Optimal stop-loss reinsurance: a dependence analysis

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Abstract

The stop-loss reinsurance is one of the most important reinsurance contracts in the insurance market. From the insurer point of view, it presents an interesting property: it is optimal if the criterion of minimizing the variance of the cost of the insurer is used. The aim of the paper is to contribute to the analysis of the stop-loss contract in one period from the point of view of the insurer and the reinsurer. Firstly, the influence of the parameters of the reinsurance contract on the correlation coefficient between the cost of the insurer and the cost of the reinsurer is studied. Secondly, the optimal stop-loss contract is obtained if the criterion used is the maximization of the joint survival probability of the insurer and the reinsurer in one period.

1. Introduction

An insurance company may decide to sign a reinsurance contract either to assume greater risks or to protect the company. This reinsurance contract transfers part of the risks assumed by the insurer to the reinsurer in exchange of giving also a part of the premiums received from policyholders. Yet, reinsurance is the most important decision that an insurance company has to consider in order to reduce its underwriting risk. Two large groups of reinsurance contracts can be distinguished: the proportional and the non-proportional reinsurance. The proportional reinsurance includes two kinds of reinsurance known as quota-share and surplus. In the former, all the risks are transferred

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in the same proportion, while in the latter the proportion may vary. As regards the non-proportional reinsurance, the stop-loss and excess-loss contracts stand out. In both cases, the reinsurance offers protection when the aggregate claims exceed a certain agreed level.

The stop-loss reinsurance has been widely studied in the actuarial literature. The actuarial literature on the optimal reinsurance can be classified using as a criterion who takes the decision. A first group may include the works that focus on the insurer point of view and then try to maximize/minimize some measure of the risk of the insurer if he/she signs a reinsurance contract. In a second group we include the papers that search for strategies that are beneficial to the two parts that participate in the contract. Borch, [7], states that "These considerations should remind us that there are two parties to a reinsurance contract, and that this parties have conflicting interests. The optimal contract must then appear as a reasonable compromise between these interests. To me the most promising line of research seems to be the study of contracts, which in different ways can be said to be optimal from the point of view of both parties". Nonetheless, most of the papers on optimal reinsurance have only considered the insurer point of view.

In the first group, a secondary criterion for classification could be the function that is maximized or minimized. Within this literature we find, for instance, the maximization of the expected utility of the insurer's wealth after reinsurance, the minimization of the probability of ruin in the short and in the long run and the minimization of the variance of the retained risk ([46]). Borch [6] (reproduced in [8]) proved that the stop-loss contract is the "most efficient" contract because, for a given net premium, it maximizes the reduction of the variance in the claim distribution of the ceding company. Daykin et al. [17], Gajek et al. [24], and Kaluszka [38] follow a similar line of research.

Gerber [27] uses the expected profit of the insurer in one period as a measure of the profitability, and the adjustment coefficient (closely connected with the probability of ruin) as a measure of security. He assumes normality and considers excess-loss, stop-loss and proportional contracts. Van Wouwe et al. [47] determine the optimal level of excess-loss reinsurance in the case that the ultimate ruin probability is taken as stability criterion. The insurer's survival probability is also considered in [40] and [29], and more recently in [43], [44], [25], [39], [13] and [41]. Guerra and Centeno[28] obtain an optimal reinsurance policy by maximizing the insurer's expected utility.

Several authors have used other kind of measures to find the optimal strategy. Van Heerwaarden et al. [46] use, as optimality criterion, the minimization of the retained risk with respect to the stop-loss order. In turn, Hoøjgaard and Taksar [30] find the optimal proportional dynamic strategy that maximizes the return function of the insurer. And in [31], the previous analysis is extended to include transactions costs. Azcue and Muler [2] consider also a dynamic choice of both the reinsurance policy and the dividend distribution strategy that maximizes the cumulative expected discounted dividend payouts. In [9], [11], [45] and [15] the optimization of a reinsurance contract under the value-at-risk and conditional tail expectation risk measures is conducted. Similarly, Zhu et al. [50] investigate optimal reinsurance strategies for an insurer with multiple lines of business using the multivariate lower-orthant Value-at-Risk. Centeno and Simões [14] provide a good summary of the classical results on optimal reinsurance and a more detailed analysis of recent results (2000-2009).

Balbás et al. [4] use a general risk measure that includes every deviation measure, every expectation bounded risk measure, and most of the coherent, convex or consistent risk measures as particular cases. In [3] the previous analysis is extended to cases where the statistical distribution of claims is not totally known, generating uncertain or ambiguous frameworks. Following the modern studies about distortion risk measures, Cui

et al. [16], Zheng and Cui [48], Zheng et al. [49] and Assa [1] use them to find the optimal reinsurance.

In spite of the above comment of Borch [7], the consideration of the interest of both the insurer and the reinsurer has not been really developed until recently. Borch [6] can be considered the first author in adopting this approach to the optimal reinsurance problem. He considers the minimization of the total variance risk for an stop-loss contract. Hürlimann [32] retakes this question and, in [33], obtains also optimal solutions under the total variance risk measure for a partial stop-loss contract.

Cai et al. [10] study the sufficient and necessary conditions for the existence of the optimal reinsurance retentions for the quota-share reinsurance and the stop-loss reinsurance under the expected value reinsurance premium principle, considering as objective function the joint survival probability and the joint profitable probability. For the joint survival probability, in [23], an extension for a combination of quota-share and stop-loss reinsurance contracts is found.

Ignatov et al. [34] and Kaishev and Dimitrova ([37] and [22]) use a different approach to joint optimality criteria. In [34] and [37], they find the parameters of the reinsurance contract that maximize the joint survival probability, when the premiums of the insurer and the reinsurer are fixed, for an excess of loss risk model when the number of claims follows a Poisson process. Salcedo-Sanz et al. [41] solve also this question using evolutionary and swarm intelligence techniques. In [37] the previous analysis is extended to include an optimal split of the premium income between the insurer and the reinsurer, given fixed retention and limiting levels. These two optimization problems are applied, with some numerical examples, to the stop-loss contract over a fixed horizon in [12]. Other optimal problems are added by Dimitrova and Kaishev [22], for an excess-loss, and by Castañer et al. [12], for an stop-loss. In [22], the authors propose a Markowitz type efficient frontier solution to the problem of optimally setting the parameters of reinsurance, so that for a given level of the probability of joint survival the expected profits of the two parties are maximized. Finally, in [12], the optimal split of the total initial reserves between the insurer and the reinsurer that maximizes the joint survival probability is considered.

The objective of this work is to contribute to the analysis of the optimal stop-loss reinsurance in one period, from the joint point of view of the insurer and the reinsurer. The contributions of this paper to the optimal reinsurance can be summarized as follows. First, using total variance risk measure, we add the analysis of the optimal reinsurance for an stop-loss contract with maximum, to the known solutions for the standard stop-loss ([6] and [32]) or the partial standard stop-loss ([33]). We also include the possibility of using the maximization of the correlation coefficient between the insurer's and reinsurer's losses. Second, we consider the maximization of the joint survival probability in one period in an stop-loss with and without maximum, and using the same hypothesis with respect to premiums as in Kaishev and Dimitrova ([37] and [22]), we obtain the optimal parameters of the stop-loss. In addition, in line with [37], we use as a criterion for the calculation of the reinsurer's premiums the maximization of the joint survival probability, given as fixed both the values of the parameters of the reinsurance contract and the initial values of the reserves of the insurer and the reinsurer. In fact, then, we propose a different way of calculating the stop-loss premium that considers not only the losses for the reinsurer, but all the other factors (loss and premium of the insurer and initial capitals of insurer and reinsurer). The solution of these two optimization problems related to the joint survival probability in one period for the stop-loss reinsurance are arguably the main findings of this paper.

