $R_2(t) = \int_t^b \frac{\Delta\tau}{r(\tau)}$.

As a special case when $\mathbb{T} = \mathbb{R}$, $\beta = 1$, $r(t) = 1$ and $p(t) = 0$, then $y^\sigma(t) = y(t)$ and equation (1.1) becomes

$$y''(t) + q(t)y(t) = 0. \tag{4.7}$$

Now, the results in Corollary 4.3 reduce to the following results obtained by Brown and Hinton [7].

4.5. Corollary. *Assume that* y *is a solution of the equation (4.7). If* $y(a) = y'(b) = 0$*, then*

$$2\int_a^b Q_1^2(t)(t-a)dt > 1, \tag{4.8}$$

where $Q_1(t) = \int_t^b q(s)ds$*. If instead* $y'(\alpha) = y(\beta) = 0$*, then*

$$2\int_a^b Q_2^2(t)(b-t)dt > 1, \tag{4.9}$$

where $Q_2(t) = \int_a^t q(s)ds$.

As a special case of Corollary 4.4 for the second order differential equation (4.7), we get the following results due to Harris and Kong [10].

4.6. Corollary. *Assume that* y *is a solution of the equation (4.7). If* $y(a) = y'(b) = 0$*, then*

$$(b-a)\sup_{a\le t\le b}\left|\int_t^b q(s)\Delta s\right| > 1. \tag{4.10}$$

If instead $y'(\alpha) = y(\beta) = 0$*, then*

$$(b-a)\sup_{a\le t\le b}\left|\int_a^t q(s)\Delta s\right| > 1. \tag{4.11}$$

Using the maximum of $|Q|$ and $|p|$ on $[a,b]_{\mathbb{T}}$ we have from Corollary 4.2 the following results for the second order difference equation

$$\Delta((\Delta y(t))^\beta + p(t)\,(\Delta y(t))^\beta + q(t)y^\beta(t+1)) = 0, \tag{4.12}$$

where $0 < \beta \le 1$ is a quotient of odd positive integers.

4.7. Corollary. *Assume that* y *is a nontrivial solution of (4.12). If* $y(a) = \Delta y(b) = 0$*, then*

$$\left[\max_{a\le\tau\le b}|p(t)| + 2^{1-\beta}\max_{a\le\tau\le b}\left|\sum_{s=t}^{b-1} q(s)\Delta s\right|\right] + \frac{2(b-a)^\beta}{(\beta+1)}\max_{a\le\tau\le b}\left|\sum_{s=t}^{b-1} q(s)\Delta s\right|$$
$$+\frac{\beta^{\frac{\beta}{\beta+1}}}{\beta+1}(b-a)\max_{a\le\tau\le b}|p(t)| \ge 1.$$

If $\Delta y(a) = y(b) = 0$, *then*

$$\left[\max_{a\le\tau\le b}|p(t)| + 2^{1-\beta}\max_{a\le\tau\le b}\left|\sum_{s=a}^{t-1} q(s)\Delta s\right|\right] + \frac{2(b-a)^{\beta}}{(\beta+1)}\max_{a\le\tau\le b}\left|\sum_{s=a}^{t-1} q(s)\Delta s\right|$$
$$+\frac{\beta^{\frac{\beta}{\beta+1}}}{\beta+1}(b-a)\max_{a\le\tau\le b}|p(t)| \ge 1.$$

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