

Compactness and local compactness of the proximal hyperspace

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Abstract

Compactness and local compactness of the hyperspace endowed with both the Vietoris topology and the Hausdorff metric topology, have been characterized by Costantini, Levi and Pelant. Our aim is to characterize these two properties for the proximal topology, which is related to both of the previous topologies.

Keywords: Hyperspace, Proximal Topology, Local Compactness, Kuratowski Convergence

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1. Introduction

The first one to characterize compactness of the Vietoris topology on the hyperspace $CL(X)$ of non-empty closed subsets of a topological space X , was Michael in [3]. He also gave a result about local compactness, but it was not correct as remarked in the paper [2]. In that paper, Costantini, Levi and Pelant studied compactness and local compactness of several hyperspace topologies. In particular they characterized compactness and local compactness of $CL(X)$ endowed both with the Vietoris topology τ_V and the Hausdorff metric topology τ_{H_d} .

Following the same spirit and using a similar technique, we characterize compactness and local compactness of $CL(X)$ endowed with the proximal topology $\tau_{\delta(d)}$. We show that both properties are equivalent to compactness of X . The choice of $\tau_{\delta(d)}$ is motivated by the fact that it is deeply connected both to τ_V and τ_{H_d} , because it can be obtained as supremum of the lower Vietoris topology and the upper Hausdorff metric topology, i.e. $\tau_{\delta(d)} = \tau_V^- \vee \tau_{H_d}^+$.

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2. Preliminaries

Let X be a metrizable space. Given a metric d on X , we denote by D_d the gap between two non-empty closed sets $E, F \in \text{CL}(X)$, defined as:

$$D_d(E, F) = \inf_{x \in E} \inf_{y \in F} d(x, y).$$

Let $x \in X$, we denote by $B_\varepsilon(x)$ the open ball of radius ε and center x . Given $A \in \text{CL}(X)$, we denote by $B_\varepsilon[A]$ the ε -expansion of A , i.e. $B_\varepsilon[A] = \bigcup_{a \in A} B_\varepsilon(a)$. It is easy to check that $B_{\frac{\varepsilon}{2}}[B_{\frac{\varepsilon}{2}}[A]] \subseteq B_\varepsilon[A]$ and $B_{\frac{\varepsilon}{3}}[\overline{B_{\frac{\varepsilon}{3}}[A]}] \subseteq B_\varepsilon[A]$ for every $\varepsilon > 0$ and every $A \in \text{CL}(X)$.

Recall that the Vietoris topology is $\tau_V = \tau_V^- \vee \tau_V^+$, where τ_V^- and τ_V^+ are generated respectively by the collection of all $V^- = \{F \in \text{CL}(X) \mid F \cap V \neq \emptyset\}$ and $V^+ = \{F \in \text{CL}(X) \mid F \subseteq W\}$, when V and W run over all the open subsets of X . The Hausdorff metric topology τ_{H_d} is generated by the Hausdorff distance on $\text{CL}(X)$ induced by d (see for instance [1]). A base for $\tau_{H_d}^+$ is constituted by the collection of all $W^{++} = \{F \in \text{CL}(X) \mid D_d(F, X \setminus W) > 0\} = \{F \in \text{CL}(X) \mid \exists \varepsilon > 0 : B_\varepsilon[F] \subseteq W\}$, when W runs through the open subsets of X . As recalled before, the proximal topology $\tau_{\delta(d)}$ is the supremum of the lower Vietoris topology and the upper Hausdorff metric topology, i.e. $\tau_{\delta(d)} = \tau_V^- \vee \tau_{H_d}^+$.

Recall that a net $(C_i)_{i \in I}$ is convergent to C with respect to the Kuratowski convergence if, and only if, it converges with respect to τ_V^- and $C \supseteq \text{Ls}_{i \in I} C_i$ where, denoted by $\mathfrak{U}(x)$ the collection of open neighbourhoods of x ,

$$\text{Ls}_{i \in I} C_i = \{x \in X \mid \forall V \in \mathfrak{U}(x) \forall i \in I \exists j_i \succeq i : V \cap C_{j_i} \neq \emptyset\} = \bigcap_{i \in I} \overline{\bigcup_{j \succeq i} C_j}.$$

Given a metrizable space X , we denote by $\mathfrak{M}(X)$ the set of all compatible metrics on X .

