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Some results on dynamic discrimination measures of order (α, β)

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Abstract

In this paper we propose two measures of discrimination of order (α, β) for residual and past lifetimes. Lower and upper bounds of the proposed measures are derived. Some bounds are obtained by considering weighted distributions and subsequently, examples are presented. Finally, characterization results of the proportional hazards and proportional reversed hazards models are given.

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1. Introduction

Discrimination measures are often useful in many applications of probability theory in comparing two probability distributions. They have great importance in information theory, reliability theory, genetics, economics, approximations of probability distributions, signal processing and pattern recognition. Several divergence measures have been proposed for this purpose. Of these the most fundamental one is Kullback-Leibler [13]. Let X and Y be two absolutely continuous random variables (rv's) representing lifetimes of two units. Let $f(x)$, $F(x)$ and $\bar{F}(x)$, respectively be the probability density function (pdf), cumulative distribution function (cdf) and survival function (sf) of X; and the corresponding functions for Y be $g(x)$, $G(x)$ and $\bar{G}(x)$. Let us to take into account that the pdf's are differentiable in their common support. Denote $\eta_X(x) = f(x)/\bar{F}(x)$ and $\eta_Y(x) = g(x)/\bar{G}(x)$ as the hazard rate functions of X and Y, respectively; and $\xi_X(x) = f(x)/F(x)$ and $\xi_Y(x) = g(x)/G(x)$, as their reversed hazard rate functions.

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Kullback and Leibler's (KL) discrimination measure, known as relative entropy, between two probability distributions with pdf's $f(x)$ and $g(x)$ is given by

$$I_{X,Y}^{KL} = \int_0^\infty f(x) \ln \frac{f(x)}{g(x)} dx. \tag{1.1}$$

The discrimination measure (1.1) is not appropriate in reliability and life-testing studies as the current age of a system needs to be included. Ebrahimi and Kirmani [11] proposed KL discrimination measure between X and Y at time t (> 0) as

$$I_{X,Y}^{KL}(t) = \int_t^\infty \frac{f(x)}{\bar{F}(t)} \ln \frac{f(x)/\bar{F}(t)}{g(x)/\bar{G}(t)} dx. \tag{1.2}$$

The measure (1.2) is also known as relative entropy of residual lifetimes $X_t^+ = [X - t|X > t]$ and $Y_t^+ = [Y - t|Y > t]$. Residual lifetime is an important concept in biology. It is defined as the remaining time to an event given that the survival time X of a patient is at least t. In several clinical studies, particularly when the associated diseases are chronic or/and incurable, it is great concern to patients to know residual lifetime. However, it is reasonable to presume that in many realistic situations, the random lifetime variable is not necessarily related to the future but can also refer to the past. For example, consider a system which is working during a specified time interval and its state is observed only at certain pre-specified inspection times. Suppose the system is inspected for the first time and it is found to be down, then the uncertainty relies in the interval $(0, t)$, it has stopped working. Let X be the failure time of the system, then the variable of interest is $X_t^- = [t - X|X < t]$. It indeed measures the time elapsed from the failure of the component given that its lifetime is less than t. The random variable X_t^- is known as past lifetime of a system. Di Crescenzo and Longobardi [6] proposed a discrimination measure between past lifetimes $X_t^- = [t - X|X < t]$ and $Y_t^- = [t - Y|Y < t]$, which is given by

$$\bar{I}_{X,Y}^{KL}(t) = \int_0^t \frac{f(x)}{F(t)} \ln \frac{f(x)/F(t)}{g(x)/G(t)} dx. \tag{1.3}$$

It is clear that $I_{X,Y}^{KL}(t) = I_{X_t^+,Y_t^+}^{KL}$ and $\bar{I}_{X,Y}^{KL}(t) = I_{X_t^-,Y_t^-}^{KL}$. Discrimination measures are used to measure mutual information concerning two variables. The measures given in (1.2) and (1.3) are respectively useful to compare the residual and past lifetimes of two biological systems, say left or right kidneys. Several researchers have studied KL discrimination measure by including the current age. In this direction we refer to Asadi *et al.* [2], Di Crescenzo and Longobardi [7] and Ebrahimi and Kirmani [10, 11]. Later the discrimination measure (1.1) was generalized, called discrimination measure of order α, as

$$I_{X,Y}^{R} = \frac{1}{\alpha - 1} \ln \int_0^\infty f^\alpha(x) g^{1-\alpha}(x) dx, \tag{1.4}$$

