A note on the embedding properties of $p$-subgroups in finite groups

Boru Zhang, Xiuyun Guo*

Department of Mathematics, Shanghai University, Shanghai 200444, P. R. China

Abstract

In this note, we prove that a finite group $G$ is $p$-supersolvable if and only if there exists a power $d$ of $p$ with $p^2 \leq d < |P|$ such that $H \cap O^p(G^*_p)$ is normal in $O^p(G)$ for all non-cyclic normal subgroups $H$ of $P$ with $|H| = d$, where $P$ is a Sylow $p$-subgroup of $G$. Moreover, we also prove that either $l_p(G) \leq 1$ and $r_p(G) \leq 2$ or else $|P \cap O^p(G)| > d$ if there exists a power $d$ of $p$ with $1 \leq d < |P|$ such that $H \cap O^p(G^*_{p^2})$ is normal in $O^p(G)$ for all non-meta-cyclic normal subgroups $H$ of $P$ with $|H| = d$.

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1. Introduction

All groups considered in this note are finite. We use conventional notions and notation, as in [9].

It is quite interesting to investigate the structure of a group by using some kind of the embedding properties of subgroups and many interesting results have been given (for example, see [1,6,8,13]). Recently, Guo and Isaacs [6] proved the following theorem by using the embedding condition “$H \cap O^p(G) \trianglelefteq O^p(G)$”.

Theorem 1.1. ([6, Theorem B]). Let $P \in Syl_p(G)$, and let $d$ be a power of $p$ such that $1 \leq d < |P|$. Assume that $H \cap O^p(G) \trianglelefteq O^p(G)$ for all subgroups $H \trianglelefteq P$ with $|H| = d$. Then either $G$ is $p$-supersolvable or else $|P \cap O^p(G)| > d$.

An interesting idea of [6] is that in the hypothesis of the theorem only the normal subgroups of order $d$ are considered, not necessarily the family of all subgroups of order $d$. Recall that a subgroup $H$ of a group $G$ is said to be $S$-semipermutable in $G$ (see [12]) if $H$ permutes with all Sylow $q$-subgroups of $G$ for the primes $q$ not dividing $|H|$. Ballester-Bolíndes et al. in their paper [1] proved an analogous result, but they only assume that $H \cap O^p(G)$ are $S$-semipermutable in $O^p(G)$ instead of assuming that these subgroups are normal in $O^p(G)$.

Theorem 1.2. ([1, Theorem 2]). Let $P \in Syl_p(G)$, and let $d$ be a power of $p$ such that $1 \leq d < |P|$. Assume that $H \cap O_p(G)$ is $S$-semipermutable in $G$ for all subgroups $H \trianglelefteq P$ with $|H| = d$. Then either $G$ is $p$-supersolvable or else $|P \cap O_p(G)| > d$.

*Corresponding Author.

Email addresses: brzhang@live.com (B. Zhang), xyguo@staff.shu.edu.cn (X. Guo)

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More recently, Yu in his paper [13] use the subgroup $G_p^*$ of a group $G$, and consider the embedding condition $O^p(G_p^*) \cap H \leq O^p(G)$ to prove the following result, where $G_p^*$ is the unique smallest normal subgroup of a group $G$ for which the corresponding factor group is abelian of exponent dividing $p - 1$.

**Theorem 1.3.** ([13, Theorem 2]). Let $P \in Syl_p(G)$, and let $d$ be a power of $p$ such that $1 \leq d < |P|$. Then $G$ is $p$-supersolvable if and only if $|P \cap O^p(G_p^*)| \leq d$ and $H \cap O^p(G_p^*) \leq O^p(G)$ for all subgroups $H \leq P$ with $|H| = d$.

Remark 2.3 and Example 2.4 in [13] show that it is better to use the embedding condition $O^p(G_p^*) \cap H \leq O^p(G)$ to investigate the $p$-supersolvability of groups. On the other hand, in all of the above results all normal subgroups of order $d$ in $P$ are considered. So we wonder whether we may reduce the number of normal subgroups of order $d$ in $P$?

In fact, we have the following results.

**Theorem 1.4.** Let $P \in Syl_p(G)$, and let $d$ be a power of $p$ such that $p^2 \leq d < |P|$. Then $G$ is $p$-supersolvable if and only if $|P \cap O^p(G_p^*)| \leq d$ and $H \cap O^p(G_p^*) \leq O^p(G)$ for all non-cyclic subgroups $H \leq P$ with $|H| = d$.

**Theorem 1.5.** Let $p$ be a prime dividing the order of a group $G$ of odd order, let $d$ be a power of $p$ such that $1 \leq d < |P|$ and $P \in Syl_p(G)$ with $|P| > d$. And let $H \cap O^p(G_p^*) \leq O^p(G)$ for all non-meta-cyclic normal subgroups $H$ in $P$ with $|H| = d$. Then either $p$-length $l_p(G) \leq 1$ and $p$-rank $r_p(G) \leq 2$ or else $|P \cap O^p(G_p^*)| > d$, where $G_p^*$ is the unique smallest normal subgroup of the group $G$ for which the corresponding factor group is abelian of exponent dividing $p^2 - 1$.

We should notice that we assume $d \geq p^2$ in Theorem 1.4. In fact, if $p = 2$ and $d = 2$, then the result is still true. Since $|P \cap O^p(G_p^*)| \leq 2$, it follows from Burnside Theorem [9, IV, 2.8] that $O^p(G_p^*)$ is 2-nilpotent, and thus $G_p^*$ is 2-nilpotent. Hence $G$ is 2-supersolvable. However, the result is not true if $p$ is odd prime and $d = p$ in Theorem 1.4. In fact, let $D$ be a non-abelian simple group such that Sylow $p$-subgroups of $D$ are cyclic of order $p$, and let $G = D \times C$ with a cyclic group $C$ of order $p$. Clearly, $G_p^* = G$ and $H \cap O^p(G_p^*)$ is normal in $O^p(G)$ for every non-cyclic normal subgroup $H$ of $P$ of order $d$, where $P$ is a Sylow $p$-subgroup. But $|P \cap O^p(G_p^*)| = p$ and $G$ is not $p$-supersolvable.

