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Estimation of $P\{X \leq Y\}$ for geometric-Poisson model

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Abstract

In this paper we estimate $R = P\{X \leq Y\}$ when X and Y are independent random variables from geometric and Poisson distribution respectively. We find maximum likelihood estimator of R and its asymptotic distribution. This asymptotic distribution is used to construct asymptotic confidence intervals. A procedure for deriving bootstrap confidence intervals is presented. UMVUE of R and UMVUE of its variance are derived and also the Bayes estimator of R for conjugate prior distributions is obtained. Finally, we perform a simulation study in order to compare these estimators.

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1. Introduction

In reliability theory the main parameter is the reliability of a system. The system fails if the applied stress X is greater than strength Y, so $R = P\{X \leq Y\}$ is a measure of system performance. Its estimation is one of the main goals and it has been widely studied in statistical literature.

The problem was first introduced by Birnbaum [4]. The estimation of R when X and Y are normally distributed has been considered by Downtown [7], Govidarajulu [9], Woodward and Kelley [26] and Owen [20]. Tong [24],[25], studied the case when X and Y were exponentially distributed. Exponential case with common location parameter was examined by Baklizi and Quader El-Masri [2]. The gamma case was studied by Constantine and Karson [5], Ismail et al. [12] and Constantine et al. [6]. Kundu and Gupta considered generalized exponetial case [16]. Kakade et al. [14] studied the exponentiated Gumbel case. Gompertz distribution was examined by Saraçoglu et al. [22], and the generalized Pareto case was considered by Rezaei et al. [21]. Kundu and Gupta [17] examined the case of Weibull distribution. Recently, the Topp-Leone distribution was studied by Genç [8]. Most of results are collected in Kotz et al. [15].

The majority of papers in this area deal with continuous probability distributions. However, there are some applications where stress and strength can have discrete distributions. For example, this is the case when the stress is the number of shocks the product undergoes and the strength is the number of shocks the product can withstand. Maiti [19] and Ahmad et al. [1] studied the geometric case. The negative binomial distribution was considered by Ivshin and Lumelskii [13] and Sathe and Dixit [23]. Belyaev and Lumelskii [3] examined the Poisson case.

In all mentioned papers both stress and strength come from the same type of distribution. In this paper we focus on the case when X and Y follow different types of distribution, namely geometric and Poisson distribution.

If we consider the stress to be the demand for some product, and the strength its supply, which are discrete in nature, then it might be convenient to model them with geometric and Poisson distributions.

Another motivating example can be the following. An employer is interviewing potential candidates for a vacant position. The number of interviews he needs to conduct until he finds suitable candidate follows geometric distribution, while the number of persons that apply for that job during a certain period of time follows Poisson distribution. Therefore R is the probability that the employer will find the right candidate.

Let X and Y be independent random variables with geometric $\mathcal{G}(p)$ and Poisson $\mathcal{P}(\lambda)$ distribution, respectively, where probability p and positive value λ are unknown parameters. Their probability mass functions are

$$P\{X = x\} = (1-p)^{x-1}p, \quad x = 1, 2, \ldots,$$

and

$$P\{Y = y\} = \frac{e^{-\lambda}\lambda^y}{y!}, \quad y = 0, 1, \ldots.$$

Then the reliability of the system is

$$\begin{aligned} R &= P\{X \le Y\} = \sum_{y=1}^{\infty}\sum_{x=1}^{y} P\{X = x, Y = y\} \\ &= \sum_{y=1}^{\infty}\sum_{x=1}^{y}(1-p)^{x-1}p\frac{e^{-\lambda}\lambda^y}{y!} = \sum_{y=1}^{\infty}\frac{e^{-\lambda}\lambda^y}{y!}(1-(1-p)^y) \\ &= 1 - e^{-\lambda} - \sum_{y=1}^{\infty}\frac{e^{-\lambda}(\lambda(1-p))^y}{y!} \\ &= 1 - e^{-\lambda} - e^{-\lambda p}\sum_{y=1}^{\infty}\frac{e^{-\lambda(1-p)}(\lambda(1-p))^y}{y!} \\ &= 1 - e^{-\lambda p}. \end{aligned} \tag{1.1}$$

In the following sections we study various estimators of R. In section 2 the maximum likelihood estimator (MLE) of R and its asymptotic distribution are derived. We use that to construct asymptotic and bootstrap confidence intervals. The uniformly minimum variance unbiased estimator (UMVUE) of R and UMVUE of its variance are obtained in section 3. Bayes estimator of R with respect to mean square error is found in section 4. In section 5 we perform a simulation study and compare the obtained estimators.

2. MLE of R and its Asymptotics

Let $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_m)$ be the samples from the distributions of random variables X and Y. Therefore, the log-likelihood function of combined sample is

$$\ln L(p,\lambda) = \big(\sum_{k=1}^{n} x_k - n\big)\ln(1-p) + n\ln p - m\lambda + \ln\lambda\sum_{k=1}^{m} y_k - \ln\prod_{k=1}^{m} y_k!\ .$$

Solving the likelihood equations with respect to p and λ we get that the MLEs for p and λ are

$$\widetilde{p} = \frac{1}{\overline{X}}, \quad \widetilde{\lambda} = \overline{Y}.$$

Using the invariance property of MLE, from (1.1) we get the MLE of R

$$\widetilde{R} = 1 - e^{-\frac{\overline{Y}}{\overline{X}}}. \tag{2.1}$$

2.1. Asymptotic Distribution. In the following two theorems we shall find the asymptotic distributions of $(\tilde{p}, \tilde{\lambda})$ and $\tilde{R}$.