where $\alpha > 0$ but $\neq 1$. Note that as α tends to 1, $I_{X,Y}^R$ reduces to $I_{X,Y}^{KL}$. As similar measure to (1.2), discrimination measure of order α between two rv's X and Y at time t can be defined by (see Asadi *et al.* [3])

$$I_{X,Y}^{R}(t) = \frac{1}{\alpha - 1} \ln \int_t^\infty \frac{f^\alpha(x)}{\bar{F}^\alpha(t)} \frac{g^{1-\alpha}(x)}{\bar{G}^{1-\alpha}(t)} dx. \tag{1.5}$$

In literature, it is also dubbed as the relative entropy of order α between X_t^+ and Y_t^+. Note that $I_{X,Y}^R(t) = I_{X_t^+,Y_t^+}^R$. Discrimination measure of order α between past lifetimes

X_t^- and Y_t^- is given by (see Asadi *et al.* [4])

$$\bar{I}^R_{X,Y}(t) = \frac{1}{\alpha-1}\ln\int_0^t \frac{f^\alpha(x)}{F^\alpha(t)}\frac{g^{1-\alpha}(x)}{G^{1-\alpha}(t)}dx. \tag{1.6}$$

Note that $\bar{I}^R_{X,Y}(t) = I^R_{X_t^-,Y_t^-}$. For more details we refer to Asadi *et al.* [3], Asadi *et al.* [4], Maya and Sunoj [14], Sunoj and Linu [18] and Sunoj and Sreejith [19]. Based on Varma's entropy (see Varma [20]) the discrimination measure of order α given in (1.4) can be further generalized as

$$I^V_{X,Y} = \frac{1}{\alpha-\beta}\ln\int_0^\infty f^\gamma(x)g^{1-\gamma}(x)dx, \tag{1.7}$$

where $\alpha \neq \beta,\ \beta \geq 1,\ \beta-1 < \alpha < \beta$ and $\gamma = \alpha+\beta-1 > 0$. We shall call it generalized discrimination measure of order (α,β), or discrimination measure of order (α,β). It is worthwhile noting that as β tends to 1, $I^V_{X,Y}$ reduces to $I^R_{X,Y}$, whereas $I^V_{X,Y}$ reduces to $I^{KL}_{X,Y}$, when both α and β tend to 1. In this paper we propose two new dynamic (time dependent) discrimination measures of order (α,β) similar to (1.5) and (1.6) with the following forms:

$$I^V_{X,Y}(t) = \frac{1}{\alpha-\beta}\ln\int_t^\infty \frac{f^\gamma(x)}{\bar{F}^\gamma(t)}\frac{g^{1-\gamma}(x)}{\bar{G}^{1-\gamma}(t)}dx \tag{1.8}$$

and

$$\bar{I}^V_{X,Y}(t) = \frac{1}{\alpha-\beta}\ln\int_0^t \frac{f^\gamma(x)}{F^\gamma(t)}\frac{g^{1-\gamma}(x)}{G^{1-\gamma}(t)}dx. \tag{1.9}$$

It is clear that $I^V_{X,Y}(t) = I^V_{X_t^+,Y_t^+}$ and $\bar{I}^V_{X,Y}(t) = \bar{I}^V_{X_t^-,Y_t^-}$. When β tends to 1, dynamic discrimination measures (1.8) and (1.9) reduce to (1.5) and (1.6), respectively. The dynamic discrimination measures (1.8) and (1.9), respectively reduces to (1.2) and (1.3) when both α and β tend to 1.

To overcome the difficulty of modeling non-experimental, non-replicated and non-random data set which usually occur in environmental and ecological studies, Rao [17] introduced the concept of weighted distributions. Let $f(x)$ be the pdf of X and $w(x)$ be a non-negative function with $\mu_w = E(w(X)) < \infty$. Also let $f_w(x),\ F_w(x)$ and $\bar{F}_w(x)$, respectively be the pdf, cdf and sf of a weighted rv X_w, where $f_w(x) = w(x)f(x)/\mu_w$, $F_w(x) = E(w(X)|X < t)F(x)/\mu_w$ and $\bar{F}_w(x) = E(w(X)|X > t)\bar{F}(x)/\mu_w$. We refer to Di Crescenzo and Longobardi [8], Gupta and Kirmani [12], Maya and Sunoj [14], Navarro *et al.* [15] and Navarro *et al.* [16] for various results and applications on weighted distributions.