The paper is organized as follows. Section 2 analyzes the expression of the covariance and the correlation coefficient and the specific expressions for different distributions of

the total cost, considering a stop-loss reinsurance with priority d with and without a maximum m. In Section 3, we find the optimal reinsurance stop-loss if the criterion is the maximization of the variance reduction due to reinsurance. In Section 4, we introduce the probability of joint survival as a measure for the solvency for a reinsurance contract with priority d and reinsurance with d and m. In Section 5, the problem of finding the optimal reinsurance stop-loss if the criterion is the maximization of the joint survival probability is solved. In addition, a number of examples are presented. Section 6 closes the paper offering some final conclusions and remarks.

2. Covariance and correlation between the cost of the insurer and the cost of the reinsurer

In the stop-loss reinsurance contract with priority $d > 0$ the random variable (r.v.) total cost of claims in one period, S, is split between the cost of the insurer, SI, and the cost of the reinsurer, SR, with $S = SI + SR$, $SR = \max\{S - d, 0\}$ and $SI = min\{S, d\}$. The distribution functions of these two r.v., $F_{SI}(s) = P[SI \leq s]$ and $F_{SR}(s) = P[SR \leq s]$, can be calculated from the distribution function of S, $F_S(s) = P[S \leq s]$,

$$(2.1) \qquad F_{SI}(s) = \begin{cases} F_S(s) & \text{if } s < d, \\ 1 & \text{if } s \geq d, \end{cases}$$

$$(2.2) \qquad F_{SR}(s) = F_S(s + d).$$

The reinsurer can calculate the reinsurance premium with several premium principles. Most of these principles are based on the expectation of the total cost assumed by the reinsurer ([21]). For instance, the net premium principle establishes that the premium is equal to the expectation of the cost. In the actuarial literature, the premium of an stop-loss contract calculated with the net premium principle is called the stop-loss premium. Let us define $\pi(d) = E[SR]$ as the stop-loss premium in a reinsurance stop-loss contract with priority d.

The r.v. cost of the reinsurer SR has the following two ordinary moments[§]:

$$(2.3) \qquad \alpha_1(SR) = E[SR] = \int_d^\infty (s - d) f_S(s) ds = \int_d^\infty (1 - F_S(s)) ds,$$

$$\alpha_2(SR) = \int_d^\infty (s - d)^2 f_S(s) ds = 2 \int_d^\infty (s - d)(1 - F_S(s)) ds.$$

Hence, the variance is

$$(2.4) \qquad V[SR] = \alpha_2(SR) - \alpha_1^2(SR) = E[SR](-2d - E[SR]) + 2 \int_d^\infty s(1 - F_S(s)) ds.$$

The expectation and the variance of the insurer cost SI can be calculated from those of S and SR, so:

$$\alpha_1(SI) = E[SI] = E[min(S, d)] = E[S] - E[SR],$$

$$V[SI] = V[S] - V[SR] - 2Cov[SI, SR],$$

[§]In order to obtain the expressions for the first two moments of the cost of the reinsurer, it is necessary to take into account that $-f_S(s)ds = d(1 - F_S(s))$ and then apply integration by parts.

being

$$Cov\left[SI, SR\right] = \int_d^\infty d(s-d)f_S(s)ds - E\left[SR\right]\left(E\left[S\right] - E\left[SR\right]\right)$$

(2.5)
$$= E[SR]\left(d - E\left[S\right] + E[SR]\right).$$

The correlation coefficient between SI and SR is

(2.6)
$$r(SI, SR) = \frac{Cov\left[SI, SR\right]}{\sqrt{V\left[SR\right]\left(V\left[S\right] - V\left[SR\right] - 2Cov\left[SI, SR\right]\right)}}.$$

In addition to the marginal analysis of the cost of the insurer and the reinsurer, we are interested in the bivariate r.v. (SI, SR). In a stop-loss reinsurance contract with priority d, the joint distribution function of the costs of the insurer and the reinsurer in one period is

(2.7)
$$P\left[SI \le x, SR \le y\right] = \begin{cases} P\left[S \le x\right] & \text{if } x < d, \\ P\left[S \le y + d\right] & \text{if } x \ge d > 0. \end{cases}$$

This r.v. (SI, SR) is comonotone ([20]) because SI and SR are increasing functions of the risk S. Then, there is a perfect positive dependence between the two marginal r.v. SI and SR and it is granted that the two parts that participate in the exchange of risk (the insurer and the reinsurer) increase their cost when the underlying risk increases. Hence, the correlation coefficient between SI and SR is the maximal one that can be attained between two random variables with the same marginal distributions, but it is not equal to one (this would be the case if one variable could be calculated as a linear function of the other, e.g. in proportional reinsurance)([18]). So, for a fixed d, $r(SI, SR)$ is the maximal one, but it is less than one in absolute value.

The stop-loss reinsurance contract can include a priority d and a maximum m, $m > d > 0$. In this case,

$$SR(d, m) = \min\left\{m - d, \max\left\{S - d, 0\right\}\right\},$$

$$SI(d, m) = \min\left\{S, d\right\} + \max\left\{S - m, 0\right\}.$$

The distribution functions of these two r.v. are

(2.8)
$$F_{SI(d,m)}(s) = \begin{cases} F_S(s) & \text{if } s < d, \\ F_S(s + m - d) & \text{if } s \ge d \end{cases}$$

and

(2.9)
$$F_{SR(d,m)}(s) = \begin{cases} F_S(s + d) & \text{if } s < m - d, \\ 1 & \text{if } s \ge m - d. \end{cases}$$

Let $\pi(d, m) = E[SR(d, m)]$ be the stop-loss premium, that is the reinsurance premium calculated with the net premium principle. It can be calculated from the premiums of a stop-loss reinsurance with priorities d and m, $\pi(d, m) = \pi(d) - \pi(m)$.

The second ordinary moment $\alpha_2(SR(d,m))$, is

$$
\begin{aligned}
\alpha_2(SR(d,m)) &= \int_d^m (s-d)^2 f_S(s)ds + \int_m^\infty (m-d)^2 f_S(s)ds \\
&= \int_d^\infty (s-d)^2 f_S(s)ds - \int_m^\infty (s-d)^2 f_S(s)ds + \int_m^\infty (m-d)^2 f_S(s)ds \\
&= \alpha_2(SR(d)) - \int_m^\infty ((s-d)^2 - (m-d)^2) f_S(s)ds \\
&= \alpha_2(SR(d)) - \alpha_2(SR(m)) - 2(m-d)\pi(m),
\end{aligned}
$$

where the last equality follows taking into account that $(s-d)^2 - (m-d)^2 = (s-m)^2 + 2(s-m)(m-d)$.

Hence, the variance $V[SR(d,m)]$, is:

$$
\begin{aligned}
V[SR(d,m)] &= \alpha_2(SR(d,m)) - \alpha_1(SR(d,m))^2 \\
&= \alpha_2(SR(d)) - \alpha_2(SR(m)) - 2(m-d)\pi(m) - (\pi(d) - \pi(m))^2 \\
&= V[SR(d)] - V[SR(m)] + 2\pi(m)(\pi(d) + d - \pi(m) - m).
\end{aligned}
$$

The covariance between the costs of the insurer and the reinsurer is:

$$
\begin{aligned}
Cov[SI(d,m), SR(d,m)] &= \int_d^m d(s-d) f_S(s)ds + \int_m^\infty (m-d)(s-m+d) f_S(s)ds \\
&= \int_d^\infty d(s-d) f_S(s)ds \\
&\quad - \int_m^\infty (d(s-d) - (m-d)(s-m+d)) f_S(s)ds \\
&= Cov[SI(d), SR(d)] - \int_m^\infty ((s-m)(2d-m)) f_S(s)ds \\
(2.10) \qquad &= Cov[SI(d), SR(d)] - (2d-m)\pi(m),
\end{aligned}
$$

where the last but one equality follows taking into account that $d(s-d) - (m-d)(s-m+d) = (s-m)(2d-m)$.

So, in order to calculate the expectation and the variance of the costs of the insurer and the reinsurer, and the covariance if the stop-loss has a maximum, we only need the expressions of a stop-loss without maximum.