2.1. Theorem. *Let the ratio $\frac{n}{m}$ converge to a positive number s when both n and m tend to infinity. Then*

$$(\sqrt{n}(\widetilde{p} - p), \sqrt{n}(\widetilde{\lambda} - \lambda)) \xrightarrow[n\to\infty]{D} \mathcal{N}_2(\mathbf{0}, J(p,\lambda)),$$

where

$$J(p,\lambda) = \begin{bmatrix} p^2(1-p) & 0 \\ 0 & s\lambda \end{bmatrix}.$$

Proof. Since

$$-E\Big(\frac{\partial^2 \ln L}{\partial p^2}\Big)=\frac{n}{p^2(1-p)}$$

and

$$-E\Big(\frac{\partial^2 \ln L}{\partial \lambda^2}\Big)=\frac{m}{\lambda},$$

from the asymptotic normality of maximum likelihood estimator (see [11]) it follows that

$$\sqrt{n}(\widetilde{p}-p)\xrightarrow[n\to\infty]{D}\mathcal{N}(0,p^2(1-p))$$

and

$$\sqrt{m}(\widetilde{\lambda}-\lambda)\xrightarrow[m\to\infty]{D}\mathcal{N}(0,\lambda).$$

Then

$$\sqrt{n}(\widetilde{\lambda}-\lambda)\xrightarrow[n\to\infty]{D}\mathcal{N}(0,s\lambda).$$

From the independence of $\widetilde{p}$ and $\widetilde{\lambda}$ we get the statement of the theorem. □

2.2. Theorem. *Let the ratio $\frac{n}{m}$ converge to a positive number s when both n and m tend to infinity. Then*

$$\sqrt{n}(\widetilde{R}-R)\overset{D}{\to}\mathcal{N}(0,e^{-2\lambda p}p^2\lambda(\lambda(1-p)+s)).$$

Proof. In order to prove this theorem we shall use the method from [11]. Since $R=R(p,\lambda)$ is the transformation such that the matrix of partial derivatives

$$B=\begin{bmatrix}\frac{\partial R}{\partial p} & \frac{\partial R}{\partial \lambda}\end{bmatrix}=\begin{bmatrix}\lambda e^{-\lambda p} & pe^{-\lambda p}\end{bmatrix}$$

has continuous elements and does not vanish in the neighbourhood of (p,λ), then we have

$$\sqrt{n}(\widetilde{R}-R)\overset{D}{\to}\mathcal{N}(0,BJB').$$

Inserting the values of B and J we get the statement of the theorem. □

Using this theorem we can construct the asymptotic confidence interval for R. Denote $\widetilde{\sigma}^2=e^{-2\widetilde{\lambda}\widetilde{p}}\widetilde{p}^2\widetilde{\lambda}(\widetilde{\lambda}(1-\widetilde{p})+s)$. Then the estimator of the variance of $\widetilde{R}$ is

$$\widetilde{Var}(\widetilde{R})=\frac{\widetilde{\sigma}^2}{n}. \tag{2.2}$$

The interval of confidence level $1-\alpha$ is given by

$$I_R=\left(\widetilde{R}-\frac{z_{1-\frac{\alpha}{2}}\widetilde{\sigma}}{\sqrt{n}},\widetilde{R}+\frac{z_{1-\frac{\alpha}{2}}\widetilde{\sigma}}{\sqrt{n}}\right), \tag{2.3}$$

where z_γ is the γth quantile from standard normal distribution.

2.2. Bootstrap-t Confidence Interval. The confidence intervals based on the asymptotic distribution do not perform very well for small sample sizes. Therefore, we propose a construction of the confidence interval based on bootstrap-t method (see [10]). The algorithm is illustrated below.

Step 1: From initial samples $\mathbf{x}=(x_1,x_2,...,x_n)$ and $\mathbf{y}=(y_1,y_2,...,y_m)$ calculate MLEs $\widetilde{p}$ and $\widetilde{\lambda}$.

Step 2: Use those estimates to generate bootstrap samples $\mathbf{x}^*$ and $\mathbf{y}^*$ and compute bootstrap sample estimates R^* of R using (2.1).

Step 3: Repeat step 2, N boot times.

Step 4: For each R_i^*, $1 \le i \le N$, calculate the following statistic

$$T_i^* = \frac{R_i^* - \widetilde{R}}{\sqrt{Var(R^*)}},$$

where $Var(R^*) = \frac{\sum_{i=1}^{N}(R_i^* - \overline{R^*})^2}{N-1}$ and $\overline{R^*} = \frac{\sum_{i=1}^{N} R_i^*}{N}$.

Step 5: For sample of T_i^* obtained in step 4, calculate sample quantiles of order $\frac{\alpha}{2}$ $(t_{\frac{\alpha}{2}})$ and $1-\frac{\alpha}{2}$ $(t_{1-\frac{\alpha}{2}})$. Then, the bootstrap-t confidence interval is given by

$$\left(\widetilde{R} - t_{1-\frac{\alpha}{2}}\sqrt{Var(R^*)}, \widetilde{R} - t_{\frac{\alpha}{2}}\sqrt{Var(R^*)}\right). \tag{2.4}$$

3. UMVUE of R

In this section we find the UMVUE of R, denoted by $\widehat{R}$, and UMVUE of the variance of $\widehat{R}$.

The complete sufficient statistics for p and λ are $T_X = \sum_{j=1}^{n} X_j$ and $T_Y = \sum_{j=1}^{m} Y_j$. The statistic T_X, as a sum of n independent identically distributed random variables with geometric distribution, has negative binomial distribution with parameters n and p, and the statistic T_Y, as a sum of m independent identically distributed random variables with Poisson distribution, has Poisson distribution with parameter $m\lambda$.