Throughout this paper, the terms decreasing and increasing are used for non-increasing and non-decreasing, respectively.

1.1. Definition *Let X and Y be two rv's with pdf's $f(x)$ and $g(x)$, respectively. Then X is said to be less than or equal to Y in likelihood ratio ordering, denoted by $X \overset{lr}{\leq} Y$, if $f(t)/g(t)$ is decreasing in t.*

The rest of the paper is arranged as follows. In Section 2, we obtain some bounds of dynamic discrimination measure of order (α,β) between residual lifetimes. Furthermore a characterization result is stated for the proportional hazard rate models through this discrimination measure. Afterward, analogous results are given for the dynamic discrimination measure of order (α,β) between past lifetimes in Section 3.

2. Residual Lifetimes

In this section we consider dynamic discrimination measure of order (α, β) between two residual lifetimes given in (1.8) and obtain some bounds which are functions of the hazard rates and/or residual entropy of order (α, β). The residual entropy of order (α, β) of a rv X at time t is defined by

$$I_X^V(t) = \frac{1}{\beta - \alpha} \ln \int_t^\infty \frac{f^\gamma(x)}{\bar{F}^\gamma(t)} dx. \tag{2.1}$$

Note that as $\beta \to 1$, $I_X^V(t)$ reduces to residual entropy of order α (see Abraham and Sankaran [1]) and it reduces to residual entropy (see Ebrahimi [9]) when both α and β tend to 1. In the following theorem we obtain lower and upper bounds of $I_{X,Y}^V(t)$ which are functions of hazard rates.

2.1. Theorem *Let $X \overset{lr}{\leq} Y$. Then*

$$\begin{aligned} (i)\ I_{X,Y}^V(t) &\geq \frac{\gamma - 1}{\alpha - \beta} \ln\Big(\frac{\eta_X(t)}{\eta_Y(t)}\Big) \ if\ \gamma > 1,\ and \\ (ii)\ I_{X,Y}^V(t) &\leq \frac{\gamma - 1}{\alpha - \beta} \ln\Big(\frac{\eta_X(t)}{\eta_Y(t)}\Big) \ if\ \gamma < 1. \end{aligned}$$

Proof. (i) As $X \overset{lr}{\leq} Y$ and $x > t$, we have $f^{\gamma-1}(t)g^{1-\gamma}(t) \geq f^{\gamma-1}(x)g^{1-\gamma}(x)$ for $\gamma > 1$. Thus, from (1.8) we immediately observe that,

$$I_{X,Y}^V(t) \geq \frac{1}{\alpha - \beta} \ln\Big(\frac{f^{\gamma-1}(t)}{\bar{F}^{\gamma-1}(t)} \frac{\bar{G}^{\gamma-1}(t)}{g^{\gamma-1}(t)}\Big) = \frac{1}{\alpha - \beta} \ln\Big(\frac{\eta_X^{\gamma-1}(t)}{\eta_Y^{\gamma-1}(t)}\Big) = \frac{\gamma - 1}{\alpha - \beta} \ln\Big(\frac{\eta_X(t)}{\eta_Y(t)}\Big).$$

Moreover, the inequality in (ii) can be yielded similarly by using $f^{\gamma-1}(t)g^{1-\gamma}(t) \leq f^{\gamma-1}(x)g^{1-\gamma}(x)$ when $\gamma < 1$.
This completes the proof of the theorem. □

Again since $\eta_X(t)/\eta_{X_w}(t) = E(w(X)|X > t)/w(t)$, Theorem 2.1. leads to the following corollary.

2.1. Corollary *Let $X \overset{lr}{\leq} X_w$. Then*

$$\begin{aligned} (i)\ I_{X,X_w}^V(t) &\geq \frac{\gamma - 1}{\alpha - \beta} \ln\Big(\frac{E(w(X)|X > t)}{w(t)}\Big) \ if\ \gamma > 1,\ and \\ (ii)\ I_{X,X_w}^V(t) &\leq \frac{\gamma - 1}{\alpha - \beta} \ln\Big(\frac{E(w(X)|X > t)}{w(t)}\Big) \ if\ \gamma < 1. \end{aligned}$$

We consider the following example as an application of the Corollary 2.1.