The distribution function of the bivariate r.v. $(SI(d,m), SR(d,m))$ is

$$
(2.11) \quad P[SI(d,m) \le x, SR(d,m) \le y] = \begin{cases} P[S \le x] & \text{if } x < d, \\ P[S \le d] & \text{if } x \ge d \text{ and } y = 0, \\ P[S \le y+d] & \text{if } x \ge d \text{ and } 0 < y < m-d, \\ P[S \le m] & \text{if } x = d \text{ and } y \ge m-d, \\ P[S \le x+m-d] & \text{if } x > d \text{ and } y \ge m-d. \end{cases}
$$

Throughout the paper, we use three approximations for the total cost in a period: gamma with two parameters, translated gamma and normal. The gamma distribution deserves special attention. It has been used in its version of two or three parameters to approximate the distribution of the total cost in a period as an alternative to the exact calculation through convolutions and to other approximations. In several papers ([5], [42], [26]), the accuracy of the translated gamma approximation and the rest of approximations has been quantified. In this sense, [35] uses the translated gamma approximation for the calculation of the stop-loss premium. In order to be self contained and to clarify the formulas that we use, we include in Section 2.1 a summary of the (translated) gamma distribution. Next, we indicate the explicit expressions of $\pi(d)$, $Cov[SI, SR]$ and $V[SR]$, which allow us calculating the coefficient of correlation for three different distributions

or approximations for the total cost in a period: gamma with two parameters, translated gamma and normal. As it is a simple calculation, we do not include the processes for obtaining these expressions.

2.1. Statistical summary. The gamma distribution with three parameters (or Pearson Type III) is also known as the translated gamma distribution, with one of its parameters interpreted as follows. If $X \sim Ga(\alpha, \beta, \gamma)$, with $\alpha > 0$, $\beta > 0$ and $\gamma \in \Re$, its density function is

$$(2.12) \quad f_X(x) = \frac{(x-\gamma)^{\alpha-1} e^{\frac{-(x-\gamma)}{\beta}}}{\beta^\alpha \Gamma(\alpha)}, x > \gamma,$$

being γ, precisely, the parameter of translation. If $\gamma = 0$, the gamma distribution with two parameters is obtained, $X \sim Ga(\alpha, \beta)$ with $\alpha > 0$ and $\beta > 0$. The standard form of the distribution is obtained if, in addition, $\beta = 1$. Then, $X \sim Ga(\alpha)$, with $\alpha > 0$.

The gamma distribution with three parameters can be calculated through a gamma distribution with two or with one parameter (the standard form). Let $X \sim Ga(\alpha, \beta, \gamma)$, if $Y = (X - \gamma)/\beta$, then, $Y \sim Ga(\alpha)$, and also, $X = Y\beta + \gamma$. If $Z = X - \gamma$, then, $Z \sim Ga(\alpha, \beta)$, and the next relations are met,

$$X = Z + \gamma, Y = \frac{Z}{\beta}.$$

Recall that the moments and measures of X, Y and Z, are related as shown in Table 1.

Table 1. Some characteristics of the gamma distribution

	$Y \sim Ga(\alpha)$	$Z \sim Ga(\alpha, \beta)$	$X \sim Ga(\alpha, \beta, \gamma)$
Mean μ_1	α	$\alpha\beta$	$\alpha\beta + \gamma$
Variance μ_2	α	$\alpha\beta^2$	$\alpha\beta^2$
μ_3	2α	$2\alpha\beta^3$	$2\alpha\beta^3$
Skewness γ_1	$\frac{2}{\sqrt{\alpha}}$	$\frac{2}{\sqrt{\alpha}}$	$\frac{2}{\sqrt{\alpha}}$

The parameters of $X \sim Ga(\alpha, \beta, \gamma)$, can be estimated by the moments' method:

$$(2.13) \quad \widehat{\alpha} = \frac{4}{\gamma_1^2(X)}, \widehat{\beta} = \frac{\mu_3(X)}{2\mu_2(X)}, \widehat{\gamma} = E[X] - \widehat{\alpha}\widehat{\beta}.$$

Taking into account Table 1, a variable $X \sim Ga(\alpha, \beta, \gamma)$, also meets the next relationship with the variable $Y \sim Ga(\alpha)$ (if the parameter α is estimated through the asymmetry of X, as in (2.13)),

$$X = \mu_1(X) + \mu_2^{0.5}(X)\frac{Y - \alpha}{\sqrt{\alpha}}.$$

Then,

$$
\begin{aligned}
P[X \le x] &= P\left[\mu_1(X) + \mu_2^{0.5}(X)\frac{Y-\alpha}{\sqrt{\alpha}} \le x\right] \\
&= P\left[Y \le \alpha + \sqrt{\alpha}\frac{x - \mu_1(X)}{\mu_2^{0.5}(X)}\right] \\
(2.14) \quad &= Ga\left(\alpha + \sqrt{\alpha}\frac{x - \mu_1(X)}{\mu_2^{0.5}(X)}; \alpha\right),
\end{aligned}
$$

being $Ga(y; \alpha) = P[Y \le y]$ with $Y \sim Ga(\alpha)$. Or alternatively,

$$
\begin{aligned}
P[X \le x] &= P[Z + \gamma \le x] = P[Z \le x - \gamma] \\
(2.15) \quad &= Ga(x - \gamma; \alpha, \beta),
\end{aligned}
$$

being $Ga(z; \alpha, \beta) = P[Z \leq z]$ with $Z \sim Ga(\alpha, \beta)$.

2.2. Gamma distribution (with two parameters). Assume $S \sim Ga(\alpha, \beta)$, with $\alpha > 0$ and $\beta > 0$. The density function and the distribution function are, respectively,

$$
\begin{aligned}
f_S(s) &= \frac{s^{\alpha-1} e^{-\frac{s}{\beta}}}{\beta^\alpha \Gamma(\alpha)}, \quad s > 0, \\
F_S(s) &= Ga(s; \alpha, \beta), \quad s > 0.
\end{aligned}
$$

Hence, in this case we have

$$
\pi(d) = \alpha\beta \left(1 - Ga(d; \alpha+1, \beta)\right) - d\left(1 - Ga(d; \alpha, \beta)\right),
$$

$$
\begin{aligned}
Cov\left[SI, SR\right] &= \left[\alpha\beta\left(1 - Ga(d; \alpha+1, \beta)\right) - d\left(1 - Ga(d; \alpha, \beta)\right)\right] \\
&\times \left[-\alpha\beta Ga(d; \alpha+1, \beta) + d\, Ga(d; \alpha, \beta)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
V\left[SR\right] &= \pi(d)\left(-2d - \pi(d)\right) - d^2\left(1 - Ga(d; \alpha, \beta)\right) \\
&+ (\alpha+1)\alpha\beta^2\left(1 - Ga(d; \alpha+2, \beta)\right).
\end{aligned}
$$

2.3. Translated gamma distribution. Assume $S \sim Ga(\alpha, \beta, \gamma)$, with $\alpha > 0$, $\beta > 0$ and $\gamma \in \Re$. The density function and the distribution function are, respectively,

$$
\begin{aligned}
f_S(s) &= \frac{(s-\gamma)^{\alpha-1} e^{-\frac{s-\gamma}{\beta}}}{\beta^\alpha \Gamma(\alpha)}, \quad s > \gamma, \\
F_S(s) &= Ga(s; \alpha, \beta, \gamma), \quad s > \gamma.
\end{aligned}
$$

For the translated gamma approximation for the distribution of the total cost, we obtain two equivalent expressions for the stop-loss premium depending on the formula used, (2.14) or (2.15). First, from (2.14) we have,

$$
(2.16) \quad \pi(d) = E\left[(S-d)_+\right] \approx \frac{\mu_2^{0.5}(S)}{\sqrt{\alpha}}\left[d' f(d'; \alpha) + (\alpha - d')(1 - Ga(d'; \alpha))\right],
$$

being $d' = \alpha + \sqrt{\alpha}\left(\frac{d - \mu_1(S)}{\mu_2^{0.5}(S)}\right)$ and $f(d'; \alpha)$, the density function of $Y \sim Ga(\alpha)$ in d'. Second, from (2.15) we have,

$$
\begin{aligned}
\pi(d) &= E\left[(S-d)_+\right] \approx \alpha\beta\left(1 - Ga(d-\gamma; \alpha+1, \beta)\right) \\
(2.17) \quad &- (d-\gamma)(1 - Ga(d-\gamma; \alpha, \beta)),
\end{aligned}
$$

Expression (2.16) can be found in [35] as a particular case of the ordinary moments of the cost of the reinsurer.