We also notice that the hypothesis “$G$ is an odd order group” in Theorem 1.5 can not be removed. In fact, let $D$ be a non-abelian simple group such that Sylow $p$-subgroups of $D$ are cyclic of order $p^m(d \geq p^m \geq 1)$, and let $G = D \times C$ with a cyclic group $C$ of order $p^m(m \geq 1)$. Clearly, $H \cap O^p(G_p^*)$ is normal in $O^p(G)$ for every non-meta-cyclic normal subgroup $H$ of $P$ of order $d$, where $P$ is a Sylow $p$-subgroup of $G$. But $|P \cap O^p(G_p^*)| = p^m \leq d$ and $G$ is not $p$-supersolvable.

Furthermore, the following example tells us that $G$ may not be $p$-supersolvable if $G$ satisfies the hypotheses in Theorem 1.5. Let $p$ be an odd prime with $p \neq 2^k - 1$ for all positive integer $k$. Write $P = (a) \times (b) \simeq C_{p^k}$. So $Aut(\Omega_1(P)) \simeq GL(2, p)$ and there exists a cyclic subgroup $T$ of order $p^2 - 1$ in $Aut(\Omega_1(P))$. Note that $p + 1$ is not a power of 2, then we can choose an automorphism $\alpha \in T$ with order $q$ such that $q|p + 1$ and $q \neq 2$. Now, considering the surjective homomorphism $\phi : Aut(P) \to Aut(\Omega_1(P))$; we can choose an automorphism $\alpha$ of $P$ such that $\phi(\alpha) = \overline{\alpha}$. Write the semidirect product $G = P \rtimes \langle \alpha \rangle$, it is clear that $H \cap O^p(G_p^*)$ is normal in $O^p(G)$ for every non-meta-cyclic normal subgroup $H$ of $P$ of order $d = p^m(m < 2n)$. Now we prove that $G$ is not $p$-supersolvable. If the action of $\overline{\pi}$ on $\Omega_1(P) = \langle a \rangle \times \langle b \rangle \simeq C_p^2$ is reducible, then it follows from $p \neq q$ and Maschke’s Theorem that $\langle a \rangle, \langle b \rangle$ are $\overline{\pi}$-invariant. It follows from $g.c.d.(p + 1, p - 1) = 2$ that $q \not| p - 1$, and therefore $\overline{\pi}$ acts trivially on both $\langle a \rangle$ and $\langle b \rangle$, that is, $\overline{\pi}$ acts trivially on $\Omega_1(P)$, a contradiction. Hence $\overline{\pi}$ acts irreducibly on $\Omega_1(P)$, implying that $\alpha$ acts irreducibly on
2. Preliminary results

In this section we list some basic and known results which will be used in our proofs.

**Definition 2.1.** ([7, Definition 1.9]). Let $p$ be prime. A group $G$ is said to be a $CS(p,n)$-group if $G$ is a $p$-group with a characteristic series

$$1 = G_0 < G_1 < \cdots < G_m = G$$

such that $|G_i/G_{i-1}| \leq p^i$ for all $i \geq 1$.

It is clear that meta-cyclic $p$-groups and $p$-groups of maximal class are both $CS(p,2)$-groups.

**Lemma 2.2.** ([3, Lemma 1.4]). Let $p$ be a prime, let $P$ be a $p$-group and let $d$ be a power of $p$ such that $p^2 \leq d < |P|$. Let $N \leq P$, where $|N| \geq d$, and suppose that every normal subgroup of $P$ that has order $d$ and is contained in $N$ is cyclic. Then $N$ is cyclic, dihedral, semidihedral or generalized quaternion.

**Lemma 2.3.** ([2, Lemma 2.4]). Let $P \leq G$, where $P$ is a $p$-group. Also, let $N \leq G$ be a $p$-subgroup with $|N| \leq |P|$ and $N \not\leq P$. Then $N < PN$, and every subgroup $H$ with $N < H \leq NP$ is non-cyclic.

**Lemma 2.4.** ([8, Lemma 2.5]). If a group $P$ of order $2^n > 2^3$ has a subgroup $M$ of order $2^{n-4}$ of maximal class, then either $P$ is of maximal class or $P/P' \cong C_2^3$, and $P'$ is cyclic.

**Lemma 2.5.** ([3, Exercise 1(b), p.114]). Let $P$ be dihedral, semidihedral or generalized quaternion, then $P$ has the only one normal subgroup $N$ of order $2^a$ for every $1 < 2^a < |P|/2$ and $N$ is cyclic.

**Lemma 2.6.** ([4, Corollary 69.5]). Let $p$ be an odd prime and $d$ be a power of $p$ such that $d \geq p^3$, and let $N$ be a normal subgroup of a $p$-group $P$ with $|N| \geq d$. If every normal subgroup of $P$ that has order $d$ and is contained in $N$ is meta-cyclic, then $N$ is a meta-cyclic group or a 3-group of maximal group.

**Lemma 2.7.** Let $p$ be a odd prime, and let $P$ be a meta-cyclic $p$-group or a 3-group of maximal class. If $N$ is normal in $P$, then $\Omega_1(N) \cong C_p \times C_p$ or $N$ is a 3-group of maximal class.

**Proof.** If $P$ is meta-cyclic, then $\Omega_1(N) \cong C_p \times C_p$. Now assume that $P$ is a 3-group of maximal class. It follows from [3, Exercise 9.1] that $N$ is a 3-group of maximal class or absolutely regular, where a $p$-group $N$ is absolutely regular if $|G/\Omega_1(G)| < p^2$ (see [3, List of definitions and notations]). If $N$ is absolutely regular, then $|\Omega_1(N)| = |N/\Omega_1(N)| \leq 3^2$, and thus $\Omega_1(N) \cong C_p \times C_p$ by [3, Lemma 1.4].