An unbiased estimator for R is $I\{X_1 \le Y_1\}$. Then

$$E(I\{X_1 \le Y_1\}|T_X = t_X, T_Y = t_Y) = P\{X_1 \le Y_1 | T_X = t_X, T_Y = t_Y\}$$

$$= \frac{P\{X_1 \le Y_1, \sum_{j=1}^{n} X_j = t_X, \sum_{j=1}^{m} Y_j = t_Y\}}{P\{\sum_{j=1}^{n} X_j = t_X, \sum_{j=1}^{m} Y_j = t_Y\}}$$

$$= \frac{\sum_{y=1}^{t_Y}\sum_{x=1}^{M} P\{X_1 = x\}P\{Y_1 = y\}P\{\sum_{j=2}^{n} X_j = t_X - x\}P\{\sum_{j=2}^{m} Y_j = t_Y - y\}}{P\{\sum_{j=1}^{n} X_j = t_X\}P\{\sum_{j=1}^{m} Y_j = t_Y\}}$$

$$= \frac{\sum_{y=1}^{t_Y}\sum_{x=1}^{M}(1-p)^{x-1}p\frac{e^{-\lambda}\lambda^y}{y!}\binom{t_X-x-1}{n-2}p^{n-1}(1-p)^{t_X-x-n+1}\frac{e^{-(m-1)\lambda}((m-1)\lambda)^{t_Y-y}}{(t_Y-y)!}}{\binom{t_X-1}{n-1}p^n(1-p)^{t_X-n}\frac{e^{-m\lambda}(m\lambda)^{t_Y}}{t_Y!}}$$

$$= \frac{\sum_{y=1}^{t_Y}\binom{t_Y}{y}(m-1)^{t_Y-y}\sum_{x=1}^{M}\binom{t_X-x-1}{n-2}}{\binom{t_X-1}{n-1}m^{t_Y}},$$

where $M = \min\{t_X - n + 1, y\}$.

Using the identity

$$\sum_{s=0}^{n}\binom{s}{c} = \binom{n+1}{c+1},$$

we get that

$$\begin{aligned}
E(I\{X_1 \le Y_1\}|T_X{=}t_X, T_Y{=}t_Y) &= \frac{\sum_{y=1}^{t_Y} \binom{t_Y}{y}(m-1)^{t_Y-y} \sum_{x=1}^{M} \binom{t_X-x-1}{n-2}}{\binom{t_X-1}{n-1} m^{t_Y}} \\
&= \frac{\sum_{y=1}^{t_Y} \binom{t_Y}{y}(m-1)^{t_Y-y} \sum_{s=t_X-M-1}^{t_X-2} \binom{s}{n-2}}{\binom{t_X-1}{n-1} m^{t_Y}} \\
&= \frac{\sum_{y=1}^{t_Y} \binom{t_Y}{y}(m-1)^{t_Y-y} \left(\sum_{s=0}^{t_X-2} \binom{s}{n-2} - \sum_{s=0}^{t_X-M-2} \binom{s}{n-2} \right)}{\binom{t_X-1}{n-1} m^{t_Y}} \\
&= \frac{\sum_{y=1}^{t_Y} \binom{t_Y}{y}(m-1)^{t_Y-y} \left(\binom{t_X-1}{n-1} - \binom{t_X-M-1}{n-1} \right)}{\binom{t_X-1}{n-1} m^{t_Y}}.
\end{aligned}$$

Using Rao-Blackwell and Lehmann-Sheffé theorems we get that the UMVUE of R is

$$\widehat{R} = 1 - \left(1 - \frac{1}{m}\right)^{T_Y} - \sum_{y=1}^{T_Y} \frac{\binom{T_Y}{y}\binom{T_X-M-1}{n-1}}{\binom{T_X-1}{n-1}} \left(1 - \frac{1}{m}\right)^{T_Y-y} \left(\frac{1}{m}\right)^{y}. \tag{3.1}$$

This formula is valid for $T_Y > 0$. If $T_Y = 0$, then $\widehat{R} = 0$.

Now, in order to find the UMVUE of variance of $\widehat{R}$, we calculate the UMVUE of R^2. An unbiased estimator for R^2 is $I\{X_1 \le Y_1, X_2 \le Y_2\}$. Then

$$
\begin{aligned}
& E(I\{X_1 \le Y_1, X_2 \le Y_2\}|T_X = t_X, T_Y = t_Y) \\
= \; & \frac{P\{X_1 \le Y_1, X_2 \le Y_2, \sum_{j=1}^{n} X_j = t_X, \sum_{j=1}^{m} Y_j = t_Y\}}{P\{\sum_{j=1}^{n} X_j = t_X, \sum_{j=1}^{m} Y_j = t_Y\}} \\
= \; & \frac{1}{P\{\sum_{j=1}^{n} X_j = t_X\} P\{\sum_{j=1}^{m} Y_j = t_Y\}} \\
\times \; & \sum_{y_1=1}^{t_Y-1} \sum_{y_2=1}^{t_Y-y_1} \sum_{x_1=1}^{M_1} \sum_{x_2=1}^{M_2} P\{X_1 = x_1\} P\{X_2 = x_2\} P\{Y_1 = y_1\} P\{Y_2 = y_2\} \\
\times \; & P\Big\{\sum_{j=3}^{n} X_j = t_X - x_1 - x_2\Big\} P\Big\{\sum_{j=3}^{m} Y_j = t_Y - y_1 - y_2\Big\} \\
= \; & \frac{1}{\binom{t_X-1}{n-1} p^n (1-p)^{t_X-n} \frac{e^{-m\lambda}(m\lambda)^{t_Y}}{t_Y!}} \\
\times \; & \sum_{y_1=1}^{t_Y-1} \sum_{y_2=1}^{t_Y-y_1} \sum_{x_1=1}^{M_1} \sum_{x_2=1}^{M_2} p(1-p)^{x_1-1} p(1-p)^{x_2-1} \frac{e^{-\lambda}\lambda^{y_1}}{y_1!} \frac{e^{-\lambda}\lambda^{y_2}}{y_2!} \\
\times \; & \binom{t_X - x_1 - x_2 - 1}{n-3} p^{n-2} (1-p)^{t_x - x_1 - x_2 - n + 2} \frac{e^{-(m-2)\lambda}((m-2)\lambda)^{t_Y - y_1 - y_2}}{(t_Y - y_1 - y_2)!} \\
(3.2) = \; & \frac{\sum_{y_1=1}^{t_Y-1} \sum_{y_2=1}^{t_Y-y_1} \binom{t_Y}{y_1+y_2} \binom{y_1+y_2}{y_1} (m-2)^{t_Y-y_1-y_2} \sum_{x_1=1}^{M_1} \sum_{x_2=1}^{M_2} \binom{t_X - x_1 - x_2 - 1}{n-3}}{\binom{t_X-1}{n-1} m^{t_Y}},
\end{aligned}
$$