From (2.4), (2.5) and (2.17) the $Cov[SI, SR]$ can be easily calculated, and the expression of the variance of SR is

$$
\begin{aligned}
V[SR] &= \pi(d)\left(-2d - \pi(d)\right) + 2\alpha\beta\gamma\left(1 - Ga(d-\gamma; \alpha+1, \beta)\right) \\
&+ (\alpha+1)\alpha\beta^2\left(1 - Ga(d-\gamma; \alpha+2, \beta)\right) + (\gamma^2 - d^2)\left(1 - Ga(d-\gamma; \alpha, \beta)\right).
\end{aligned}
$$

2.4. Normal distribution. Assume $S \sim N(\mu, \sigma)$, with $\mu = E[S]$ and $\sigma^2 = V[S] > 0$. The density and distribution functions are, respectively, in terms of the distribution of $N(0, 1)$,

$$
\begin{aligned}
f_S(s) &= \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(s-\mu)^2}{2\sigma^2}}, \\
F_S(s) &= \Phi\left(\frac{s-\mu}{\sigma}\right),
\end{aligned}
$$

and then,

$$\pi(d) = \sigma\phi\left(\frac{d-\mu}{\sigma}\right) + (\mu - d)\left(1 - \Phi\left(\frac{d-\mu}{\sigma}\right)\right),$$

$$Cov\left[SI, SR\right] = \left[\sigma\phi\left(\frac{d-\mu}{\sigma}\right) + (\mu-d)\left(1 - \Phi\left(\frac{d-\mu}{\sigma}\right)\right)\right]$$
$$\times \left[\sigma\phi\left(\frac{d-\mu}{\sigma}\right) - (\mu-d)\left(\Phi\left(\frac{d-\mu}{\sigma}\right)\right)\right]$$

and

$$V\left[SR\right] = -\sigma(d-\mu)\phi\left(\frac{d-\mu}{\sigma}\right) - \pi(d)^2 + \left((\mu-d)^2 + \sigma^2\right)\left(1 - \Phi\left(\frac{d-\mu}{\sigma}\right)\right).$$

3. Optimal reinsurance if the criterion is the maximization of the variance reduction

Following [8], we can choose as measures of risk the variance or the probability of ruin. These two measures have different properties, and correspond to a different idea. If we use the variance we exclusively focus on the randomness of the cost of claims and we disregard the premiums (and then the security loadings) and the initial reserves of the insurer and the reinsurer. These two factors can be also taken into account if we use the probability of ruin as an alternative risk measure.

If our objective is to find an optimal contract, we can not only rely on the insurer's risk measure. We have to keep in mind Borch's statement that there are two parties to a reinsurance contract, and that these parties have conflicting interests. The optimal contract must then appear as a reasonable compromise between the interest of the insurer and the reinsurer and thus, it has to be found undertaking a joint analysis of this two parties.

In this section we perform a first analysis of the optimal reinsurance choosing the variance as measure of risk and maximizing the variance reduction defined as the difference between the variance of the loss and the sum of the variance of the insurer and the reinsurer, $V[S] - (V[SI(d)] + V[SR(d)])$ or $V[S] - (V[SI(d,m)] + V[SR(d,m)])$ if the contract includes a maximum m. By definition, in the first case this difference equals $2Cov[SI(d), SR(d)]$, and in the second case equals $2Cov[SI(d,m), SR(d,m)]$. Then, we choose the reinsurance parameters as those that maximize the covariance between the costs of the insurer and the reinsurer.

We consider first a stop-loss reinsurance contract with priority $d > 0$. The maximization program, from (2.5), is

(3.1) $\max\limits_{d} \pi(d)\left(d - E[S] + \pi(d)\right)$ subject to $0 < d$.

As the covariance is a continuous function of d and the limits when d tends to 0 and to infinity are zero, the covariance has a maximum for at least one finite, positive value of d ([8]).

3.1. Proposition. The optimal point of program (3.1) is a value of d such that the following conditions are fulfilled:

$$\pi(d)\left(2F_S(d) - 1\right) + (d - E[S])\left(F_S(d) - 1\right) = 0,$$

$$\pi(d) < \frac{2F_S(d)\left(1 - F_S(d)\right)^2}{f_S(d)},$$

$$d > 0.$$

Proof. The first order condition of optimality is

$$[\pi(d)(d - E[S] + \pi(d))]' = 0.$$

Considering that $\pi(d) = \int_d^\infty (1 - F_S(s))\,ds$, this condition is

$$(3.2) \qquad \pi(d)F_S(d) + (F_S(d) - 1)(d - E[S] + \pi(d))$$
$$= \pi(d)(2F_S(d) - 1) + (d - E[S])(F_S(d) - 1) = 0.$$

The second order condition for the maximization is

$$(3.3) \qquad [\pi(d)(d - E[S] + \pi(d))]'' = f_S(d)(d - E[S] + \pi(d))$$
$$+ (F_S(d) - 1)\,2f_S(d) < 0.$$

Isolating $(d - E[S] + \pi(d))$ from (3.2) and substituting in (3.3), the condition

$$\pi(d) < \frac{2F_S(d)(1 - F_S(d))^2}{f_S(d)}$$

is obtained. $\qquad\qquad\square$

3.2. Corollary. If S has a symmetric density function, $d = E[S]$ is the only finite point that fulfils condition (3.2).

Proof. If S has a symmetric density function, $F_S(E[S]) = 0.5$, then (3.2) is fulfilled if and only if $d = E[S]$. $\qquad\square$

Note: An equivalent expression to (3.2) can be found in Borch (1974) as well as the value of d that fulfils this condition when S follows an exponential distribution.

As an alternative, instead of maximizing the variance reduction in absolute value, we could apply the criterion of maximizing the coefficient of correlation. In this case, the conditions that must fulfil the optimal point are complex but easy to obtain. In order to be concise we only include in the paper (without proof) the necessary conditions.

3.3. Proposition. The optimal point of program

$$(3.4) \qquad \max_d r(SI, SR) = \frac{Cov\,[SI, SR]}{\sqrt{V\,[SR]\,V\,[SI]}} \text{ subject to } 0 < d$$

fulfil the necessary condition

$$2\frac{Cov[SI, SR]'}{Cov[SI, SR]} = \frac{V[SI]'}{V[SI]} + \frac{V[SR]'}{V[SR]},$$

being

$$Cov[SI, SR]' = \pi(d)(2F_S(d) - 1) + (d - E[S])(F_S(d) - 1),$$
$$V[SI]' = -2(F_S(d) - 1)(d - E[S] + \pi(d)),$$
$$V[SR]' = -2\pi(d)F_S(d).$$

3.4. Example. We assume that the total cost of a period has the following characteristics: $E\,[S] = 1$, $V\,[S] = 2$ and skewness $\gamma_1(S) = \frac{3}{\sqrt{2}}$. In Table 2 we show the maximum points and the maximum values of the covariance and the coefficient of correlation that are obtained using the gamma, the translated gamma and the normal approximations.