**Lemma 2.8.** Let $p$ be a prime and $d$ be a power of $p$ such that $p^3 \leq d$, and let $P$ be a $p$-group. Also, let $N$ and $P_1$ be normal subgroups of $P$ with $N \cong C_p \times C_p$ and $N \not\leq P_1$. If $P_1$ contains a non-meta-cyclic normal subgroup of order $d$ of $P$, then there exists a non-meta-cyclic normal subgroup $H$ of order $d$ of $P$ such that $N < H \leq P_1$.

**Proof.** Let $H_1$ be a non-meta-cyclic normal subgroup of order $d$ of $P$ with $H_1 \leq P_1$. If $N \not\leq H_1$, then $|N \cap H_1| = 1$ or $p$ by $N \cong C_p \times C_p$, that is, $|N : N \cap H_1| = p^2$ or $p$. First, we assume that $|N : N \cap H_1| = p$. Since $N \cap H_1$ is normal in $P$, there exists a maximal subgroup $M$ of $H_1$ such that $M \leq P$ and $N \cap H_1 \leq M$, and so $H = NM$ is normal in $P$ and $|H| = d$. Noting that $H_1$ is non-meta-cyclic, we have that $H$ is non-cyclic. It follows from $N \not\leq M$ and $N \cong C_p^2$ that $\Omega_1(H) > \Omega_1(M) \geq p^2$. Thus $H$ is non-meta-cyclic by Lemma 2.7 and $H \leq P_1$. Now assume that $|N : N \cap H_1| = p^2$ and take a subgroup $M_1$ of $H_1$. Then $\Omega_1(M_1)$ is a $p$-group of maximal class.
with $|M_1| = d/p^2$ and $M_1 \subseteq P$. Then $H = NM_1$ is a normal subgroup of $P$ with $|H| = d$. Noticing that $N \cong C_p \times C_p$ and $N \cap M = 1$, we see $|\Omega_1(H)| \geq |\Omega_1(N)||\Omega_1(M)| \geq p^3$. Hence $H$ is non-meta-cyclic by Lemma 2.7, as we wanted. 

**Lemma 2.9.** ([7, Lemma 2.2]). Let $P$ be a $p$-group. If $P$ has a meta-cyclic maximal subgroup and $P$ is not isomorphic to $C_p^n$, then $P$ is a $CS(p, 2)$-group.

**Lemma 2.10.** ([7, Lemma 3.2]). Let an odd order group $A$ act on a $CS(p, 2)$-group $P$. Then $P$ is centralized by $O^p(A_{p^2})$.

**Lemma 2.11.** Let $G$ be a group and let $p$ be a prime of $|G|$. If $G_{p^2}$ is $p$-nilpotent, then $G$ is $p$-solvable with $l_p(G) \leq 1$ and $r_p(G) \leq 2$.

**Proof.** Since $G_{p^2}$ is $p$-nilpotent, $G$ is $p$-solvable of $l_p(G) \leq 1$. We see $G$ has a chief series

$$1 = K_0 < \cdots < H_0 = O^p(G_{p^2}) < H_1 < \cdots < H_n = G_{p^2} < \cdots < G$$

Noticing that $O^p(G_{p^2}) \leq C_G(H_{i+1}/H_i)$ ($0 \leq i \leq n - 1$), we have $A_G(H_{i+1}/H_i) \simeq G/C_G(H_{i+1}/H_i) \in \mathfrak{D}_p \mathfrak{U}_{p-1}$, where $\mathfrak{D}_p$ is the formation consisting of all $p$-groups and $\mathfrak{U}_{p-1}$ is the formation consisting of all abelian groups with exponent dividing $p^2 - 1$. Since $O_p(A_G(H_{i+1}/H_i)) = 1$ by [5, A, Lemma 13.6], it follows that $A_G(H_{i+1}/H_i) \in \mathfrak{U}_{p-1}$, and so $A_G(H_{i+1}/H_i)$ is abelian with exponent dividing $p^2 - 1$.

Write $|H_{i+1}/H_i| = p^m$. By the faithful and irreducible action of the abelian group $A_G(H_{i+1}/H_i)$ on $H_{i+1}/H_i$, we see that $A_G(H_{i+1}/H_i)$ is cyclic and $m$ is the smallest positive integer such that $|A_G(H_{i+1}/H_i)|$ divides $p^m - 1$ by [9, II, Lemma 3.10], and thus $m \leq 2$ since the exponent of $A_G(H_{i+1}/H_i)$ divides $p^2 - 1$. Then $r_p(G) \leq 2$. □

3. **Proof of Theorem 1.4**

**Lemma 3.1.** Let $p$ be a prime, and let $P \in \text{Syl}_p(G)$, where $G$ is a group. If $P$ is a cyclic group, then either $G$ is $p$-supersolvable or else $P \cap O^p(G_p^n) = P$.

**Proof.** Without loss of generality, we assume $P \cap O^p(G_p^n) < P$. If $P \cap O^p(G_p^n) = 1$, then $G_p^n$ is $p$-nilpotent, and thus $G$ is $p$-supersolvable. So $1 < P \cap O^p(G_p^n) < P$, then it follows from [11, Theorem 2.1] that $G$ is $p$-supersolvable. □

**Proof of Theorem 1.4.** Note that $G$ is $p$-supersolvable if and only if $G_p^n$ is $p$-nilpotent, and so we only need to prove the sufficient. Now assume that $G$ is a counterexample of minimal order. Then $G$ is not $p$-supersolvable. In particular, $G_p^n$ is not $p$-nilpotent, and therefore $N = P \cap O^p(G_p^n) > 1$. For convenience, we write

$$\mathfrak{H} = \{H \trianglelefteq P \mid H \text{ is a non-cyclic subgroup with } |H| = d\}$$

and

$$\mathfrak{Y} = \{Y < P \mid N \not\trianglelefteq Y\}.$$ 

It is easy to see that $H \cap O^p(G_p^n) \leq G$ for all $H \in \mathfrak{H}$. We proceed in a number of steps to derive a contradiction.