where $M_1 = \min\{y_1, t_X - n + 1\}$ and $M_2 = \min\{y_2, t_X - n + 2 - x_1\}$. Using similar technique as when finding $\widehat{R}$, we get that

$$
\begin{aligned}
& \sum_{x_1=1}^{M_1} \sum_{x_2=1}^{M_2} \binom{t_X - x_1 - x_2 - 1}{n-3} = \sum_{x_1=1}^{M_1} \sum_{s=t_X - x_1 - M_2 - 1}^{t_X - x_1 - 2} \binom{s}{n-3} \\
= \; & \sum_{x_1=1}^{M_1} \left(\binom{t_X - x_1 - 1}{n-2} - \binom{t_X - x_1 - M_2 - 1}{n-2} \right) \\
= \; & \sum_{s=0}^{t_X-2} \binom{s}{n-2} - \sum_{s=0}^{t_X - M_1 - 2} \binom{s}{n-2} - \sum_{x_1=1}^{M_1} \binom{t_X - x_1 - M_2 - 1}{n-2} \\
= \; & \binom{t_X - 1}{n-1} - \binom{t_X - M_1 - 1}{n-1} - \sum_{x_1=1}^{M_1} \binom{t_X - x_1 - M_2 - 1}{n-2}.
\end{aligned}
$$

Inserting this into (3.2) and using Rao-Blackwell and Lehmann-Sheffé theorems we get that the UMVUE of R^2 is

$$\widehat{R^2} = \frac{1}{\binom{T_X-1}{n-1} m^{T_Y}} \sum_{y_1=1}^{T_Y-1} \sum_{y_2=1}^{T_Y-y_1} \binom{T_Y}{y_1+y_2} \binom{y_1+y_2}{y_1} (m-2)^{T_Y-y_1-y_2}$$
$$\times \left(\binom{T_X-1}{n-1} - \binom{T_X-M_1-1}{n-1} - \sum_{x_1=1}^{M_1} \binom{T_X-x_1-M_2-1}{n-2} \right). \tag{3.3}$$

This formula is valid for $T_Y > 1$. If $T_Y \leq 1$, then $\widehat{R^2} = 0$.

Finally, we obtain the UMVUE of variance of $\widehat{R}$ using the following theorem.

3.1. Theorem. *The UMVUE of $Var(\widehat{R})$ is given by*

$$\widehat{Var}(\widehat{R}) = (\widehat{R})^2 - \widehat{R^2}, \tag{3.4}$$

where $\widehat{R}$ and $\widehat{R^2}$ are given by (3.1) *and* (3.3).

The proof follows from general result obtained in [18] and [13].

4. Bayes Estimator of R

In this section we shall find the Bayes estimator of R with respect to mean square error. Let us suppose that p and λ have conjugate prior distributions, beta $\mathcal{B}(a,b)$, $a, b \in \mathbb{N}$, and gamma $\Gamma(\alpha, \beta)$, $\alpha \in \mathbb{N}, \beta > 0$, with the following joint density:

$$\pi(p, \lambda) = \frac{p^{a-1}(1-p)^{b-1}}{B(a,b)} \frac{\lambda^{\alpha-1}\beta^{\alpha}e^{-\beta\lambda}}{\Gamma(\alpha)}, \; p \in (0,1), \lambda > 0.$$

Then the joint posterior density given the sample $(\mathbf{x}, \mathbf{y})$, or, equivalently, given the sufficient statistics (t_X, t_Y) is

$$\pi(p, \lambda | t_X, t_Y) = K p^{a-1+n} (1-p)^{t_X-n+b-1} \lambda^{\alpha-1+t_Y} e^{-\lambda(\beta+m)}, \; p \in (0,1), \lambda > 0,$$

where

$$K = \left(\int_0^1 \int_0^{\infty} p^{a-1+n} (1-p)^{t_X-n+b-1} \lambda^{\alpha-1+t_Y} e^{-\lambda(\beta+m)} d\lambda \, dp \right)^{-1}$$

is the proportionality constant.