2.1. Example *Let X be a rv following Pareto distribution with pdf*

$$f(x|a, b) = \frac{ab^a}{x^{a+1}},\ x > b > 0,\ a > 1.$$

Consider the weight function $w(x) = x$. Here $X \overset{lr}{\leq} X_w$, because the expression $f_w(x)/f(x) = ((a-1)/ab)x$ is an increasing function in x for $a > 1$. The dynamic discrimination measure of order (α, β) between X and X_w can be obtained by

$$\begin{aligned} I_{X,X_w}^V(t) &= \frac{\gamma - 1}{\alpha - \beta} \ln\Big(\frac{a}{a-1}\Big) + \frac{1}{\alpha - \beta} \ln\Big(\frac{a}{\gamma + a - 1}\Big) \\ &= \frac{\gamma - 1}{\alpha - \beta} \ln\Big(\frac{E(w(X)|X > t)}{w(t)}\Big) + \frac{1}{\alpha - \beta} \ln\Big(\frac{a}{\gamma + a - 1}\Big). \end{aligned} \tag{2.2}$$

Therefore from (2.2), Corollary 2.1. can be verified.

In the following theorem we present upper and lower bounds for $I^V_{X,Y}(t)$, which are the functions of the hazard rate and residual entropy of order (α, β) given in (2.1).

2.2. Theorem *Let $g(x)$ be a decreasing function in x. Then*

$$(i)\ I^V_{X,Y}(t) \leq -I^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln(\eta_Y(t))\ if\ \gamma > 1,\ and$$
$$(ii)\ I^V_{X,Y}(t) \geq -I^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln(\eta_Y(t))\ if\ \gamma < 1.$$

Proof: The proof is straightforward. Hence omitted. □

With reference to this fact that the hazard rate function can be written as $\eta_{X_w}(t) = (w(t)\eta_X(t))/E(w(X)|X > t)$, the next corollary follows as a direct consequence of the Theorem 2.2.

2.2. Corollary *Let $f_w(x)$ be a decreasing function in x. Then*

$$(i)\ I^V_{X,X_w}(t) \leq -I^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln\left(\frac{w(t)\eta_X(t)}{E(w(X)|X>t)}\right)\ if\ \gamma > 1,\ and$$
$$(ii)\ I^V_{X,X_w}(t) \geq -I^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln\left(\frac{w(t)\eta_X(t)}{E(w(X)|X>t)}\right)\ if\ \gamma < 1.$$

The following example illustrates the Corollary 2.2.

2.2. Example *Consider the rv X and the weighted rv X_w as described in Example* 2.1. *Also $f_w(x)$ is decreasing in x. The dynamic discrimination measure of order (α, β), obtained in Example* 2.1. *can be written as*

$$I^V_{X,X_w}(t) = -I^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln\left(\frac{w(t)\eta_X(t)}{E(w(X)|X>t)}\right) + \frac{1}{\alpha-\beta}\ln\left(\frac{\gamma-1+a\gamma}{\gamma-1+a}\right), \tag{2.3}$$

provided $\gamma-1+a\gamma > 0$. From (2.3) *we easily obtain the inequalities given in the Corollary 2.2.*

In the next result, we consider three rv's X_1, X_2 and X_3, and obtain a lower bound of $I^V_{X_1,X_3}(t) - I^V_{X_2,X_3}(t)$.

2.3. Theorem *Let X_1, X_2, X_3 be three rv's with pdf's $f_1(x)$, $f_2(x)$, $f_3(x)$; sf's $\bar{F}_1(x)$, $\bar{F}_2(x)$, $\bar{F}_3(x)$ and hazard rate functions $\eta_{X_1}(x)$, $\eta_{X_2}(x)$, $\eta_{X_3}(x)$, respectively. Also let $X_1 \overset{lr}{\leq} X_2$. Then the inequality*

$$I^V_{X_1,X_3}(t) - I^V_{X_2,X_3}(t) \geq \frac{\gamma}{\alpha-\beta}\ln\left(\frac{\eta_{X_1}(t)}{\eta_{X_2}(t)}\right)$$

holds for $\gamma > 0$.