**Step 1.** $P$ is not cyclic, dihedral, semidihedral or generalized quaternion.

If $P$ is cyclic, then, by Lemma 3.1, $G$ is $p$-supersolvable, a contradiction. Now assume that $P$ is dihedral, semidihedral or generalized quaternion. If $N$ is a cyclic subgroup, then it follows from Burnside’s theorem [9, IV, 2.8] and $p = 2$ that $O^p(G_p^n)$ is $2$-nilpotent, and thus $G_p^n$ is $2$-nilpotent, a contradiction. Thus, by Lemma 2.5, we may assume that $N$ is a non-cyclic maximal subgroup of $P$ and $|N| = d$. In this case $P = D_{2^n}$ ($n \geq 3$), $Q_{2^n}$ ($n \geq 4$) or $SD_{2^n}$, and thus there exists a non-cyclic maximal subgroup $N_1$ of $P$ such that $N \not\subseteq N_1$. For convenience, we write $M_1 = N \cap N_1$ and have

$$M_1 = N \cap O^p(G_p^n) \cap N_1 \cap O^p(G_p^n) \leq G.$$
Since \(|P : M_1|=2^2\), \(M_1\) is cyclic by Lemma 2.5. It follows that \(O^p(G_p^*)\) is 2-supersolvable and therefore \(O^p(G_p^*)\) is 2-nilpotent. Hence \(G_p^*\) is 2-nilpotent, a contradiction.

Step 2. \(\mathfrak{N} \neq \emptyset\).

Suppose not, that is, all normal subgroups of \(P\) with order \(d\) are cyclic. Now by Lemma 2.2, \(P\) is cyclic, dihedral, semidihedral or generalized quaternion, in contradiction to Step 1.

Step 3. \(O^p(G_p^*) = 1\).

Write \(D = O^p(G_p^*)\) and \(\overline{G} = G/D\), and note that \(O^p(\overline{G}_p^*) = \overline{O^p(G_p^*)}\) by [13, Lemma 2.9]. It follows from Dedekind’s lemma that \(O^p(G_p^*)\cap DH = D(O^p(G_p^*)\cap H)\) for \(H \in \mathfrak{N}\). In addition, both \(D\) and \(O^p(G_p^*)\cap H\) are normalized by \(O^p(G)\), we see that \(O^p(G)\) normalizes \(O^p(G_p^*)\cap D\), or equivalently, \(\overline{O^p(G)}\) normalizes \(\overline{O^p(G_p^*)}\cap \overline{H}\). Since \(PD\cap O^p(G_p^*) = D(P\cap U) = DN\) and \(|N| \leq d\), we see that \(|\mathcal{P} \cap \overline{O^p(G_p^*)}| \leq d\). Then \(\overline{G}\) satisfies the hypotheses, and therefore \(\overline{G}\) is \(p\)-supersolvable. It is clear that the subgroups of \(\overline{G}\) corresponding to the members of \(\mathfrak{N}\) are exactly the subgroups \(\overline{H}\) for \(H \in \mathfrak{N}\). Hence \(\overline{G}\) is \(p\)-supersolvable. Furthermore, we see that \(G\) is \(p\)-supersolvable, which is a contradiction. So we conclude that \(D = 1\).

Step 4. \(N\) is normal in \(G\). In fact, \(G\) is \(p\)-solvable and \(P \leq G\).

Since \(H \cap O^p(G_p^*) \leq O^p(G)\) for \(H \in \mathfrak{N}\) and \(O^p(G_p^*) \leq O^p(G)\), we see that \(H \cap O^p(G_p^*) \leq O^p(G_p^*)\) for \(H \in \mathfrak{N}\). Then \(G_p^*\) satisfies the hypotheses of [8, Theorem 3.2], and thus \(G_p^*\) is \(p\)-supersolvable. Hence \(G\) is \(p\)-solvable. Noticing that \(O^p(G_p^*) = 1\) and \(G_p^*\) is \(p\)-supersolvable, we have \(P \leq G_p^*\) by [9, VI, 6.6]. Then it follows from \(P \in \text{Syl}_p(G_p^*)\) and \(G_p^* \leq G\) that \(P\) is normal in \(G\). So \(N = P \cap O^p(G_p^*)\) is normal in \(G\) by \(O^p(G_p^*) \leq G\).

Step 5. There exists a maximal subgroup \(Y \in \mathcal{F}\) with \(L = N \cap Y\) is not normal in \(G\) and \(L\) is cyclic.

If \(N \leq \Phi(P)\), then it follows from Tate’s theorem [9, IV, 4.7] that \(O^p(G_p^*)\) is \(p\)-nilpotent, and therefore \(G_p^*\) is \(p\)-nilpotent, a contradiction. Thus there exists a maximal subgroup \(Y\) of \(P\) with \(N \nsubseteq Y\).

Next we prove that there exists \(Y \in \mathcal{F}\) such that \(L = N \cap Y\) is not normal in \(G\). If not, then \(L = N \cap Y\) is normal in \(G\) and \(|N : L| = p\) for all \(Y \in \mathcal{F}\). So \(G_p^* \leq C_G(N/L)\). Noticing that \(N/L\) is a normal Sylow \(p\)-subgroup of \(O^p(G_p^*)/L\), we see \(N/L \leq Z(O^p(G_p^*)/L)\), and therefore \(O^p(G_p^*)/L\) is \(p\)-nilpotent by Burnside’s theorem [9, IV, 2.6]. Hence \(O^p(O^p(G_p^*)) < O^p(G_p^*)\), a contradiction.

Finally, we prove that \(L\) is cyclic. If \(L\) is non-cyclic, then there exists \(H \in \mathcal{N}\) such that \(L < H \leq Y\). So \(L = H \cap L = H \cap Y \cap N = H \cap N = H \cap P \cap O^p(G_p^*) = H \cap O^p(G_p^*) \leq G\), which is a contradiction.