3. The main result

In the sequel, in the definition of compactness and local compactness we require the space also to be Hausdorff.

Our main result is the following theorem.

3.1. Theorem. *Let X be a metrizable space, let $C \in \text{CL}(X)$ and $d \in \mathfrak{M}(X)$. Then $\text{CL}(X)$ is $\tau_{\delta(d)}$ -locally compact at C if, and only if, there exists $\varepsilon > 0$ such that $\overline{B_\varepsilon[C]}$ is compact.*

As a consequence we can characterize both local compactness and compactness of X , using conditions on $\text{CL}(X)$. Moreover we can also characterize compactness of $(\text{CL}(X), \tau_{\delta(d)})$, showing that it is equivalent to local compactness of $(\text{CL}(X), \tau_{\delta(d)})$.

3.2. Corollary. *Let X be a metrizable space. X is locally compact if, and only if, $\text{CL}(X)$ is $\tau_{\delta(d)}$ -locally compact at $\{x\}$, for every $x \in X$.*

3.3. Corollary. *Let X be a metrizable space. The following are equivalent:*

- (1) X is compact;
- (2) $(\text{CL}(X), \tau_{\delta(d)})$ is locally compact;
- (3) $(\text{CL}(X), \tau_{\delta(d)})$ is locally compact at X .

Proof. The implications (1) \Rightarrow (2) and (3) \Rightarrow (1) follow from Theorem 3.1, while (2) \Rightarrow (3) is obvious. ■

The following result was proved in [2, Theorem 11].

3.4. Theorem. *Let X be a regular space and let $C \in \text{CL}(X)$. Then $\text{CL}(X)$ is τ_V -locally compact at C if, and only if, there exists an open set $A \subseteq X$ such that $C \subseteq A$ and \overline{A} is compact.*

Combining the previous result and our Theorem 3.1, we obtain as a consequence the equivalence of local compactness and compactness of proximal and Vietoris topologies.

3.5. Corollary. *Let X be a metrizable space. The following are equivalent:*

- (1) X is compact;
- (2) $(\text{CL}(X), \tau_{\delta(d)})$ is compact;
- (3) $(\text{CL}(X), \tau_V)$ is compact;
- (4) $(\text{CL}(X), \tau_{\delta(d)})$ is locally compact;
- (5) $(\text{CL}(X), \tau_V)$ is locally compact;

Proof. The equivalence between (1) \Leftrightarrow (3) \Leftrightarrow (5) has been proved in [2, Corollary 13]. If X is compact, then $(\text{CL}(X), \tau_{\delta(d)})$ is compact since $\tau_{\delta(d)} = \tau_V$ and this proves the implication (1) \Rightarrow (2). The implication (2) \Rightarrow (4) is obvious. Finally the equivalence (1) \Leftrightarrow (4) follows from Corollary 3.3. ■

3.6. Remark. Note that (4) \Rightarrow (5) can be easily proved in a direct way, in order to explicitly use the condition that characterize local compactness of $\tau_{\delta(d)}$ and τ_V . Indeed, if $(\text{CL}(X), \tau_{\delta(d)})$ is locally compact at C , then by Theorem 3.1 there exists $\varepsilon > 0$ such that $\overline{B_\varepsilon[C]}$ is compact. Then $V = B_\varepsilon[C]$ is an open set containing C and with compact closure. Hence $(\text{CL}(X), \tau_V)$ is locally compact at C by Theorem 3.4.

In [2] it has been proved that local compactness of the Vietoris hyperspace is in general a strictly stronger condition than local compactness of the Hausdorff hyperspace. As a consequence of Corollary 3.5, local compactness of the proximal hyperspace has the same behaviour. Finally it has been proved in [1, Theorem 3.2.4] that compactness of the Hausdorff hyperspace is equivalent to compactness of X , and by Corollary 3.5, this is equivalent to compactness of the proximal hyperspace.

4. Proof our main result

To prove our main theorem we need several preliminary results. The following remarks are of easy verification.

4.1. Remark. Let X be a metrizable space and let $d \in \mathfrak{M}(X)$. If C is closed, then C^+ is $\tau_{\delta(d)}$ -closed. If K is compact, then $(X \setminus K)^{++} = (X \setminus K)^+$ and therefore $(X \setminus K)^+$ is $\tau_{\delta(d)}$ -open and K^- is $\tau_{\delta(d)}$ -closed.