Table 2. Optimal points and maximum values of covariance and correlation coefficient. Stop-loss contract

	d^*	$Cov[SI(d^*), SR(d^*)]$	d^*	$r[SI(d^*), SR(d^*)]$
gamma	2.19654	0.326122	1.3598	0.499926
translated gamma	1.89158	0.324196	1.27352	0.490588
normal	1	0.31831	1	0.466942

If the stop-loss reinsurance contract has also a maximum m, the maximization program, from (2.10) is

$$(3.5) \qquad \max_{d,m} \pi(d)\,(d - E[S] + \pi(d)) - (2d - m)\,\pi(m) \text{ subject to } 0 < d < m$$

3.5. Proposition. The optimal point of program (3.5) is a value of $(d, m) \in \Re_+^2$ such that the following conditions are fulfilled:

$$\pi(d)F_S(d) + (F_S(d) - 1)\,(d - E[S] + \pi(d)) - 2\pi(m) = 0,$$
$$\pi(m) - (2d - m)\,(F_S(d) - 1) = 0,$$
$$f_S(d)\,(2\pi(d) + d - E[S]) + 2F_S(d)\,(F_S(d) - 1) < 0,$$

$$f_S(d)\,(2\pi(d) + d - E[S]) + 2F_S(d)\,(F_S(d) - 1) < \frac{4(F_S(m) - 1)^2}{mf_S(m) + 2\,(F_S(m) - 1)},$$
$$0 < d < m.$$

Proof. The first order condition of optimality is

$$\begin{cases} \frac{\partial[\pi(d)(d - E[S] + \pi(d)) - (2d - m)\pi(m)]}{\partial d} = 0, \\[2ex] \frac{\partial[\pi(d)(d - E[S] + \pi(d)) - (2d - m)\pi(m)]}{\partial m} = 0. \end{cases}$$

Considering that $\pi(d) = \int_d^\infty (1 - F_S(s))\,ds$, this condition is

$$\begin{cases} \pi(d)F_S(d) + (F_S(d) - 1)\,(d - E[S] + \pi(d)) - 2\pi(m) = 0, \\[2ex] \pi(m) - (2d - m)\,(F_S(d) - 1) = 0. \end{cases}$$

The inequalities are obtained applying the second order condition for a maximum. $\square$

Using the values of Example 3.4, we numerically show that there is no solution of the program (3.5), although with the normal distribution the point $(d, m) = (2.38, 3.63)$ is a local optimum that fulfils the conditions included in Proposition 3.5. For illustration, in Figure 1, the covariance for the normal distribution is plotted.
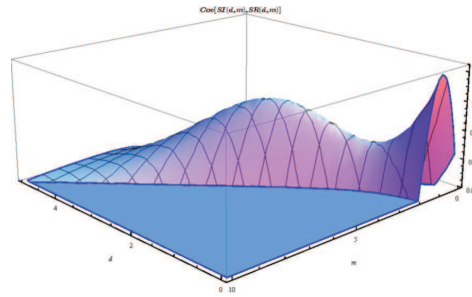
Figure 1. Covariance between the costs of the insurer and the reinsurer with a normal distribution

4. Survival probabilities in one period

The survival probability is one of the most important measures of the solvency of an insurer/reinsurer. The survival probability in one period of an insurer considering only the underwriting risk, can be calculated knowing the distribution of the cost of the insurer, the reserves at the beginning of the period and the premium earned by the insurer to cover the insured risk. If a stop-loss reinsurance contract is agreed, the survival probability of the insurer is obviously different and needs to be calculated again with the new parameters; but, as in this case, if the payment of the claims depends on the two parts, the joint survival probability of insurer and reinsurer is also a quantity of interest.

Let $PT > 0$ be the premium earned by the insurer in the period; let $PR > 0$ be the reinsurer's premium; let $uI \geq 0$ and $uR \geq 0$ be the initial reserves of the insurer and the reinsurer, respectively. It is then possible to incorporate in the model an economic constraint: the reinsurer's premium must be less than the premium earned by the insurer in the period, $0 < PR < PT$.

The survival probability is in fact a particular case of a family of probabilities regarding the technical result at the end of the period. Let $\varphi(u, P, \alpha)$ be the probability that the technical result (initial capital (u) plus earned premiums (P) minus aggregated claims (S)) of an insurer is greater or equal to α,

$$\varphi(u, P, \alpha) = P\left[u + P - S \geq \alpha\right].$$

The technical result has a natural maximum value, $u+P$, that is attained when no claims occur during the period. As $\varphi(u, P, \alpha) = 0$ for $\alpha > u + P$, and $\varphi(u, P, \alpha) = 1$ for $\alpha < 0$, we can consider that $0 \leq \alpha \leq u + P$.

Survival probability $\phi(\cdot)$ is a particular case of $\varphi(\cdot)$ that is obtained considering $\alpha = 0$.

4.1. Stop-loss reinsurance with priority d. Probabilities regarding the technical result of the insurer, $\varphi_I(uI, d, PR, PT, \alpha)$, are

$$
\begin{aligned}
\varphi_I(uI, d, PR, PT, \alpha) &= P\left[uI + PT - PR - SI \geq \alpha\right] \\
&= P\left[SI \leq uI + PT - PR - \alpha\right] \\
&= F_{SI}(uI + PT - PR - \alpha)
\end{aligned}
$$

and from (2.1),

$$(4.1) \quad \varphi_I(uI, d, PR, PT, \alpha) = \begin{cases} F_S(uI + PT - PR - \alpha) & \text{if } uI + PT - PR - \alpha < d, \\ 1 & \text{if } uI + PT - PR - \alpha \geq d. \end{cases}$$

The probabilities regarding the technical result of the reinsurer, $\phi_R(uR, d, PR, \alpha)$, are

$$\begin{aligned} \varphi_R(uR, d, PR, \alpha) &= P\left[uR + PR - SR - \alpha \geq 0\right] = P\left[SR \leq uR + PR - \alpha\right] \\ &= F_{SR}(uR + PR - \alpha) \end{aligned}$$

and from (2.2),

(4.2) $\varphi_R(uR, d, PR, \alpha) = F_S(uR + PR + d - \alpha).$

The joint probabilities regarding the technical result of both the insurer and the reinsurer, $\varphi_{I,R}(uI, uR, d, PR, PT, \alpha_1, \alpha_2)$, are

$$\begin{aligned} &\varphi_{I,R}(uI, uR, d, PR, PT, \alpha_1, \alpha_2) \\ &\quad = P\left[SI \leq uI + PT - PR - \alpha_1, SR \leq uR + PR - \alpha_2\right] \end{aligned}$$

and from (2.7),

(4.3) $\varphi_{I,R}(uI, uR, d, PR, PT, \alpha_1, \alpha_2)$

$$= \begin{cases} F_S(uI + PT - PR - \alpha_1) & \text{if } uI + PT - PR - \alpha_1 < d, \\ F_S(uR + PR + d - \alpha_2) & \text{if } uI + PT - PR - \alpha_1 \geq d. \end{cases}$$

The joint survival probability of the insurer and the reinsurer $\phi_{I,R}(uI, uR, d, PR, PT)$ is obtained when both α_1 and α_2 are equal to zero,

$$\phi_{I,R}(uI, uR, d, PR, PT) = \varphi_{I,R}(uI, uR, d, PR, PT, 0, 0).$$

4.2. Stop-loss reinsurance with priority d and maximum m. The joint probabilities of the insurer, $\varphi_I(uI, d, m, PR, PT, \alpha)$, are

$$\varphi_I(uI, d, m, PR, PT, \alpha) = F_{SI(d,m)}(uI + PT - PR - \alpha)$$

and from (2.8)

$$\varphi_I(uI, d, m, PR, PT, \alpha)$$
$$= \begin{cases} F_S(uI + PT - PR - \alpha) & \text{if } uI + PT - PR - \alpha < d, \\ F_S(uI + PT - PR - \alpha + m - d) & \text{if } uI + PT - PR - \alpha \geq d. \end{cases}$$

The joint probabilities of the reinsurer, $\varphi_R(uR, d, m, PR, \alpha)$, are

$$\varphi_R(uR, d, m, PR, \alpha) = F_{SR(d,m)}(uR + PR - \alpha)$$

and from (2.9)

$$\varphi_R(uR, d, m, PR, \alpha)$$
$$= \begin{cases} F_S(uR + PR + d - \alpha) & \text{if } uR + PR - \alpha < m - d, \\ 1 & \text{if } uR + PR - \alpha \geq m - d. \end{cases}$$