Step 6. \(Y\) is a cyclic, dihedral, semidihedral or generalized quaternion group.

Let \(Y\) and \(L\) be as in Step 5. If there exists a subgroup \(S\) in \(Y\) such that \(S \in \mathfrak{N}\), then, since \(|L| > |N| \leq d = |S|\), there exists \(H \in \mathfrak{N}\) such that \(L < H \leq LS \leq Y\) by Lemma 2.3. In this case, we have \(L = H \cap L = H \cap Y \cap N = H \cap N = H \cap P \cap O^p(G_p^*) = H \cap O^p(G_p^*) \leq G\), in contradiction to Step 5. So every normal subgroup of \(P\) that has order \(d\) is contained in \(Y\) is cyclic. By Lemma 2.2, the Step 6 is true.

Step 7. The final contradictions.

If \(Y\) is a cyclic maximal subgroup of \(P\), then it follows from [2, Lemma 2.1(b)] and Step 1 that \(O^p(G_p^*)\) acts trivially on \(P\), and therefore \(G_p^*\) is \(p\)-nilpotent, a contradiction. Now assume that \(Y\) is a dihedral, semidihedral or generalized quaternion group. If \(|Y| = d\), then \(Y \in \mathfrak{N}\), and therefore \(L = N \cap Y = P \cap O^p(G_p^*) \cap Y\) is normal in \(G\), in contradiction to Step 5. The remaining case is \(|Y| > d\). In this case, since \(Y\) is of maximal class, we see
that $|Y : Y'| = 2^2$. Furthermore, by $|Y| > d \geq |N|$ and $|N : L| = 2$, we see that $L \leq Y'$ by Lemma 2.5. It follows from Lemma 2.4 that $P'$ is cyclic, and therefore $L \leq Y' \leq P'$ is normal in $G$ by $P \unlhd G$, in contradiction to Step 5, which is the final contradiction. So the proof is complete.

Now we present some application of Theorem 1.4.

**Lemma 3.2.** Let $P \in \text{Syl}_p(G)$ with $|P| > p^3$. If $P$ has exactly one non-cyclic maximal subgroup $M$ and $M \trianglelefteq G_p$, then $G$ is $p$-supersolvable.

**Proof.** It is easy to see that the hypotheses are inherited by $G/O_{p'}(G_p^*)$ and $P^{G_p^*}$, so we can assume that $O_{p'}(G_p^*) = 1$. If $P^{G_p^*} < G$, then $P^{G_p^*}$ is $p$-supersolvable by induction. It follows from $O_p(G_p^*) = 1$ and [9, VI, 6.6] that $P$ is normal in $P^{G_p^*}$, and thus $P = P^{G_p^*}$. Since $G_p^* \leq G$ and $P \in \text{Syl}_p(G^*)$, we see that $P \unlhd G$. Noticing that there exists a cyclic maximal subgroup in $P$, we see, by [2, Lemma 2.1], that $O_p(G_p^*)$ acts trivially on $P$. Thus $G_p^*$ is $p$-nilpotent, and therefore $G$ is $p$-supersolvable. Now we can assume that $P^{G_p^*} = G$, and in particular, $G_p^* = G$. Then it follows from [8, Lemma 4.1] that $G$ is $p$-supersolvable.

**Lemma 3.3.** Let a Sylow $p$-subgroup $P$ of $G$ be a non-cyclic subgroup with $|P| > p^3$. If every non-cyclic maximal subgroup of $P$ is normal in $G_p^*$, then $G$ is $p$-supersolvable.

**Proof.** By Lemma 3.2, we can assume that $P$ has two distinct non-cyclic maximal subgroups. Then $P$ is normal in $G_p^*$. In addition, $G_p^*$ is normal in $G$ and $P \in \text{Syl}_p(G_p^*)$. Thus $P$ is normal in $G$. Since $|P| > p^3$, we see, by [2, Theorem A], that $O_p(G_p^*)$ acts trivially on $P$. Then $G_p^*$ is $p$-nilpotent, and therefore $G$ is $p$-supersolvable.

**Corollary 3.4.** Let $P$ be a non-cyclic Sylow $p$-subgroup of $G$ with $|P| > p^3$, and suppose for every non-cyclic maximal subgroup $H$ of $P$ that $H \cap O_p(G_p^*) \leq O_p(G)$. Then $G$ is $p$-supersolvable.

**Proof.** Assume that $G$ is not $p$-supersolvable. Applying Theorem 1.4 with $d = |P|/p$, we deduce that $O_p(G_p^*) = G_p^*$, and thus every non-cyclic maximal subgroup of $P$ is normal in $G_p^*$. It follows from Lemma 3.3 that $G$ is $p$-supersolvable, a contradiction.

**Corollary 3.5.** Let $p$ be an odd prime and $P \in \text{Syl}_p(G)$, where $P$ is non-cyclic. Let $d$ be a power of $p$ such that $p^2 \leq d < |P|$, and let $\mathfrak{H}$ be the set of all non-cyclic normal subgroups $H$ of $P$ with $|H| = d$. Assume that $H \cap O_p(G_p^*) \leq O_p(G)$ for all $H \in \mathfrak{H}$. If $N_G(H)$ is $p$-supersolvable for all $H \in \mathfrak{H}$, then $G$ is $p$-supersolvable.