Denote, for simplicity, $A = a+n-1$, $B = t_X - n + b - 1$, $C = \alpha - 1 + t_Y$ and $D = \beta + m$. Since $R = 1 - e^{-\lambda p}$, we get that $p = -\frac{\ln(1-R)}{\lambda}$. Using the transformation of random variables (p, λ) to (R, λ) we get

$$\begin{aligned}
\pi(r, \lambda | t_X, t_Y) &= \pi\big(p(r,\lambda), \lambda(r,\lambda) | t_X, t_Y\big) \begin{vmatrix} \frac{\partial p}{\partial r} & \frac{\partial p}{\partial \lambda} \\ \frac{\partial \lambda}{\partial r} & \frac{\partial \lambda}{\partial \lambda} \end{vmatrix} \\
&= \pi\big(p(r,\lambda), \lambda(r,\lambda) | t_X, t_Y\big) \begin{vmatrix} \frac{1}{\lambda}\frac{1}{1-r} & \frac{\ln(1-r)}{\lambda^2} \\ 0 & 1 \end{vmatrix} \\
&= K \left(-\frac{\ln(1-r)}{\lambda} \right)^A \left(1 + \frac{\ln(1-r)}{\lambda} \right)^B \lambda^C e^{-\lambda D} \frac{1}{\lambda} \frac{1}{1-r} \\
&= K \left(-\ln(1-r)\right)^A \left(1 + \frac{\ln(1-r)}{\lambda} \right)^B \lambda^{C-A-1} \frac{e^{-\lambda D}}{1-r}, \\
&\quad r \in (0,1), \lambda > -\ln(1-r).
\end{aligned}$$

Then the marginal posterior density of R is

$$\pi_R(r|t_X, t_Y) = K\int_{-\ln(1-r)}^{\infty}(-\ln(1-r))^A \sum_{j=0}^{B}\binom{B}{j}\left(\frac{\ln(1-r)}{\lambda}\right)^j \lambda^{C-A-1}\frac{e^{-\lambda D}}{1-r}d\lambda$$

$$= K\sum_{j=0}^{B}\binom{B}{j}(-1)^j\frac{(-\ln(1-r))^{A+j}}{1-r}\int_{-\ln(1-r)}^{\infty}\lambda^{C-A-j-1}e^{-\lambda D}d\lambda$$

$$= K\sum_{j=0}^{B}\binom{B}{j}(-1)^j\frac{(-\ln(1-r))^{A+j}}{(1-r)D^{C-A-j}}\int_{-D\ln(1-r)}^{\infty}t^{C-A-j-1}e^{-t}dt, \quad r \in (0,1).$$

The Bayes estimator $\check{R}$ of R for mean square loss function is the posterior mean. After some calculations (see Appendix) we obtain

$$\check{R} = 1 - K\left[I_{\{C-A>0\}}\sum_{j=0}^{\min\{C-A-1,B\}}W_1 + I_{\{0\le C-A\le B\}}W_2 + I_{\{C-A<B\}}\sum_{j=\max\{0,C-A+1\}}^{B}W_3\right], \tag{4.1}$$

where

$$W_1 = (-1)^j\binom{B}{j}\frac{(C-1)!}{\binom{C-1}{A+j}}\sum_{i=0}^{C-A-j-1}\frac{\binom{A+j+i}{i}}{D^{C-A-j-i}(D+1)^{A+j+i+1}},$$

$$W_2 = (-1)^{C-A-1}\binom{B}{C-A}C!\left[\ln D + \sum_{i=1}^{\infty}\frac{(-1)^i}{i}\left(D^i\binom{C+i}{i} - \binom{C}{i}\right)\right],$$

$$W_3 = \binom{B}{j}(C-1)!\sum_{i=1}^{A-C+j}(-1)^{i+j+1}\frac{D^{i-1}\binom{C+i-1}{i}}{(D+1)^{C+i}\binom{A-C+j}{i}} + \frac{(-1)^{A-C+1}}{D^{C-A-j}}$$

$$\times \binom{B}{j}\binom{A+j}{C}C!\left[\ln D + \sum_{i=1}^{\infty}\frac{(-1)^i}{i}\left(D^i\binom{A+j+i}{i} - \binom{A+j}{i}\right)\right].$$

It is possible to generalize this estimator for real values of the hyperparametres, but it would be much more complicated and not practical for presentation.

5. Simulation Study

In this section we perform a simulation study for various sample sizes and different values of unknown parameters.

For fixed values of n, m, p and λ we do the following procedure. We choose a sample and calculate the MLE and its variance using (2.1) and (2.2), and the UMVUE and its variance using (3.1), (3.3) and (3.4). Since we do not know the prior distributions and to get better comparison with other types of estimates, we obtain Bayes estimates using non-informative Jeffreys' priors where $\pi(p) \sim p^{-1}(1-p)^{-\frac{1}{2}}$ and $\pi(\lambda) \sim \lambda^{-\frac{1}{2}}$. We find the estimates from posterior distribution for R using Monte Carlo method with 5000 replicates.

We also calculate 95% asymptotic confidence interval using (2.3) and 95% bootstrap-t confidence interval using (2.4) with $N = 1000$ boot times.

This procedure is repeated for 500 samples and the averages for each estimate are calculated.

In table 1 we present point estimates for R and their standard errors. In table 2 we present 95% asymptotic and bootstrap-t confidence intervals as well as 95% Bayes credible intervals based on a Monte Carlo method mentioned above. The coverage percentages of these intervals (the percentage of intervals that contain true value of R) are also shown.

In table 1 we can notice that in most cases the UMVUE has the value closest to R as expected due to its unbiasedness. However, its standard error is the largest. For most values of R the standard error of Bayes estimate is the smallest, while for larger values of R, the standard error of MLE has that property. In the last case ($R = 0.7981$), the standard error of Bayes estimate is even larger than the UMVUE one.

From table 2 we can see that in almost all cases the asymptotic intervals have the worst coverage percentages, which is expected because we have small sample sizes, while Bayes credible intervals and bootstrap-t confidence intervals both perform very well.