Proof. Since $X \setminus C$ is open, $\text{CL}(X) \setminus C^+ = (X \setminus C)^-$ is $\tau_{\delta(d)}$ -open.

If K is compact and F is closed, then $F \cap K = \emptyset$ implies $D_d(F, K) > 0$. Therefore $(X \setminus K)^+ = \{F \mid F \subseteq X \setminus K\} = \{F \mid D_d(F, K) > 0\} = (X \setminus K)^{++}$. Moreover $\text{CL}(X) \setminus K^- = (X \setminus K)^+ = (X \setminus K)^{++}$ which is $\tau_{\delta(d)}$ -open. ■

4.2. Remark. Let X be a metrizable space and let $d \in \mathfrak{M}(X)$. Then $(\text{CL}(X), \tau_{\delta(d)})$ is T_2 .

Proof. Let $A, C \in \text{CL}(X)$ such that $A \neq C$. We may suppose there exists $a \in A \setminus C$. Since $a \notin \bigcap_{\varepsilon > 0} \overline{B_\varepsilon[C]} = C$, there exists $\varepsilon > 0$ such that $a \in X \setminus \overline{B_\varepsilon[C]}$. Take $\delta > 0$ such that $B_\delta(a) \subseteq X \setminus \overline{B_\varepsilon[C]}$. We claim that $B_\varepsilon[C] \subseteq X \setminus \overline{B_{\frac{\delta}{2}}(a)}$. Indeed let on the contrary $x \in B_\varepsilon[C] \cap \overline{B_{\frac{\delta}{2}}(a)}$. There exists $y \in B_{\frac{\delta}{2}}(x) \cap B_{\frac{\delta}{2}}(a)$, hence $d(x, a) \leq d(x, y) + d(y, a) < \delta$. Then $x \in B_\delta(a) \subseteq X \setminus \overline{B_\varepsilon[C]}$, impossible.

Hence $B_\varepsilon[C] \subseteq X \setminus \overline{B_{\frac{\delta}{2}}(a)}$, that is $C \in (X \setminus \overline{B_{\frac{\delta}{2}}(a)})^{++}$ which is $\tau_{\delta(d)}$ -open. Moreover $A \in B_{\frac{\delta}{2}}(a)^-$ which is $\tau_{\delta(d)}$ -open and $B_{\frac{\delta}{2}}(a)^- \cap (X \setminus \overline{B_{\frac{\delta}{2}}(a)})^{++} \subseteq B_{\frac{\delta}{2}}(a)^- \cap (X \setminus \overline{B_{\frac{\delta}{2}}(a)})^+ \subseteq B_{\frac{\delta}{2}}(a)^- \cap (X \setminus B_{\frac{\delta}{2}}(a))^+ = \emptyset$. ■

The following result gives a sufficient condition for compactness of a collection $\mathcal{K} \subseteq \text{CL}(X)$.

4.3. Proposition. *Let X be a metrizable space. Let $\mathcal{K} \subseteq \text{CL}(X)$ and $d \in \mathfrak{M}(X)$. If \mathcal{K} is $\tau_{\delta(d)}$ -closed, and for every $F \in \text{CL}(X)$, for every $\varepsilon > 0$ and for every open cover \mathcal{U} of $\overline{B_\varepsilon[F]}$, there exists a finite open subcover \mathcal{F} such that*

$$\mathcal{K} \cap \overline{B_\varepsilon[F]}^- \subseteq \bigcup_{U \in \mathcal{F}} U^-,$$

then \mathcal{K} is $\tau_{\delta(d)}$ -compact.

Proof. Let $(C_j)_{j \in J}$ be a net in \mathcal{K} . By Remark 4.2, we have to prove that it has a convergent subnet. By [1, Theorem 5.2.11], there exists a subnet $(C_{j_i})_{i \in I}$ which is K -convergent to a set $C \in \text{CL}(X) \cup \{\emptyset\}$. Note that since $\{\emptyset\} = \emptyset^{++}$, \emptyset is isolated with respect to $\tau_{\delta(d)}$ and therefore $C \in \text{CL}(X)$. Moreover $(C_{j_i})_{i \in I}$ is τ_V^- -convergent to C . We want to prove that $C_{j_i} \rightarrow C$ with respect to $\tau_{H_d}^+$, and this would also imply that $C \in \mathcal{K}$ since \mathcal{K} is $\tau_{\delta(d)}$ -closed.