**Proof.** If $|P \cap O_p(G_p^*)| \leq d$, then $G$ is $p$-supersolvable by Theorem 1.4. Now we can assume that $|P \cap O_p(G_p^*)| > d$. In this case, if there exists $H \in \mathfrak{H}$ such that $H \leq O_p(G_p^*)$, then $H \leq O_p(G)$, and thus $H \leq O^p(G) = G$. Hence $G = N_G(H)$ is $p$-supersolvable. Now we may assume that $N = P \cap O_p(G_p^*)$ is cyclic by Lemma 2.2. Let $L$ be a subgroup of $N$ with order $d/p$. Since $P$ is non-cyclic, there exists $H \in \mathfrak{H}$ such that $L \leq H$ by Lemma 2.2 and [8, Lemma 2.4], and thus $L = N \cap H$. Noticing that $L = N \cap H = H \cap O_p(G_p^*)$ is normal in $O_p(G)$, we have that $L$ is normal in $G$. It follows from [11, Theorem 2.1] that $O_p(G_p^*)$ is $p$-supersolvable, and therefore $N \leq O_p(G_p^*)$. In addition, $O_p(G_p^*)$ is normal in $G$ and $N \in \text{Syl}_p(O^p(G_p^*))$. Then $N$ is normal in $G$. Hence, by [2, lemma 2.1], $O_p(G_p^*)$ acts trivially on $N$. Furthermore, we see that $G_p^*$ is $p$-nilpotent and $G$ is $p$-supersolvable. The proof of the corollary is complete.

4. **Proof of Theorem 1.5**

**Lemma 4.1.** Let $G$ be a group of odd order and $P$ be a Sylow $p$-subgroup of $G$. If $P$ is a meta-cyclic group or 3-group of maximal class, then $l_p(G) \leq 1$ and $r_p(G) \leq 2$. 

Proof. We proceed by induction on $|G|$. It is easy to see that the hypotheses are inherited by $G/O_p'(G)$. So we assume that $O_p'(G) = 1$. Then $O_p(G) \neq 1$ since $G$ is $p$-solvable. By Lemma 2.7, we see that $O_p(G)$ is a 3-group of maximal class or $\Omega_1(O_p(G)) \leq C_p \times C_p$. If $O_p(G)$ is of maximal class, then $O_p(G)$ is a $CS(p, 2)$-group. Hence $O_p^*(G_{p^2})$ acts trivially on $O_p(G)$ by Lemma 2.10. It follows from Hall-Higman lemma [10, Theorem 3.21] that $O_p^*(G_{p^2}) \leq C_{C_p}O_p(G) \leq O_p(G)$, and thus $G_{p^2}^*$ is a $p$-group. Furthermore, by Lemma 2.11, $l_p(G) \leq 1$ and $r_p(G) \leq 2$. Now we assume that $\Omega_1(O_p(G)) \leq C_p \times C_p$. It follows from Lemma 2.10 that $O_p^*(G_{p^2})$ act trivially on $\Omega_1(O_p(G))$, and thus $O_p^*(G_{p^2})$ act trivial on $O_p(G)$ by [9, IV, 5.12]. So $l_p(G) \leq 1$ and $r_p(G) \leq 2$ by using the arguments above. □

Proof of Theorem 1.5. Suppose that $G$ is a counterexample of minimal order. Then $|P \cap O_p^*(G_{p^2})| \leq d$ and $l_p(G) \leq 1$ or $r_p(G) \leq 2$. By Lemma 2.11, we see that $G_{p^2}^*$ is not $p$-nilpotent, and $N = P \cap O_p^*(G_{p^2}) > 1$. For convenience, we write

$$\mathfrak{H} = \{ H \trianglelefteq P \mid H \text{ is a non-meta-cyclic subgroup with } |H| = d \}$$

and

$$\mathfrak{J} = \{ Y < P \mid N \not\subseteq Y \}.$$ 

It is easy to see $H \cap O_p^*(G_{p^2}) \leq G$ for all $H \in \mathfrak{H}_1$. We proceed in a number of steps to derive a contradiction.

Step 1. $O_p'(G) = 1$.

Write $D = O_p'(G)$ and $G = G/D$. We argue that $G$ satisfies the hypotheses of the theorem. The subgroups of $G$ corresponding to the members of $\mathfrak{H}_1$ are exactly the subgroups $\mathfrak{H}$ for $H \in \mathfrak{H}_1$, and since $O_p(G) = G_p(G)$ and $O_p^*(G_{p^2}) = G_{p^2}(G)$, we must show that $O_p(D)$ normalizes $O_p^*(G_{p^2}) \cap \mathfrak{H}$. On the other hand, $O_p(D) \cap \mathfrak{H} = (O_p^*(G_{p^2}) \cap H)/D$ by [13, Lemma 2.8]. Then $O_p^*(G_{p^2})$ normalizes $O_p(D) \cap H$. Since $D$ and $O_p^*(G_{p^2}) \cap H$ are normalized by $O_p(G)$, this shows that $G$ satisfies the hypotheses, as claimed.

If $D > 1$, then $l_p(G) \leq 1$ or $r_p(G) \leq 2$, and thus $|P \cap O_p^*(G_{p^2})| > d$ by the minimality of $G$. Hence $|PD \cap O_p^*(G_{p^2})| > d|D|$. Since $PD \cap U = D(P \cap O_p^*(G_{p^2})) = DN$, we see that $|N| > d$, which is a contradiction with $|N| \leq d$. So we conclude that $D = 1$.

Step 2. $d \geq p^3$.

If $d \leq p^2$, then $|N| \leq p^2$. Since $G$ is an odd order group, we see that $G$ is $p$-solvable. Then it follows from [9, VI, 6.6] that $l_p(O_p^*(G_{p^2})) \leq 1$. In addition, $O_p'(G_{p^2}) = 1$ since $O_p'(G) = 1$. Thus $N \leq O_p^*(G_{p^2})$, and therefore $N \leq G$. It follows that $O_p^*(G_{p^2})$ acts trivially on $N$ by Lemma 2.10, and so $O_p^*(G_{p^2})$ is $p$-nilpotent by Burnside’s theorem [9, IV, 2.6]. Hence $G_{p^2}^*$ is $p$-nilpotent, a contradiction.

Step 3. $\mathfrak{H}_1 \neq \emptyset$.

Suppose not, that is, all subgroups of $P$ with order $d$ are meta-cyclic. Now by Lemma 2.6, $P$ is a meta-cyclic group or a 3-group of maximal class. Then it follows from Step 2 and Lemma 4.1 that $l_p(G) \leq 1$ and $r_p(G) \leq 2$, a contradiction.