Proof. Given $X_1 \overset{lr}{\leq} X_2$. Therefore, $f_2(x)/f_1(x)$ is an increasing function in x. Thus from (1.8), we get

$$I^V_{X_1,X_3}(t) \geq \frac{1}{\alpha-\beta}\ln\int_t^\infty \frac{f_2^\gamma(x)f_1^\gamma(t)}{f_2^\gamma(t)\bar{F}_1^\gamma(t)}\frac{f_3^{1-\gamma}(x)}{\bar{F}_3^{1-\gamma}(t)}dx,$$

which leads to the required inequality. □

2.1. Remark *Let X_1, X_2 and X_3 be three rv's as described in the Theorem 2.3. with $X_2 \overset{lr}{\leq} X_3$. Then*

$$\begin{aligned}(i)\ I^V_{X_1,X_2}(t) - I^V_{X_1,X_3}(t) &\leq -\frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{\eta_{X_2}(t)}{\eta_{X_3}(t)}\Big) \text{ if } \gamma>1, \text{ and}\\ (ii)\ I^V_{X_1,X_2}(t) - I^V_{X_1,X_3}(t) &\geq -\frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{\eta_{X_2}(t)}{\eta_{X_3}(t)}\Big) \text{ if } \gamma<1.\end{aligned}$$

In the following we shall here derive examples to verify the inequalities stated in the Theorem 2.3. and Remark 2.1.

2.3. Example *Let X_1 and X_2 be two independent rv's following exponential distributions with means $1/\sigma_1$ and $1/\sigma_2$, respectively, where $\sigma_1, \sigma_2 > 0$ and $\sigma_1 > \sigma_2$. It is easy to verify that $X_1 \overset{lr}{\leq} X_2$. With further assumption, $X_3 = \min(X_1, X_2)$, it can be written*

$$(2.4)\ I^V_{X_1,X_3}(t) - I^V_{X_2,X_3}(t) = \frac{\gamma}{\alpha-\beta}\ln\Big(\frac{\eta_{X_1}(t)}{\eta_{X_2}(t)}\Big) + \frac{1}{\alpha-\beta}\ln\Big(\frac{\sigma_1+\sigma_2-\sigma_1\gamma}{\sigma_1+\sigma_2-\sigma_2\gamma}\Big),$$

provided $\sigma_1+\sigma_2-\sigma_1\gamma > 0$ and $\sigma_1+\sigma_2-\sigma_2\gamma > 0$. From (2.4) we get

$$I^V_{X_1,X_3}(t) - I^V_{X_2,X_3}(t) \geq \frac{\gamma}{\alpha-\beta}\ln\Big(\frac{\eta_{X_1}(t)}{\eta_{X_2}(t)}\Big).$$

Hence, the Theorem 2.3. is verified.

2.4. Example *Let X_2 and X_3 be two independent rv's with pdf's $f_2(x|a_2,b_2) = a_2 b_2^{a_2}/x^{a_2+1}, x > b_2 > 0, a_2 > 0$ and $f_3(x|a_3,b_3) = a_3 b_3^{a_3}/x^{a_3+1}$, $x > b_3 > 0$, $a_3 > 0$, respectively, where $b_2 > b_3$. It can be shown that $X_2 \overset{lr}{\leq} X_3$. Moreover, consider another rv $X_1 = \min(X_2, X_3)$. Then*

$$I^V_{X_1,X_2}(t) - I^V_{X_1,X_3}(t) = -\frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{\eta_{X_2}(t)}{\eta_{X_3}(t)}\Big) + \frac{1}{\alpha-\beta}\ln\Big(\frac{a_2\gamma+a_3}{a_3\gamma+a_2}\Big).$$

Hence the inequalities given in Remark 2.1. follow.

Proportional hazards rate model was introduced by Cox in 1972 in order to estimate the effects of different covariates influencing the times to the failures of a system. Since then this model is extensively used in biomedical applications and reliability engineering. We refer to Cox and Oakes [5] for various applications of this model. In the following we obtain a characterization result of the proportional hazard rates models through the dynamic discrimination measure of order (α, β) given in (1.8). Assume that the survival functions of the rv's X and Y are related by

$$\bar{F}(t) = (\bar{G}(t))^{\theta},\ t > 0, \tag{2.5}$$

where $\theta > 0$ is called proportionality constant.