On the contrary, suppose there exists W open such that $C \in W^{++}$ but $C_{j_i} \notin W^{++}$ frequently. Since $D_d(C, X \setminus W) > 0$, there exists $\varepsilon > 0$ such that $\overline{B_\varepsilon[X \setminus W]} \subseteq X \setminus C$. Since $C \supseteq \text{Ls}_{i \in I} C_{j_i}$, for every $x \in \overline{B_\varepsilon[X \setminus W]} \subseteq X \setminus C \subseteq X \setminus \text{Ls}_{i \in I} C_{j_i}$, there exist a neighbourhood V_x of x , and an index $i_x \in I$ such that $V_x \cap C_{j_i} = \emptyset$ for every $i \geq i_x$. Since $\overline{B_\varepsilon[X \setminus W]} \subseteq \bigcup_{x \in \overline{B_\varepsilon[X \setminus W]}} V_x$, by hypothesis there exist $x_1, \dots, x_n \in \overline{B_\varepsilon[X \setminus W]}$ such that

$$\mathcal{K} \cap \overline{B_\varepsilon[X \setminus W]}^- \subseteq \bigcup_{k=1}^n V_{x_k}^-.$$

Let $i_0 \in I$ such that $i_0 \geq i_{x_k}$ for $k = 1, \dots, n$. Then for every $i \geq i_0$, $C_{j_i} \cap V_{x_k} = \emptyset$ for every $k = 1, \dots, n$, that is $C_{j_i} \notin \bigcup_{k=1}^n V_{x_k}^-$. Since $C_{j_i} \in \mathcal{K}$, then $C_{j_i} \notin \overline{B_\varepsilon[X \setminus W]}^-$ for every $i \geq i_0$, that is $C_{j_i} \cap \overline{B_\varepsilon[X \setminus W]} = \emptyset$.

On the other hand, $C_{j_i} \notin W^{++}$ frequently, so that there exists $k \geq i_0$, such that for every $\delta > 0$, $B_\delta[C_{j_k}] \cap (X \setminus W) \neq \emptyset$. In particular for $\delta = \varepsilon$, there exist $y \in C_{j_k}$, $z \in X \setminus W$ such that $d(z, y) < \varepsilon$. But then $y \in C_{j_k} \cap \overline{B_\varepsilon[X \setminus W]} \neq \emptyset$, a contradiction. ■

4.4. Lemma. *Let X be a metrizable space. Let $K \in \text{CL}(X)$ and $d \in \mathfrak{M}(X)$. If K is compact and there exists $\delta > 0$ such that $B_\delta(x)$ is compact for every $x \in K$, then $\overline{B_{\frac{\delta}{2}}[K]}$ is compact.*

Proof. Of course $K \subseteq \bigcup_{x \in K} B_{\frac{\delta}{2}}(x)$. Since K is compact there exist $x_1, \dots, x_n \in K$ such that $K \subseteq \bigcup_{i=1}^n B_{\frac{\delta}{2}}(x_i)$. Then $\overline{B_{\frac{\delta}{2}}[K]} \subseteq \bigcup_{i=1}^n \overline{B_{\frac{\delta}{2}}(x_i)}$ and therefore $\overline{B_{\frac{\delta}{2}}[K]} \subseteq \bigcup_{i=1}^n B_\delta(x_i) = \bigcup_{i=1}^n \overline{B_\delta(x_i)}$ which is compact. Hence $\overline{B_{\frac{\delta}{2}}[K]}$ is compact. ■

4.5. Lemma. *Let X be a metrizable space. Let $K \in \text{CL}(X)$ and $d \in \mathfrak{M}(X)$. If K is compact and there exists $\delta > 0$ such that $\overline{B_\delta(x)}$ is compact for every $x \in K$, then $\overline{B_{\frac{\delta}{2}}[K]}^+$ is $\tau_{\delta(d)}$ -compact.*