Table 1. Point estimates for R and their standard errors

samples		parameters		reliability	MLE		UMVUE		Bayes	
n	m	p	λ	R	$\widetilde{R}$	$\sigma(\widetilde{R})$	$\widehat{R}$	$\widehat{\sigma}(\widehat{R})$	$\check{R}$	$\sigma(\check{R})$
		0.5	0.5	0.2212	0.2226	0.0813	0.2176	0.0825	0.2133	0.0778
		0.25	1	0.2212	0.2324	0.0742	0.2215	0.0746	0.2257	0.0719
		0.3	1	0.2592	0.2634	0.0809	0.2527	0.0821	0.2552	0.0782
		0.5	1	0.3935	0.3986	0.1003	0.3929	0.1047	0.3824	0.0966
		0.8	1	0.5507	0.5398	0.1024	0.5454	0.1069	0.5146	0.1007
10	15	0.4	2	0.5507	0.5550	0.1045	0.5478	0.1106	0.5353	0.1023
		0.67	1.5	0.6340	0.6415	0.0972	0.6343	0.1023	0.6162	0.0975
		0.5	2	0.6340	0.6369	0.1011	0.6346	0.1067	0.6137	0.1006
		0.8	1.5	0.6988	0.6918	0.0891	0.6987	0.0964	0.6648	0.0915
		0.6	2	0.6988	0.7019	0.0927	0.7008	0.0982	0.6767	0.0945
		0.8	2	0.7981	0.7868	0.0737	0.7940	0.1043	0.7611	0.0790
		0.5	0.5	0.2212	0.2236	0.0762	0.2228	0.0774	0.2174	0.0735
		0.25	1	0.2212	0.2229	0.0625	0.2202	0.0625	0.2190	0.0612
		0.3	1	0.2592	0.2648	0.0706	0.2605	0.0712	0.2596	0.0689
		0.5	1	0.3935	0.3914	0.0899	0.3921	0.0923	0.3818	0.0872
		0.8	1	0.5507	0.5597	0.0970	0.5573	0.0998	0.5324	0.0949
20	15	0.4	2	0.5507	0.5561	0.0881	0.5547	0.0910	0.5436	0.0864
		0.67	1.5	0.6340	0.6260	0.0895	0.6312	0.0920	0.6093	0.0884
		0.5	2	0.6340	0.6374	0.0864	0.6392	0.0891	0.6226	0.0854
		0.8	1.5	0.6988	0.6894	0.0834	0.6977	0.0848	0.6712	0.0835
		0.6	2	0.6988	0.6975	0.0813	0.7021	0.0832	0.6812	0.0813
		0.8	2	0.7981	0.7895	0.0668	0.7975	0.0668	0.7725	0.0688
		0.5	0.5	0.2212	0.2256	0.0680	0.2240	0.0688	0.2200	0.0660
		0.25	1	0.2212	0.2277	0.0579	0.2225	0.0580	0.2241	0.0569
		0.3	1	0.2592	0.2665	0.0649	0.2616	0.0654	0.2617	0.0636
		0.5	1	0.3935	0.3965	0.0817	0.3948	0.0836	0.3872	0.0796
		0.8	1	0.5507	0.5449	0.0865	0.5500	0.0887	0.5297	0.0850
20	20	0.4	2	0.5507	0.5469	0.0823	0.5491	0.0848	0.5358	0.0809
		0.67	1.5	0.6340	0.6350	0.0802	0.6352	0.0823	0.6201	0.0798
		0.5	2	0.6340	0.6348	0.0798	0.6341	0.0822	0.6215	0.0793
		0.8	1.5	0.6988	0.6951	0.0738	0.7010	0.0749	0.6791	0.0744
		0.6	2	0.6988	0.6924	0.0751	0.6949	0.0770	0.6779	0.0754
		0.8	2	0.7981	0.7960	0.0593	0.8018	0.0593	0.7809	0.0614
		0.5	0.5	0.2212	0.2214	0.0432	0.2207	0.0434	0.2192	0.0427
		0.25	1	0.2212	0.2226	0.0364	0.2205	0.0364	0.2212	0.0362
		0.3	1	0.2592	0.2605	0.0410	0.2585	0.0411	0.2586	0.0406
		0.5	1	0.3935	0.3960	0.0522	0.3953	0.0527	0.3923	0.0517
		0.8	1	0.5507	0.5489	0.0552	0.5509	0.0558	0.5426	0.0548
50	50	0.4	2	0.5507	0.5494	0.0528	0.5483	0.0535	0.5449	0.0524
		0.67	1.5	0.6340	0.6355	0.0513	0.6348	0.0518	0.6294	0.0512
		0.5	2	0.6340	0.6327	0.0514	0.6338	0.0521	0.6273	0.0512
		0.8	1.5	0.6988	0.6964	0.0472	0.6997	0.0475	0.6908	0.0474
		0.6	2	0.6988	0.6952	0.0482	0.6962	0.0487	0.6892	0.0483
		0.8	2	0.7981	0.7974	0.0379	0.7977	0.0379	0.7912	0.0385

6. Conclusion

In this paper we considered the estimation of the probability $P\{X \leq Y\}$ when X and Y are two independent random variables from geometric and Poisson distribution respectively. We determined MLE, UMVUE and Bayes point estimator. The asymptotic and bootstrap-t confidence intervals were constructed.

A simulation study was performed. The obtained point estimates were compared and in most cases UMVUEs have the smallest bias, while Bayes estimates have the smallest standard error. Comparison of interval estimates was also done and we concluded that bootstrap-t and Bayes intervals had notably higher coverage percentages than asymptotic ones.