Step 4. $N$ is non-meta-cyclic and is normal in $G$.

Suppose that $N$ is meta-cyclic, that is, $N$ is a cyclic group or a meta-cyclic group with $d(N) = 2$, where $d(N)$ is a minimal number of generators of $N$. If $N$ is cyclic, and let $A$ be a subgroup of $N$ with order $p$, then $A$ is normal in $P$ by $N \leq P$, and therefore there exists $H \in \mathfrak{H}_1$ such that $A \leq H$ by Lemma 2.8 and $H \cap N \neq 1$. Hence, by $H \cap N = H \cap P \cap O_p^*(G_{p^2}) \leq G$ and [11, Theorem 2.1], $O_p^*(G_{p^2})$ is $p$-supersolvable. Furthermore, it follows from $O_p'(G) = 1$ and [9, VI, 6.6] that $N$ is normal in $O_p^*(G_{p^2})$. By Lemma 2.9 and 2.10, we see that $O_p^*(G_{p^2})$ centralizes $N$, and thus $O_p^*(G_{p^2})$ is $p$-nilpotent and $G_{p^2}^*$ is $p$-nilpotent, a contradiction. Now we assume that $N$ is a metacyclic subgroup of $P$ with $d(G) = 2$. Then $\Omega_1(N) \simeq C_p \times C_p$, and thus, by Lemma 2.8, there exists $H \in \mathfrak{H}_1$ such that $\Omega_1(N) \leq H$ and $H \cap N \neq 1$. Hence $T = H \cap N = H \cap P \cap O_p^*(G_{p^2}) \leq G$. Noticing
that $\Omega_1(N) = \Omega_1(T) \text{ char } T$, we have that $\Omega_1(N)$ is normal in $G$, and therefore $O^p(G^{*}_{p^2})$ centralizes $\Omega_1(N)$ by Lemma 2.10. Since $p$ is odd, we see that $O^p(G^{*}_{p^2})$ centralizes $N$ by [9, IV, 5.12]. Then $O^p(G^{*}_{p^2})$ is $p$-nilpotent by Burnside’s Theorem [9, IV, 2.6], and thus $G^{*}_{p^2}$ is $p$-nilpotent, a contradiction.

Hence $N$ is non-meta-cyclic, and thus there exists $H \in \mathcal{H}_1$ such that $N \subseteq H$. We see

$$N = N \cap H = O^p(G^{*}_{p^2}) \cap P \cap H \unlhd G.$$  

Step 5. There exists a maximal subgroup $Y \in \mathcal{Y}$ such that $N \nsubseteq Y$.

If $N \subseteq \Phi(P)$, then it follows from Tate's theorem [9, IV, 4.7] that $O^p(G^{*}_{p^2})$ is $p$-nilpotent, and therefore $G^{*}_{p^2}$ is $p$-nilpotent, a contradiction. Thus there exists a maximal subgroup $Y$ of $P$ with $N \nsubseteq Y$.

Step 6. For any $Y \in \mathcal{Y}$, $L = N \cap Y$ is not normal in $G$ and $L$ is meta-cyclic.

First, we prove that $L = N \cap Y$ is not normal in $G$ for any $Y \in \mathcal{Y}$. If not, then there exists $Y \in \mathcal{Y}$ such that $L = N \cap Y \subseteq G$. Since $|N : L| = p$ for all $Y \in \mathcal{Y}$, $G^{*}_{p^2} \leq C_G(N/L)$. In addition, $N/L$ is a normal Sylow $p$-subgroup of $O^p(G^{*}_{p^2})/L$, then $N/L \leq Z(O^p(G^{*}_{p^2})/L)$, and therefore $O^p(G^{*}_{p^2})/L$ is $p$-nilpotent by Burnside’s theorem [9, IV, 2.6]. Hence $O^p(O^p(G^{*}_{p^2})) < O^p(G^{*}_{p^2})$, a contradiction.

Next, we prove that $L$ is meta-cyclic. If $L$ is non-meta-cyclic, then there exists $H \in \mathcal{H}_1$ such that $L < H \leq Y$. So

$$L = H \cap L = H \cap Y \cap N = H \cap N = H \cap P \cap O^p(G^{*}_{p^2}) = H \cap O^p(G^{*}_{p^2}) \subseteq G,$$

which is a contradiction.

Step 7. $N \simeq C_p \times C_p \times C_p$.

If not, then, since $L$ is a meta-cyclic maximal subgroup of $N$, we see that $N$ is a $CS(p, 2)$-group by Lemma 2.9, and thus $N$ is centralized by $O^p(G^{*}_{p^2})$ by Lemma 2.10. Hence $G^{*}_{p^2}$ is $p$-nilpotent, a contradiction.

Step 8. The final contradiction.