2.4. Theorem *The dynamic discrimination measure $I^V_{X,Y}(t)$ is independent of t, for $\gamma\theta - \gamma + 1 > 0$, if and only if $F(x)$ and $G(x)$ have proportional hazard rate models.*

Proof. Assume that $F(x)$ and $G(x)$ have proportional hazard rate models, that is, (2.5) holds. Thus using (2.5) in (1.8) we obtain

$$I^V_{X,Y}(t) = \frac{1}{\alpha-\beta}\ln\Big(\frac{\theta^{\gamma}}{\theta\gamma-\gamma+1}\Big), \tag{2.6}$$

provided $\theta\gamma - \gamma + 1 > 0$. Note that (2.6) is free from t. Next we assume that $I^V_{X,Y}(t) = c_1$, where c_1 is a non-zero constant free from t. Therefore, we have

$$\int_t^\infty \frac{f^\gamma(x)}{\bar{F}^\gamma(t)} \frac{g^{1-\gamma}(x)}{\bar{G}^{1-\gamma}(t)} dx = exp\{(\alpha - \beta)c_1\} = c_2(\neq 1), \ say. \tag{2.7}$$

Differentiating (2.7) with respect to t, we get

$$\gamma\phi^{\gamma-1}(t) + (1-\gamma)\phi^\gamma(t) = c_2^{-1}, \tag{2.8}$$

where $\phi(t) = \eta_Y(t)/\eta_X(t)$. We also assume that $\phi(t)$ is a differentiable function. By differentiating from (2.8) with respect to t, we compute

$$\gamma(\gamma-1)\phi'(t)\phi^{\gamma-2}(t)[1-\phi(t)] = 0, \tag{2.9}$$

where $\phi'(t) = \frac{d\phi}{dt}$. Therefore, from (2.9), either $\phi'(t) = 0$, or $\phi(t) = 1$, since $\gamma \neq 1$ and $\phi(t) \neq 0$. Note that $\phi(t) = 1$ implies $f(x) = g(x)$, which leads to $c_1 = 0$. But it is assumed that $c_1 \neq 0$. Hence, $\phi(t) = 1$ is not a feasible choice. Thus we have $\phi'(t) = 0$, that is, there exists a constant $\theta(> 0)$ such that $\eta_F(t) = \theta\eta_G(t)$.
This completes the proof of the theorem. □

2.5. Example *We consider a series system of n components with lifetimes X_i, $i = 1, \ldots, n$, which are identically, independently distributed having exponential distribution with mean lifetime $1/\sigma$. The lifetime of the system is $Z = \min(X_1, \ldots, X_n)$. It is easy to see that $\bar{F}_Z(x) = (\bar{F}_{X_i}(x))^n$, that is, Z and X_i satisfy the proportional hazard rates models. Here by using (2.6), $I^V_{Z,X_i}(t)$ can be obtained as*

$$I^V_{Z,X_i}(t) = \frac{1}{\alpha-\beta} \ln\Big(\frac{n^\gamma}{n\gamma - \gamma + 1}\Big),$$

which is independent of t. Conversely, assuming

$$I^V_{Z,X_i}(t) = \frac{1}{\alpha-\beta} \ln \int_t^\infty \frac{f_Z^\gamma(x)}{\bar{F}_Z^\gamma(t)} \frac{f_{X_i}^{1-\gamma}(x)}{\bar{F}_{X_i}^{1-\gamma}(t)} dx = constant$$

and along the lines (Equation 2.7. onwards) of the proof of the Theorem 2.4. it can be shown that $\bar{F}_Z(x) = (\bar{F}_{X_i}(x))^n$.

3. Past Lifetimes

Due to duality it is natural to study the dynamic discrimination measure of order (α, β) between past lifetimes given in (1.9). In this section we derive some of its bounds which are functions of the reversed hazard rates and/or past entropy of order (α, β). Note that proofs of the theorems stated for past lifetime case have analogous methodology with the residual lifetime case, hence they are omitted. The past entropy of order (α, β) of a rv X at time t is given by

$$\bar{I}^V_X(t) = \frac{1}{\beta-\alpha} \ln \int_0^t \frac{f^\gamma(x)}{F^\gamma(t)} dx. \tag{3.1}$$

We have the following theorem regarding upper and lower bounds of $\bar{I}^V_{X,Y}(t)$, which are functions of reversed hazard rates.