The joint probabilities of the insurer and the reinsurer are

$$\begin{aligned} &\varphi_{I,R}(uI, uR, d, m, PR, PT, \alpha_1, \alpha_2) \\ &\quad = P\left[SI \leq uI + PT - PR - \alpha_1, SR \leq uR + PR - \alpha_2\right] \end{aligned}$$

and from (2.11)

$$(4.4) \quad \varphi_{I,R}(uI, uR, d, m, PR, PT, \alpha_1, \alpha_2)$$

$$= \begin{cases} F_S(uI{+}PT{-}PR{-}\alpha_1) & \text{if } uI{+}PT{-}PR{-}\alpha_1 < d, \\ F_S(d) & \text{if } uI{+}PT{-}PR{-}\alpha_1 \geq d \text{ and } uR{+}PR{-}\alpha_2 = 0, \\ F_S(uR{+}PR{+}d{-}\alpha_2) & \text{if } uI{+}PT{-}PR{-}\alpha_1 \geq d \text{ and } 0 < uR{+}PR{-}\alpha_2 < m{-}d, \\ F_S(m) & \text{if } uI{+}PT{-}PR{-}\alpha_1 = d \text{ and } uR{+}PR{-}\alpha_2 \geq m{-}d, \\ F_S(uI{+}PT{-}PR{-}\alpha_1{+}m{-}d) & \text{if } uI{+}PT{-}PR{-}\alpha_1 > d \text{ and } uR{+}PR{-}\alpha_2 \geq m{-}d. \end{cases}$$

The joint survival probability of the insurer and the reinsurer is obtained when both α_1 and α_2 are equal to zero,

$$\phi_{I,R}(uI, uR, d, m, PR, PT) = \varphi_{I,R}(uI, uR, d, m, PR, PT, 0, 0).$$

5. Optimal joint survival probability in one period

In this section, we are interested in solving two different optimization problems related with the joint survival probability of the insurer and the reinsurer in one period.

In the first optimization problem, the reinsurance premium is fixed (as it is the total premium PT) and so are the initial values of the reserves of the insurer and the reinsurer. In addition, the parameters of the reinsurance maximize the joint survival probability. This probability is a function of the parameters of the reinsurance, d or d and m. Propositions 5.1 and 5.7 solve this problem. In this case, the insurer has a fixed amount of money available to purchase the reinsurance protection and we look for the most efficient stop-loss contract since it offers the lowest risk (measured by the joint probability of ruin) for this given value of the reinsurer premium. This idea of finding the parameters of the reinsurance contract that maximize the joint survival probability when the premiums of the insurer and the reinsurer are fixed, can also be found in [37] and [22], where the authors consider an excess of loss risk model when the number of claims follows a Poisson process. The assumptions of our model are totally different but, in Proposition 5.1 and 5.7 we consider the same maximization problem.

It is usually considered that PR is a function of the parameters of the stop-loss reinsurance (d, m) and the total cost S. In that instance, the reinsurer would apply for the calculation of the premium some of the usual criteria, for instance, the expected value, variance and standard deviation principles (for more details see [36]). We adopt as a criterion for the calculation of the reinsurer's premiums the maximization of the joint survival probability, given as fixed both the values of the parameters of the reinsurance contract and the initial values of the reserves of the insurer and the reinsurer. Then, in the second optimization problem, the joint survival probability is considered to be a function of the reinsurance premium, PR. Propositions 5.5 and 5.9 tackle this problem.

5.1. Proposition. In a stop-loss reinsurance with priority d, the program

$$\max_d \phi_{I,R}(uI, uR, d, PR, PT) \text{ subject to } 0 < d$$

has as a maximum value $\phi_{I,R}^*(uI, uR, PR, PT) = F_S(uI + uR + PT)$, being the optimal point $d^*(uI, uR, PR, PT) = uI + PT - PR$.

Proof. The joint survival probability to be maximized, (4.3), is a step function built with the distribution function of the total cost. Since $F_S(x)$ is increasing in x and $uI + PT - PR < d < uR + PR + d$, for all $d > uI + PT - PR$, $F_S(uI + PT - PR) \leq F_S(uR + PR + uI + PT - PR) = F_S(uR + uI + PT)$, then it is immediate that $\phi_{I,R}^*(uI, uR, PR, PT)$ is attained at $d^*(uI, uR, PR, PT) = uI + PT - PR$. $\qquad \square$

5.2. Remark (Proposition 5.1). In Figure 2, we plot the two-step function indicating the argument of the distribution function of the total cost in (4.3), as a function of d.
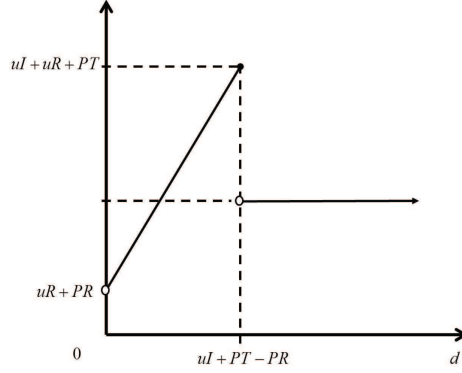

Figure 2. the argument of the distribution function of the total cost in (4.3) as a function of d

5.3. Remark (Proposition 5.1). For this optimal reinsurance, in which the maximum joint survival probability of the insurer and the reinsurer is obtained, the individual survival probability of the insurer (4.1) is $\phi_I(uI, uI + PT - PR, PR, PT) = 1$, whereas the individual survival probability of the reinsurer (4.2) is $\phi_R(uR, uI + PT - PR, PR) = F_S(uI + uR + PT) = \phi^*_{I,R}(uI, uR, PR, PT)$. Hence, the insurer, with this optimal reinsurance, increases his/her individual survival probability (compared to the absence of reinsurance) in $(1 - P[S \leq uI + PT]) > 0$.

5.4. Remark (Proposition 5.1). If the initial capitals of the insurer and the reinsurer are zero, then the maximum joint survival probability is obtained when the priority d is equal to the net premium of the insurer.

5.5. Proposition. In a stop-loss reinsurance with priority d, the program

$$\max_{PR} \phi_{I,R}(uI, uR, d, PR, PT) \text{ subject to } 0 < PR < PT$$

only provides a solution if $uI < d < uI + PT$, being in that case the maximum value $\phi^*_{I,R}(uI, uR, d, PT) = F_S(uI + uR + PT)$, which is reached for $PR^*(uI, uR, d, PT) = uI + PT - d$.

Proof. It is developed in a similar way as in Proposition 5.1. Since $F_S(x)$ is increasing in x, if $d \in (uI, uI + PT)$, for all $0 < PR \leq uI + PT - d$, $F_S(uR + uI + PT - d + d) = F_S(uR + uI + PT) \geq F_S(uR + PR + d)$ and for all $uI + PT - d < PR < PT$, $F_S(uI + uR + PT) > F_S(uI + PT - PR)$. If $d > uI + PT$, for all $0 < PR < PT$, $F_S(uI + PT - PR)$ does not have a maximum. If $d < uI$, for all $0 < PR < PT$, $F_S(uR + PR + d)$ does not have a maximum. Then, the program provides a solution only if $uI < d < uI + PT$ and $\phi^*_{I,R}(uI, uR, d, PT)$ is attained at $PR^*(uI, uR, d, PT) = uI + PT - d$. $\square$

5.6. Remark (Proposition 5.5). In Figure 3, we plot the two-step function indicating the argument of the distribution function of the total in (4.3), as a function of PR when $uI < d < uI + PT$.
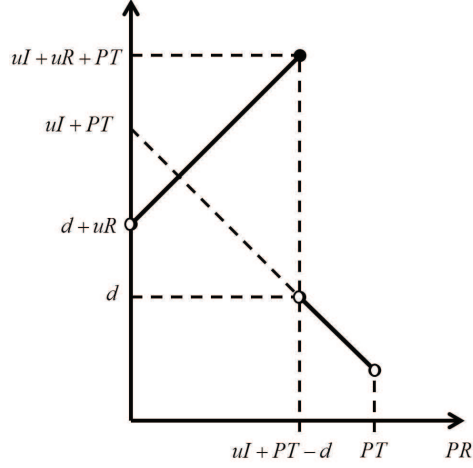
Figure 3. The argument of the distribution function of the total cost in (4.3) as a function of PR when $uI < d < uI + PT$