Proof. The set $\overline{B_{\frac{\delta}{2}}[K]}^+$ is $\tau_{\delta(d)}$ -closed by Remark 4.1. Let $C \in \text{CL}(X)$, $\varepsilon > 0$ and let \mathcal{U} be an open cover of $\overline{B_\varepsilon[C]}$. By Proposition 4.3, we have to find a finite open subcover \mathcal{F}

such that $\overline{B_{\frac{\delta}{2}}[K]^+} \cap \overline{B_{\varepsilon}[C]^-} \subseteq \bigcup_{U \in \mathcal{F}} U^-$. If $\overline{B_{\varepsilon}[C]} \cap \overline{B_{\frac{\delta}{2}}[K]} = \emptyset$ then $\overline{B_{\varepsilon}[C]^-} \cap \overline{B_{\frac{\delta}{2}}[K]^+}$ is empty. Suppose $\overline{B_{\varepsilon}[C]} \cap \overline{B_{\frac{\delta}{2}}[K]} \neq \emptyset$. Note that $\overline{B_{\varepsilon}[C]} \cap \overline{B_{\frac{\delta}{2}}[K]}$ is compact since it is closed and it is contained in $\overline{B_{\frac{\delta}{2}}[K]}$ which is compact by Lemma 4.4. There exist $U_1, \dots, U_n \in \mathcal{U}$ such that $\overline{B_{\varepsilon}[C]} \cap \overline{B_{\frac{\delta}{2}}[K]} \subseteq \bigcup_{i=1}^n U_i$. Therefore $\overline{B_{\varepsilon}[C]^-} \cap \overline{B_{\frac{\delta}{2}}[K]^+} \subseteq (\overline{B_{\varepsilon}[C]} \cap \overline{B_{\frac{\delta}{2}}[K]})^- \subseteq \bigcup_{i=1}^n U_i^-$. ■

4.6. Proposition. *Let X be a metrizable space and $d \in \mathfrak{M}(X)$. Let V, V_1, \dots, V_k be any non-empty set. If $\mathcal{V} = \bigcap_{i=1}^k V_i^- \cap V^+$ is $\tau_{\delta(d)}$ -compact and non-empty, then V is compact.*

Proof. Since $\mathcal{V} \neq \emptyset$, for every $i = 1, \dots, k$ we can fix $x_i \in V_i \cap V$.

We first prove that V is closed. Otherwise there should exist $(y_n)_{n \in \mathbb{N}}$ such that $y_n \in V$ and $y_n \rightarrow y \in X \setminus V$. Since $\mathcal{V} \neq \emptyset$, for every $i = 1, \dots, k$ we can find $x_i \in V_i \cap V$. For every $n \in \mathbb{N}$ set $C_n = \{x_1, \dots, x_k, y_n\}$ and $C = \{x_1, \dots, x_k, y\}$. Note that $C \notin \mathcal{V}$ because $y \in C \setminus V$. Moreover $C_n \in \mathcal{V}$ for every $n \in \mathbb{N}$. We prove that $C_n \rightarrow_{\tau_{\delta(d)}} C$ in order to have a contradiction, since \mathcal{V} is $\tau_{\delta(d)}$ -compact and hence closed.

Let $\mathcal{U} = \bigcap_{i=1}^p U_i^- \cap U^{++}$ be a $\tau_{\delta(d)}$ -neighbourhood of C . Let $i \in \{1, \dots, p\}$; we distinguish two cases. If there exists $j \in \{1, \dots, k\}$ such that $x_j \in U_i$, then $x_j \in C_n \cap U_i$, hence $C_n \in U_i^-$ for every $n \in \mathbb{N}$. If $x_j \notin U_i$ for every $j = 1, \dots, k$, since $C \cap U_i \neq \emptyset$, then $y \in C \cap U_i$. Take $\varepsilon > 0$ such that $B_{\varepsilon}(y) \in U_i$. Then $y_n \in B_{\varepsilon}(y)$ eventually and therefore $C_n \in U_i^-$ eventually. Since $C \in U^{++}$, there exists $\delta > 0$ such that $B_{\delta}[C] \subseteq U$. Eventually $y_n \in B_{\frac{\delta}{2}}(y)$ and therefore $B_{\frac{\delta}{2}}[C_n] \subseteq B_{\delta}[C] \subseteq U$, that is $C_n \in U^{++}$. Hence $C_n \in \bigcap_{i=1}^p U_i^- \cap U^{++}$ eventually, that is $C_n \rightarrow_{\tau_{\delta(d)}} C$.