3.1. Theorem *Let $X \overset{lr}{\leq} Y$. Then*

$$(i)\ \bar{I}^V_{X,Y}(t) \leq \frac{\gamma-1}{\alpha-\beta} \ln\Big(\frac{\xi_X(t)}{\xi_Y(t)}\Big) \ if\ \gamma > 1,\ and$$

$$(ii)\ \bar{I}^V_{X,Y}(t) \geq \frac{\gamma-1}{\alpha-\beta} \ln\Big(\frac{\xi_X(t)}{\xi_Y(t)}\Big) \ if\ \gamma < 1.$$

Note that $\xi_X(t)/\xi_{X_w}(t) = E(w(X)|X < t)/w(t)$. An immediate corollary of this theorem is the following, which, in the weighted rv case can be useful result.

3.1. Corollary *Let* $X \overset{lr}{\leq} X_w$. *Then*

$$(i)\ \bar{I}^V_{X,X_w}(t) \leq \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{E(w(X)|X<t)}{w(t)}\Big)\ \textit{if}\ \gamma>1,\ \textit{and}$$
$$(ii)\ \bar{I}^V_{X,X_w}(t) \geq \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{E(w(X)|X<t)}{w(t)}\Big)\ \textit{if}\ \gamma<1.$$

The next example describes the results stated in the Corollary 3.1.

3.1. Example *For a rv X with pdf*

$$f(x|a) = ax^{a-1}, \quad 0<x<1, \quad a>0. \tag{3.2}$$

Consider the weight function $w(x) = x^b,\ b>0$. *The pdf of* X_w *can be obtained as*

$$f_w(x) = (b+a)x^{b+a-1}, \quad 0<x<1.$$

Therefore, it can be checked that $X \overset{lr}{\leq} X_w$. *Now the expression of* $\bar{I}^V_{X,X_w}(t)$ *is computed by*

$$\bar{I}^V_{X,X_w}(t) = \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{a}{b+a}\Big) + \frac{1}{\alpha-\beta}\ln\Big(\frac{a}{a-b\gamma+b}\Big), \tag{3.3}$$

where $a - b\gamma + b > 0$. *Thus, from* (3.3) *we can easily obtain the inequalities given in the Corollary 3.1.*

In the following result we obtain upper and lower bounds of $\bar{I}^V_{X,Y}(t)$, which are functions of the reversed hazard rate as well as past entropy of order (α,β).

3.2. Theorem *Let* $g(x)$ *be an increasing function in* x. *Then*

$$(i)\ \bar{I}^V_{X,Y}(t) \leq -\bar{I}^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln(\xi_Y(t))\ \textit{if}\ \gamma>1,\ \textit{and}$$
$$(ii)\ \bar{I}^V_{X,Y}(t) \geq -\bar{I}^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln(\xi_Y(t))\ \textit{if}\ \gamma<1.$$

The Theorem 3.2. leads to the following corollary as, $\xi_{X_w}(t) = w(t)\xi_X(t)/E(w(X)|X < t)$.

3.2. Corollary *Let* $f_w(x)$ *be increasing in* x. *Then*

$$(i)\ \bar{I}^V_{X,X_w}(t) \leq -\bar{I}^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{w(t)\xi_X(t)}{E(w(X)|X<t)}\Big)\ \textit{if}\ \gamma>1,\ \textit{and}$$
$$(ii)\ \bar{I}^V_{X,X_w}(t) \geq -\bar{I}^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{w(t)\xi_X(t)}{E(w(X)|X<t)}\Big)\ \textit{if}\ \gamma<1.$$

In this part of paper we state the following example to illustrate the Corollary 3.2.

3.2. Example *Let X be a rv with pdf given by* (3.2). *Consider weight function* $w(x) = x$. *Then*

$$\bar{I}^V_{X,X_w}(t) = -\bar{I}^V_X(t) - \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{w(t)\xi_X(t)}{E(w(X)|X<t)}\Big) + \frac{1}{\alpha-\beta}\ln\Big(\frac{a\gamma-\gamma+1}{a-\gamma+1}\Big),$$

provided $a\gamma - \gamma + 1 > 0$ *and* $a - \gamma + 1 > 0$. *Hence, the results in Corollary* 3.2. *follow.*

Furthermore, we consider three rv's $X_1,\ X_2$ and X_3 in the following theorem and obtain an upper bound of $\bar{I}^V_{X_1,X_3}(t) - \bar{I}^V_{X_2,X_3}(t)$.