5.7. Proposition. In a stop-loss reinsurance with priority d and maximum m, the program

$$\max_{(d,m)} \phi_{I,R}(uI, uR, d, m, PR, PT) \text{ subject to } 0 < d < m$$

has a maximum value $\phi_{I,R}^*(uI, uR, PR, PT) = F_S(uI + uR + PT)$. This maximum is attained at the non-convex set

$$\left\{(d,m) \in \Re_+^2 \mid d \leq uI + PT - PR \text{ and } m = uR + PR + d\right\}$$
$$\cup \left\{(d,m) \in \Re_+^2 \mid d = uI + PT - PR \text{ and } m > uR + PR + d\right\}$$

Proof. The joint survival probability to be maximized now is (4.4), a piecewise function built with the distribution function of the total cost. Since $F_S(x)$ is increasing in x, for all $(d,m) \in \Re_+^2$ such that $d \leq uI + PT - PR$ and $m > uR + PR + d$, $F_S(uR + PR + d) \leq F_S(uR + PR + uI + PT - PR) = F_S(uR + uI + PT)$. For all $(d,m) \in \Re_+^2$ such that $d < uI + PT - PR$ and $m \leq uR + PR + d$, $F_S(uI + PT - PR + m - d) \leq F_S(uI + PT - PR + uR + PR) = F_S(uI + uR + PT)$. Taking into account that $F_S(uI + uR + PT) > F_S(uI + PT - PR)$, the proof is completed. $\square$

5.8. Remark (Proposition 5.7). In Figure 4, we plot the step function indicating the argument of the distribution function of the total cost in (4.4) as a function of d and m and its level curves. For $PT = 1$, $PR = 0.4$ and $uI = uR = 0$, the maximum value is 1 and the set of optimal points are $\{d \leq 0.6 \text{ and } m = 0.4 + d\} \cup \{d = 0.6 \text{ and } m > 0.4 + d\}$.
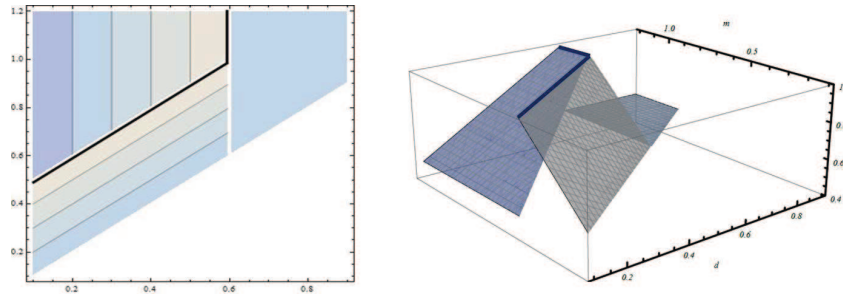
Figure 4. The argument of the distribution function of the total cost in (4.4) as a function of d and m (right graph) and its level curves (left graph) (for $PT = 1$, $PR = 0.4$ and $uI = uR = 0$)

5.9. Proposition. In a stop-loss reinsurance with priority d and maximum m, the program

$$\max_{PR} \phi_{I,R}(uI, uR, d, m, PR, PT) \text{ subject to } 0 < PR < PT$$

only provides solutions if one of the two following conditions is fulfilled: $uI < d < uI + PT$ and $m \geq uI + uR + PT$ (first condition) or $m < uI + uR + PT$ and $PT + uR > m - d > uR$ (second condition).

In that case, the maximum value is $\phi_{I,R}^*(uI, uR, d, m, PT) = F_S(uI + uR + PT)$, being the optimal premiums of the reinsurer

$$PR^*(uI, uR, d, m, PT) = \begin{cases} uI + PT - d & \text{if } uI < d < uI + PT \text{ and } m \geq uI + uR + PT, \\ m - d - uR & \text{if } m < uI + uR + PT \text{ and } PT + uR > m - d > uR. \end{cases}$$

Proof. Taking into account (4.4) and that $0 < PR < PT$, lets first consider the case that $d \in (uI, uI + PT)$. If $uI + PT - d < m - d - uR$, for all $0 < PR \leq uI + PT - d$, $F_S(uR + uI + PT - d + d) = F_S(uR + uI + PT) \geq F_S(uR + PR + d)$ and for all $uI + PT - d < PR < PT$, $F_S(uI + uR + PT) > F_S(uI + PT - PR)$. If $uI + PT - d = m - d - uR$, for all $0 < PR \leq uI + PT - d$, $F_S(m) = F_S(uI + uR + PT) > F_S(uR + PR + d)$ and for all $uI + PT - d < PR < PT$, $F_S(uI + uR + PT) > F_S(uI + PT - PR)$.

Secondly, lets consider that $(m - d) \in (uR, uR + PT)$ and $uI + PT - d > m - d - uR$, for all $0 < PR \leq m - d - uR$, $F_S(uI + PT - m + d + uR + m - d) = F_S(uI + PT + uR) > F_S(uR + PR + d)$ and for all $PR > m - d - uR$, $F_S(uI + uR + PT) > F_S(uI + PT - PR + m - d) > F_S(uI + PT - PR)$.

It is then easy to demonstrate that for all the other possibles values of d and m, the maximum does not exist. $\square$

5.10. Remark (Proposition 5.9)**.** In Figure 5, the argument of the distribution function of the total cost in (4.4) is plotted as a function of PR for the values d and m for which the joint survival probability has a maximum. It can be divided into three cases depending on whether $uI + PT - d$ is less, equal or greater than $m - d - uR$.
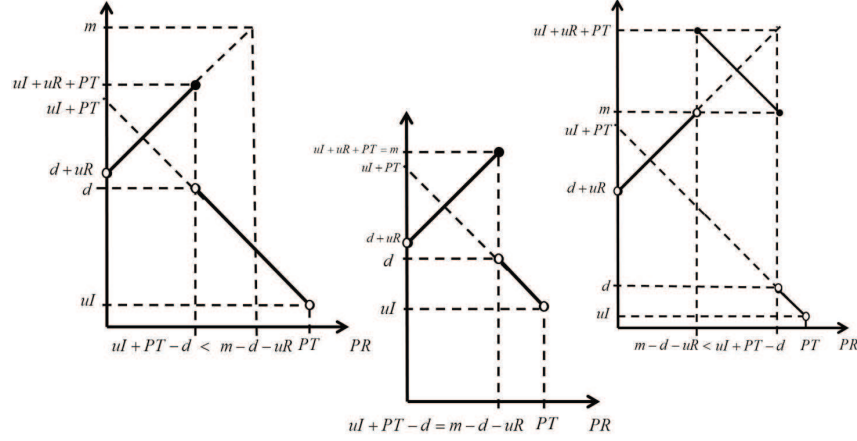
Figure 5. The argument of the distribution function of the total cost in (4.4) as a function of PR when $uI + PT - d \lessgtr m - d - uR$. The graph on the left considers $uI + PT - d < m - d - uR$; the graph on the middle considers $uI + PT - d = m - d - uR$ and the graph on the right considers $uI + PT - d > m - d - uR$.

From Propositions 5.1, 5.5, 5.7 and 5.9, the maximum joint survival probability (considering the constraints), when it exists, is equal to

$$F_S \left(uI + uR + PT \right).$$

From the first definition of ruin in a bivariate risk process ([12]), the joint survival probability equals to the minimum between the survival probability of the insurer and the survival probability of the reinsurer, and this is also true at the optimal points. Then, at the optimal points, the survival probability of the insurer or the reinsurer must be equal to $F_S \left(uI + uR + PT \right)$, and the other must be greater than this value. Table 3 includes the values of the survival probability of the insurer and the reinsurer at the points that maximize the joint survival probability.