We now prove that V is compact. Otherwise, there should exist $(a_n)_{n \in \mathbb{N}}$ in V with no cluster point. That is for every $x \in X$ there exists a neighbourhood V_x of x and $\nu_x \in \mathbb{N}$ such that for every $n \geq \nu_x$, $a_n \notin V_x$. For every $n \in \mathbb{N}$ set $C_n = \{x_1, \dots, x_k, a_n\}$. We will prove that $(C_n)_{n \in \mathbb{N}}$ has no cluster point in order to have a contradiction since \mathcal{V} is compact. Let $C \in \text{CL}(X)$.

- If $C \subseteq \{x_1, \dots, x_k\}$, then $(\bigcup_{i=1}^k V_{x_i})^{++}$ is a $\tau_{\delta(d)}$ -neighbourhood of C . On the other hand if $\nu \geq \max\{\nu_{x_1}, \dots, \nu_{x_k}\}$, then for every $n \geq \nu$, $a_n \notin V_{x_i}$ for every $i = 1, \dots, k$. Then $C_n \notin (\bigcup_{i=1}^k V_{x_i})^{++}$ eventually.
- If there exists $x_0 \in C \setminus \{x_1, \dots, x_k\}$, let W be a neighbourhood of x_0 such that $W \cap \{x_1, \dots, x_k\} = \emptyset$. Then $C_n \notin (W \cap V_{x_0})^-$ for every $n \geq \nu_{x_0}$, while $C \in (W \cap V_{x_0})^-$. ■

We are now able to finally prove our main result.

Proof of Theorem 3.1.

\Rightarrow) Suppose that $(\text{CL}(X), \tau_{\delta(d)})$ is locally compact at C . There exists a neighbourhood $\mathcal{V} = \bigcap_{i=1}^n V_i^- \cap V^{++}$ of C such that $\overline{\mathcal{V}}$ is $\tau_{\delta(d)}$ -compact. For every $i = 1, \dots, n$, let $x_i \in V_i \cap C$ and let $\delta > 0$ such that $B_{\delta}[C] \subseteq V$. Set $\mathcal{U} = \bigcap_{i=1}^n \{x_i\}^- \cap \overline{B_{\frac{\delta}{3}}[C]^+}$. Note that \mathcal{U} is $\tau_{\delta(d)}$ -closed by Remark 4.1. If $T \in \mathcal{U}$, then $B_{\frac{\delta}{3}}[T] \subseteq B_{\frac{\delta}{3}}(\overline{B_{\frac{\delta}{3}}[C]}) \subseteq B_{\delta}[C] \subseteq V$, and this implies $\mathcal{U} \subseteq \mathcal{V}$. Hence \mathcal{U} is $\tau_{\delta(d)}$ -compact being $\tau_{\delta(d)}$ -closed and contained in the $\tau_{\delta(d)}$ -compact set $\overline{\mathcal{V}}$. By Proposition 4.6, $\overline{B_{\frac{\delta}{3}}[C]}$ is compact.

\Leftarrow) Since C is closed and contained in the compact set $\overline{B_{\varepsilon}[C]}$, it is compact. Moreover for every $x \in C$, $\overline{B_{\varepsilon}(x)}$ is compact since it is contained in $\overline{B_{\varepsilon}[C]}$. By Lemma 4.5, $\overline{B_{\frac{\varepsilon}{2}}[C]^+}$ is $\tau_{\delta(d)}$ -compact. Moreover $C \in B_{\frac{\varepsilon}{2}}[C]^{++} \subseteq \overline{B_{\frac{\varepsilon}{2}}[C]^+}$. The

set $B_{\frac{\varepsilon}{2}}[C]^{++}$ is a $\tau_{\delta(d)}$ -neighbourhood of C and its closure in $\text{CL}(X)$ is compact since it is contained in $\overline{B_{\frac{\varepsilon}{2}}[C]^+}$, which is $\tau_{\delta(d)}$ -compact. Hence $(\text{CL}(X), \tau_{\delta(d)})$ is locally compact at C . ■

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