3.3. Theorem *Let there be three rv's $X_1,\ X_2,\ X_3$ with pdf's $f_1(x),\ f_2(x),\ f_3(x)$; cdf's $F_1(x),\ F_2(x),\ F_3(x)$ and reversed hazard rate functions $\xi_{X_1}(x),\ \xi_{X_2}(x),\ \xi_{X_3}(x)$, respectively. Also let $X_1 \overset{lr}{\leq} X_2$. Then for $\gamma > 0$,*

$$\bar{I}^V_{X_1,X_3}(t) - \bar{I}^V_{X_2,X_3}(t) \leq \frac{\gamma}{\alpha-\beta}\ln\left(\frac{\xi_{X_1}(t)}{\xi_{X_2}(t)}\right).$$

3.1. Remark *Consider three rv's $X_1,\ X_2$ and X_3 as described in Theorem 3.3. and $X_2 \overset{lr}{\leq} X_3$. Then*

$$\begin{aligned}(i)\ \bar{I}^V_{X_1,X_2}(t) - \bar{I}^V_{X_1,X_3}(t) &\geq -\frac{\gamma-1}{\alpha-\beta}\ln\left(\frac{\xi_{X_2}(t)}{\xi_{X_3}(t)}\right)\ \text{if } \gamma > 1, \text{ and}\\ (ii)\ \bar{I}^V_{X_1,X_2}(t) - \bar{I}^V_{X_1,X_3}(t) &\leq -\frac{\gamma-1}{\alpha-\beta}\ln\left(\frac{\xi_{X_2}(t)}{\xi_{X_3}(t)}\right)\ \text{if } \gamma < 1.\end{aligned}$$

As an application of the Theorem 3.3 and Remark 3.1, the upcoming example is presented

3.3. Example *Let X_1 and X_2 be two independent rv's with pdf's*

$$f_1(x|a_1) = a_1 x^{a_1-1},\quad 0 < x < 1,\quad a_1 > 0$$

and

$$f_2(x|a_2) = a_2 x^{a_2-1},\quad 0 < x < 1,\quad a_2 > 0,$$

where $a_1 < a_2$. It can be shown that $X_1 \overset{lr}{\leq} X_2$. Consider another rv $X_3 = \max(X_1, X_2)$. Then the inequality of the Theorem 3.3. is provided as,

$$\bar{I}^V_{X_1,X_3}(t) - \bar{I}^V_{X_2,X_3}(t) = \frac{\gamma}{\alpha-\beta}\ln\left(\frac{\xi_{X_1}(t)}{\xi_{X_2}(t)}\right) + \frac{1}{\alpha-\beta}\ln\left(\frac{a_1+a_2-a_1\gamma}{a_1+a_2-a_2\gamma}\right),$$

where $a_1 + a_2 - a_1\gamma > 0$ and $a_1 + a_2 - a_2\gamma > 0$.

3.4. Example *Let X_2 and X_3 be two independent rv's with pdf's*

$$f_2(x|a_2) = a_2 x^{a_2-1},\quad 0 < x < 1,\quad a_2 > 0$$

and

$$f_3(x|a_3) = a_3 x^{a_3-1},\quad 0 < x < 1,\quad a_3 > 0,$$

where $a_2 < a_3$. It is easy to see that $X_2 \overset{lr}{\leq} X_3$. Consider another rv $X_1 = \max(X_2, X_3)$. Then

$$\bar{I}^V_{X_1,X_2}(t) = \bar{I}^V_{X_1,X_3}(t) + \frac{1-\gamma}{\alpha-\beta}\ln\left(\frac{\xi_{X_2}(t)}{\xi_{X_3}(t)}\right) + \frac{1}{\alpha-\beta}\ln\left(\frac{a_2\gamma+a_3}{a_3\gamma+a_2}\right). \tag{3.4}$$

From (3.4), Remark 3.1. can be verified.

We now conclude this article by presenting a characterization result of proportional reversed hazard rates models through the dynamic discrimination measure of order (α, β) given in (1.9). Suppose the cdf's of two rv's X and Y satisfy the following relation:

$$F(t) = (G(t))^{\theta},\ t > 0, \tag{3.5}$$

where $\theta > 0$.

3.5. Theorem *The dynamic past discrimination measure of order (α, β) $\bar{I}^V_{X,Y}(t)$ is independent of t, for $\gamma\theta - \gamma + 1 > 0$, if and only if $F(x)$ and $G(x)$ have proportional reversed hazard rates models.*

It is worthwhile to mention that if we consider a parallel system of n components instead of series system in Example 2.5 the result in the theorem can be verified.

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