Table 3. ϕ_I and ϕ_R at the optimal points for the different optimization problems

	Φ_I	Φ_R
$d^* = uI + PT - PR$ (Prop. 5.1)	1	$F_S(uI + uR + PT)$
$PR^* = uI + PT - d,$ if $uI < d < uI + PT$ (Prop. 5.5)	1	$F_S(uI + uR + PT)$
$\{(d,m) \in \Re_+^2 \mid d \leq uI + PT - PR \text{ and } m = uR + PR + d\}$ (Prop. 5.7)	$F_S(uI + uR + PT)$	1
$\{(d,m) \in \Re_+^2 \mid d = uI + PT - PR \text{ and } m > uR + PR + d\}$ (Prop. 5.7)	$F_S(m),$ $m > uI + uR + PT$	$F_S(uI + uR + PT)$
$PR^* = uI + PT - d,$ if $uI < d < uI + PT$ and $m \geq uI + uR + PT$ (Prop. 5.9)	$F_S(m),$ $m > uI + uR + PT$	$F_S(uI + uR + PT)$
$PR^* = m - d - uR,$ if $m < uI + uR + PT$ and $PT + uR > m - d > uR$ (Prop. 5.9)	$F_S(uI + uR + PT)$	1

5.11. Example. Using the data for the total cost in Example 3.4, assume first that a stop-loss contract with priority d is agreed and that the initial reserves of the insurer and the reinsurer are zero. The premium fixed by the insurer is 1.8 (so if the criterion is the expected value, the security loading applied by the insurer is 80%). The premium earned by the reinsurer is fixed and equal to $PR = 0.5, ..., 1.5$. In Table 4, we calculate the priority that maximizes the joint survival probability, using Proposition 5.1, and the difference between the premium earned by the reinsurer and the expectation of its cost, $PR - E[SR(d^*)]$, if the gamma (G), the translated gamma (TG) or the normal approximations (N), are used. In Table 4, we also include the net security premium for the insurer, that is given by $1.8 - PR - E[SI(d^*)]$. These two quantities included in Table 4, permit us to calculate the security loading of the reinsurer and the insurer (for the insurer it is the net loading) included in the optimal strategy. These security loadings are shown in Table 5. In Table 6, we calculate the maximal joint survival probability (that equals to the survival probability of the reinsurer (Remark 5.3)), and the increase in the survival probability of the insurer if the optimal reinsurance is agreed, when the gamma, the translated gamma or the normal approximations, are used.

Table 4. Priority, security premium for the reinsurer and net security premium for the insurer if the joint survival probability is maximized for several fixed reinsurer's premiums

		$PR - E[SR(d^*)]$			$1.8 - PR - E[SI(d^*)]$		
PR	d^*	G	TG	N	G	TG	N
0.5	1.3	0.1013	0.0820	0.0732	0.6987	0.7180	0.7268
0.6	1.2	0.1750	0.1518	0.1302	0.6250	0.6482	0.6698
0.7	1.1	0.2466	0.2195	0.1844	0.5534	0.5805	0.6156
0.8	1	0.3161	0.2847	0.2358	0.4839	0.5153	0.5642
0.9	0.9	0.3831	0.3474	0.2844	0.4169	0.4526	0.5156
1	0.8	0.4474	0.4072	0.3302	0.3526	0.3928	0.4698
1.1	0.7	0.5087	0.4641	0.3732	0.2913	0.3359	0.4268
1.2	0.6	0.5667	0.5178	0.4134	0.2333	0.2822	0.3866
1.3	0.5	0.6209	0.5679	0.4509	0.1791	0.2321	0.3491
1.4	0.4	0.6706	0.6143	0.4858	0.1294	0.1857	0.3142
1.5	0.3	0.7151	0.6565	0.5181	0.0849	0.1435	0.2819

Table 5. Security loadings of the insurer and the reinsurer if the joint survival probability is maximized for several fixed reinsurer's premiums

		$\frac{100(PR - E[SR(d^*)])}{E[SR(d^*)]}$			$\frac{100(1.8 - PR - E[SI(d^*)])}{E[SI(d^*)]}$		
PR	d^*	G	TG	N	G	TG	N
0.5	1.3	25.42	19.61	17.14	116.18	123.38	126.81
0.6	1.2	41.17	33.88	27.71	108.70	117.50	126.34
0.7	1.1	54.40	45.67	35.76	101.24	111.76	127.08
0.8	1	65.31	55.25	41.80	93.78	106.31	129.46
0.9	0.9	74.11	62.85	46.20	86.31	101.18	134.13
1	0.8	80.97	68.70	49.29	78.81	96.44	142.29
1.1	0.7	86.04	72.99	51.34	71.26	92.24	156.26
1.2	0.6	89.49	75.89	52.55	63.61	88.82	181.17
1.3	0.5	91.42	77.57	53.11	55.83	86.63	231.32
1.4	0.4	91.94	78.18	53.14	47.82	86.69	366.30
1.5	0.3	91.12	77.83	52.76	39.44	91.69	1559.80

Table 6. Maximal joint survival probability and the increase in the survival probability of the insurer

	G	TG	N
$\phi^*_{I,R} = \phi_R = F_S(1.8)$	0.8202875	0.7955186	0.7141962
$1 - P\left[S \leq 1.8\right]$	0.1797125	0.2044814	0.2858038

As it is reflected in Table 6, obviously, the maximal joint survival probability ($\phi^*_{I,R} = \phi_R = F_S(1.8)$) and the increase in the survival probability of the insurer due to the optimal reinsurance ($1 - P\left[S \leq 1.8\right]$), is always the same and is independent of the specific optimal combination of the reinsurer's premium and priority. Hence, from the point of view of the joint survival probability, the reinsurer survival probability and the insurer survival probability, all the alternative combinations of the reinsurer's premium and priority included in Table 5 are indifferent. The differences in the security loading applied by the reinsurer and the net security loading of the insurer do not modify the optimal survival probabilities.

Assume now that the insurer and the reinsurer have positive initial reserves, and that the reinsurer's premium is 0.5 and the total premium is 1.8. From Proposition 5.1, the optimal priority is $d^* = uI + 1.3$, and the maximum joint survival probability is $F_S(uI + uR + 1.8) = \phi^*_{I,R}$. Table 7 includes the optimal priority and the maximum joint survival probability for several combinations of initial capitals, using the translated gamma approximation.

Table 7. d^* and $\phi^*_{I,R}$ as functions of initial capitals, for $PR = 0.5$ and $PT = 1.8$

uI/uR		0.25	0.5	0.75	1
0.25	d^*	1.55	1.55	1.55	1.55
	$\phi^*_{I,R}$	0.855824	0.8788329	0.8981223	0.9143059
0.5	d^*	1.8	1.8	1.8	1.8
	$\phi^*_{I,R}$	0.8788329	0.8981223	0.9143059	0.9278928
0.75	d^*	2.05	2.05	2.05	2.05
	$\phi^*_{I,R}$	0.8981223	0.9143059	0.9278928	0.9393062
1	d^*	2.3	2.3	2.3	2.3
	$\phi^*_{I,R}$	0.9143059	0.9278928	0.9393062	0.9488984

Table 7 shows that when different combinations of initial capitals are considered for a specific uI, the optimal priority does not vary if uR is increased. This result is due to the fact that d^* does not depend on the initial capital of the reinsurer. However, the joint survival probability does change with increasing values.

6. Concluding remarks

In the stop-loss reinsurance contract, the cost of the claims of both the insurer and the reinsurer are related. This work contributes to the analysis of the optimal stop-loss reinsurance in one period, from the joint point of view of the insurer and the reinsurer and then, incorporating the aforementioned relation.

Several optimal problems with two different objective functions are studied. First, using the total variance risk measure, we analyze the optimal reinsurance parameters

(retention and maximum) that maximize the covariance (and also the coefficient of correlation) between the cost of the claims of the insurer and the reinsurer. Second, two optimal problems with the same objective function, the joint survival probability of the insurer and the reinsurer in one period, are solved. The maximum joint survival probability always exists if the reinsurance premium is fixed, and is equal to the probability that the total cost is less than, or equal, to the sum of the total premium and the two initial capitals. This maximum is attained for a unique value of the priority or for a non-convex set of priority and maximum if the reinsurance contract includes a maximum. If we consider that the parameters of the reinsurance contract are fixed, the optimal reinsurance premium and the maximum joint survival probability do not always exist, and in case they exist, the maximum is exactly the same as in the first problem. These findings can be of great help for the insurer and reinsurer in their decision making process.

Acknowledgements

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