It is easy to see that $G^{*}_{p^2}/N$ is $p$-nilpotent. If $N \leq \Phi(G)$, then $G^{*}_{p^2}$ is $p$-nilpotent, a contradiction. Hence there exists a maximal subgroup $M$ of $G$ such that $N \nsubseteq M$. It is easy to see that $N$ is a minimal normal subgroup of $G$. If not, there is nothing to be proved. Then $G = NM$ and $N \cap M = 1$. It follows that $P = N(P \cap M)$ by Dedekind’s lemma. For convenience, write $S = P \cap M$. Noticing that there exists a maximal subgroup $P_1$ of $P$ such that $S \leq P_1$ and $N \nsubseteq P_1$. Write $K = N \cap P_1$ is normal in $P$ and $K \simeq C_p \times C_p$ by Step 7. If there exists $H_1 \in \mathcal{H}_1$ such that $H_1 \leq P_1$, then, by Lemma 2.8, there exists $H \in \mathcal{H}_1$ such that $K \leq H \leq P_1$, and thus $K = N \cap P_1 \cap H = H \cap O^p(G^{*}_{p^2}) \leq G$, which contradicts Step 6. Then it follows from Lemma 2.6 and Lemma 4.1 that $P_1$ is a meta-cyclic group of $d(P_1) = 2$ or a $3$-group of maximal class. If $P_1$ is meta-cyclic of $d(P_1) = 2$, then $\Omega_1(P_1) \simeq C_p \times C_p$, and therefore $\Omega_1(S) \leq \Omega_1(P_1) = K \leq N$. In addition, we know that $\Omega_1(S) \leq S \leq M$ and $N \cap M = 1$. Then $\Omega_1(S) = 1$, and thus $S = 1$. Hence $N = P$, which is a contradiction with $|N| \leq d < |P|$. Now we assume that $P_1$ is a $3$-group of maximal class. Since $p^3 = |N| \leq d < |P|$, we see that $|P_1| \geq p^3$. If $|P_1| \geq 3^4$, then $K \leq \Phi(P_1)$ by [3, Exercise 9.1]. It follows from Dedekind’s lemma that $P_1 = (P_1 \cap N)S$ and $P_1 = S$, which is a contradiction with $P = NS > P$. Now we assume that $|P_1| = 3^3$ and $|P| = 3^4$. Then it follows from $p^3 = |N| \leq d < |P| = p^4$ that $d = p^3$. Hence $P_1 \in \mathcal{H}_1$. Furthermore, we see that $K = N \cap P_1 = P_1 \cap O^p(G^{*}_{p^2}) \leq G$, which is a contradiction with Step 6. This final contradiction completes the proof.  

Now we may present some applications of Theorem 1.5.

**Lemma 4.2.** Let $G$ be a group of odd order and $P \in \text{Syl}_p(G)$ with $|P| > p^4$. If $P$ has exactly one non-meta-cyclic maximal subgroup $M$ and $M \unlhd G$, then $t_p(G) \leq 1$ and $r_p(G) \leq 2$. 

Proof. We proceed by induction on $|G|$. It is clear that the hypotheses are inherited by $G/O_{p'}(G)$ and $P^G$, so we can assume that $O_{p'}(G) = 1$. If $P^G < G$, then $l_p(P^G) = 1$ by induction. In addition, $O_{p'}(G) = 1$, then $P$ is normal in $P^G$. Since $P^G \leq G$ and $P \in Syl_p(P^G)$, we see $P = P^G \leq G$. Notice that $P$ has a meta-cyclic maximal subgroup and $|P| > p^4$. Then $P$ is a $CS(p, 2)$-group by Lemma 2.11, and thus $P$ is centralized by $O^p(G^*_{p^2})$ by Lemma 2.10. Then it follows from Burnside’s theorem [9, IV, 2.6] that $O^p(G^*_{p^2})$ is $p$-nilpotent. Hence $G^*_{p^2}$ is $p$-nilpotent, and therefore $l_p(G) \leq 1$ and $r_p(G) \leq 2$ by Lemma 2.11.

Now we can assume that $P^G = G$, and in particular, $G^*_{p^2} = G$. Applying Theorem 1.5, we may assume that $d = |P|/p$ and $|P/O^p(G^*_{p^2})| > d$, and therefore $O^p(G^*_{p^2}) = G^*_{p^2}$. Since $M$ is the unique non-meta-cyclic maximal subgroup of $P$, we see that $M$ has a meta-cyclic maximal subgroup by [7, Lemma 2.3]. On the other hand, $M$ is a $CS(p, 2)$-group since $|M| > p^3$. Then, by Lemma 2.10, $O^p(G^*_{p^2}) = G$ acts trivially on $M$. Thus $P$ is abelian and $P \simeq C_{p^m} \times C_p \times C_{p}(m \geq 3)$. Let $N = N_G(P)$. We see that $N/P$ acts on the $P$ and centralizes $M$. It follows from Fitting’s lemma [10, Lemma 4.28] and $P \simeq C_{p^m} \times C_p \times C_p$ that the action of $N/P$ on $P$ is trivial, and therefore $P \leq Z(N)$. So $G$ is $p$-nilpotent by Burnside’s theorem [9, IV, 2.6], and thus $l_p(G) \leq 1$ and $r_p(G) \leq 2$ by Lemma 2.11. □

Lemma 4.3. Let $G$ be a group of odd order, and let $P$ be a Sylow $p$-subgroup of $G$ with $|P| > p^4$. If every non-meta-cyclic maximal subgroup of $P$ is normal in $G$, then $l_p(G) \leq 1$ and $r_p(G) \leq 2$.

Proof. We proceed by induction on $|G|$. It is easy to see that the hypotheses are inherited by $G/O_{p'}(G)$, so we can assume that $O_{p'}(G) = 1$. It follows from Lemma 2.6 and 4.1 that $P$ has a non-meta-cyclic maximal subgroup. By Lemma 4.2, we can assume that $P$ has two distinct non-meta-cyclic maximal subgroups, and therefore $P$ is normal in $G$. Since $|P| > p^4$, $O^p(G^*_{p^2})$ acts trivially on $P$ by [7, Theorem A]. Hence $G^*_{p^2}$ is $p$-nilpotent, and thus $l_p(G) \leq 1$ and $r_p(G) \leq 2$. □

Corollary 4.4. Let $G$ be an odd order group and $P$ be a Sylow $p$-subgroup of $G$ with $|P| > p^4$, and suppose for every non-cyclic maximal subgroup $H$ of $P$ that $H \cap U \leq U$, where $U = O^p(G)$. Then $l_p(G) \leq 1$ and $r_p(G) \leq 2$.

Proof. Applying Theorem 1.5 with $d = |P|/p$, we deduce that $O^p(G^*_{p^2}) = G^*_{p^2}$, and thus every non-cyclic maximal subgroup of $P$ is normal in $G$. It follows from Lemma 4.3 that $l_p(G) \leq 1$ and $r_p(G) \leq 2$, a contradiction. □

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