Table 2. Interval estimates for R and their coverage percentages

samples n	m	parameters p	λ	reliability R	asymptotic CI	cov.	bootstrap CI	cov.	Bayes CI	cov.
10	15	0.5	0.5	0.2212	(0.06, 0.38)	91.4	(0.09, 0.40)	93.2	(0.09, 0.39)	94.0
		0.25	1	0.2212	(0.09, 0.38)	92.0	(0.12, 0.41)	93.4	(0.11, 0.38)	94.0
		0.3	1	0.2592	(0.11, 0.42)	93.0	(0.14, 0.45)	95.2	(0.12, 0.42)	94.8
		0.5	1	0.3935	(0.20, 0.60)	91.4	(0.22, 0.61)	93.8	(0.21, 0.58)	94.2
		0.8	1	0.5507	(0.34, 0.74)	92.6	(0.32, 0.76)	91.6	(0.32, 0.71)	93.8
		0.4	2	0.5507	(0.35, 0.76)	89.6	(0.36, 0.76)	93.2	(0.33, 0.73)	93.8
		0.67	1.5	0.6340	(0.45, 0.83)	90.6	(0.44, 0.81)	94.8	(0.41, 0.79)	94.8
		0.5	2	0.6340	(0.44, 0.83)	92.8	(0.44, 0.83)	96.4	(0.40, 0.79)	94.6
		0.8	1.5	0.6988	(0.52, 0.87)	92.2	(0.50, 0.84)	94.6	(0.47, 0.83)	93.4
		0.6	2	0.6988	(0.52, 0.88)	90.2	(0.51, 0.86)	94.2	(0.47, 0.84)	94.4
		0.8	2	0.7981	(0.64, 0.93)	92.8	(0.62, 0.90)	94.0	(0.58, 0.89)	94.4
20	15	0.5	0.5	0.2212	(0.07, 0.37)	91.6	(0.09, 0.38)	93.0	(0.09, 0.38)	93.4
		0.25	1	0.2212	(0.10, 0.35)	94.6	(0.12, 0.36)	95.2	(0.11, 0.35)	95.4
		0.3	1	0.2592	(0.13, 0.40)	93.6	(0.14, 0.42)	93.8	(0.14, 0.41)	94.4
		0.5	1	0.3935	(0.22, 0.57)	93.0	(0.22, 0.57)	94.8	(0.22, 0.56)	94.0
		0.8	1	0.5507	(0.35, 0.73)	93.6	(0.33, 0.74)	92.0	(0.34, 0.71)	94.8
		0.4	2	0.5507	(0.38, 0.73)	93.2	(0.39, 0.73)	95.0	(0.37, 0.71)	94.4
		0.67	1.5	0.6340	(0.45, 0.80)	95.2	(0.44, 0.78)	97.4	(0.43, 0.77)	96.4
		0.5	2	0.6340	(0.47, 0.81)	94.2	(0.46, 0.79)	95.6	(0.45, 0.78)	95.8
		0.8	1.5	0.6988	(0.53, 0.85)	93.4	(0.50, 0.83)	94.2	(0.50, 0.82)	95.0
		0.6	2	0.6988	(0.54, 0.86)	93.8	(0.52, 0.84)	95.6	(0.51, 0.83)	94.6
		0.8	2	0.7981	(0.66, 0.92)	92.4	(0.63, 0.89)	93.0	(0.62, 0.89)	93.6
20	20	0.5	0.5	0.2212	(0.09, 0.36)	92.8	(0.10, 0.37)	94.0	(0.11, 0.36)	93.8
		0.25	1	0.2212	(0.12, 0.34)	94.0	(0.13, 0.36)	93.8	(0.13, 0.35)	94.2
		0.3	1	0.2592	(0.14, 0.39)	93.0	(0.16, 0.41)	92.4	(0.15, 0.40)	93.4
		0.5	1	0.3935	(0.24, 0.56)	94.2	(0.24, 0.56)	95.2	(0.24, 0.55)	95.4
		0.8	1	0.5507	(0.37, 0.71)	93.8	(0.36, 0.70)	95.0	(0.36, 0.69)	94.8
		0.4	2	0.5507	(0.40, 0.71)	92.6	(0.39, 0.71)	93.8	(0.38, 0.69)	93.2
		0.67	1.5	0.6340	(0.48, 0.79)	95.0	(0.47, 0.78)	95.8	(0.46, 0.77)	96.8
		0.5	2	0.6340	(0.48, 0.79)	91.6	(0.47, 0.78)	93.2	(0.50, 0.77)	94.0
		0.8	1.5	0.6988	(0.55, 0.84)	91.6	(0.53, 0.82)	93.0	(0.52, 0.81)	94.0
		0.6	2	0.6988	(0.55, 0.84)	90.8	(0.53, 0.83)	93.4	(0.52, 0.81)	92.6
		0.8	2	0.7981	(0.68, 0.91)	91.8	(0.66, 0.89)	94.6	(0.65, 0.89)	95.2
50	50	0.5	0.5	0.2212	(0.14, 0.31)	95.4	(0.14, 0.31)	95.8	(0.14, 0.31)	95.8
		0.25	1	0.2212	(0.15, 0.30)	94.6	(0.16, 0.31)	95.0	(0.16, 0.30)	95.0
		0.3	1	0.2592	(0.18, 0.34)	92.6	(0.19, 0.35)	94.2	(0.18, 0.34)	94.2
		0.5	1	0.3935	(0.29, 0.50)	93.2	(0.30, 0.50)	94.2	(0.29, 0.50)	94.0
		0.8	1	0.5507	(0.44, 0.66)	96.0	(0.44, 0.65)	95.2	(0.43, 0.65)	95.6
		0.4	2	0.5507	(0.45, 0.65)	93.6	(0.45, 0.65)	94.8	(0.44, 0.65)	94.4
		0.67	1.5	0.6340	(0.53, 0.74)	93.6	(0.53, 0.73)	93.4	(0.53, 0.73)	93.2
		0.5	2	0.6340	(0.53, 0.73)	94.0	(0.53, 0.73)	95.4	(0.52, 0.72)	94.4
		0.8	1.5	0.6988	(0.60, 0.79)	93.6	(0.60, 0.78)	94.0	(0.59, 0.78)	94.4
		0.6	2	0.6988	(0.60, 0.79)	94.2	(0.60, 0.78)	94.2	(0.59, 0.78)	94.8
		0.8	2	0.7981	(0.72, 0.87)	93.0	(0.72, 0.86)	93.6	(0.71, 0.86)	94.4

Appendix

$$\begin{aligned}
\check{R} &= E(R|t_X,t_Y) = 1 - E(1-R|t_X,t_Y) = 1 - \int_0^1 (1-r)\pi_R(r|t_X,t_Y)dr \\
&= 1 - \int_0^1 (1-r)K\sum_{j=0}^{B}\binom{B}{j}(-1)^j\frac{(-\ln(1-r))^{A+j}}{(1-r)D^{C-A-j}}\int_{-D\ln(1-r)}^{\infty} t^{C-A-j-1}e^{-t}dtdr \\
&= 1 - K\sum_{j=0}^{B}\binom{B}{j}\frac{(-1)^j}{D^{C-A-j}}\int_0^1 (-\ln(1-r))^{A+j}\int_{-D\ln(1-r)}^{\infty} t^{C-A-j-1}e^{-t}dtdr \\
&= 1 - K\sum_{j=0}^{B}\binom{B}{j}\frac{(-1)^j}{D^{C-A-j}}\int_0^{\infty} s^{A+j}e^{-s}\int_{Ds}^{\infty} t^{C-A-j-1}e^{-t}dtds. \qquad (.1)
\end{aligned}$$

We need to calculate the integral $L_q(z) = \int\limits_z^\infty t^{q-1}e^{-t}dt$, $z>0$, $q \in \mathbb{Z}$. Depending on q we have the following three possibilities:

(1) $q > 0$

$$L_q(z) = \Gamma(q,z) = (q-1)!e^{-z}\sum_{i=0}^{q-1}\frac{z^i}{i!},$$

where $\Gamma(q,z)$ is the incomplete gamma function.

(2) $q = 0$

$$L_q(z) = -\mathrm{Ei}(-z) = -\gamma - \ln z + \sum_{i=1}^{\infty}(-1)^{i+1}\frac{z^i}{i \cdot i!},$$

where $\mathrm{Ei}(x)$ is the exponetial integral and γ is Euler's constant.

(3) $q < 0$

Using integration by parts $|q|$ times we get

$$L_q(z) = e^{-z}\sum_{i=1}^{-q}(-1)^{i+1}\frac{z^{i+q-1}}{(-q)!}(-q-i)! + \frac{(-1)^{-q}}{(-q)!}L_0(z).$$

Thus, the summands in (.1) can be expressed as

$$\binom{B}{j}\frac{(-1)^j}{D^{C-A-j}}\int\limits_0^\infty s^{A+j}e^{-s}L_{C-A-j}(Ds)ds,$$

and depending on j, we have three types of summands:

(1) $j < C - A$

$$\begin{aligned}
W_1 &= \binom{B}{j}\frac{(-1)^j}{D^{C-A-j}}\int\limits_0^\infty s^{A+j}e^{-s}(C-A-j-1)!e^{-Ds}\sum_{i=0}^{C-A-j-1}\frac{(Ds)^i}{i!}ds \\
&= (-1)^j\binom{B}{j}\frac{(C-A-j-1)!}{D^{C-A-j}}\sum_{i=0}^{C-A-j-1}\frac{D^i}{i!}\int\limits_0^\infty s^{A+j+i}e^{-(D+1)s}ds \\
&= (-1)^j\binom{B}{j}\frac{(C-1)!}{\binom{C-1}{A+j}}\sum_{i=0}^{C-A-j-1}\frac{\binom{A+j+i}{i}}{D^{C-A-j-i}(D+1)^{A+j+i+1}}.
\end{aligned}$$

This type of summand appears in (.1) whenever $C - A > 0$.

(2) $j = C - A$

$$\begin{aligned}
W_2 &= \binom{B}{C{-}A}(-1)^{C-A}\int\limits_0^\infty s^Ce^{-s}(-\gamma - \ln(Ds) + \sum_{i=1}^{\infty}(-1)^{i+1}\frac{(Ds)^i}{i \cdot i!})ds \\
&= \binom{B}{C{-}A}(-1)^{C-A-1}\Big((\gamma + \ln D)\int\limits_0^\infty s^Ce^{-s}ds + \int\limits_0^\infty \ln s\, s^Ce^{-s}ds \\
&+ \sum_{i=1}^{\infty}(-1)^i\frac{D^i}{i \cdot i!}\int\limits_0^\infty s^{C+i}e^{-s}ds\Big) \\
&= \binom{B}{C{-}A}(-1)^{C-A-1}\Big((\gamma + \ln D)C! + \psi(C+1)C!
\end{aligned}$$

$$+ \quad \sum_{i=1}^{\infty}(-1)^i \frac{D^i}{i \cdot i!}(C+i)!\Big)$$

$$= \binom{B}{C{-}A}(-1)^{C-A-1}\Big((\gamma + \ln D)C! + C!\Big(-\gamma - \sum_{i=1}^{\infty}\frac{(-1)^i}{i}\binom{C}{i}\Big)$$

$$+ \quad \sum_{i=1}^{\infty}(-1)^i \frac{D^i}{i \cdot i!}(C+i)!\Big)$$

$$= (-1)^{C-A-1}\binom{B}{C{-}A}C!\left[\ln D + \sum_{i=1}^{\infty}\frac{(-1)^i}{i}\left(D^i\binom{C{+}i}{i} - \binom{C}{i}\right)\right],$$

where $\psi(x)$ is digamma function.

This type of summand appears in (.1) whenever $0 \le C - A \le B$.

(3) $j > C - A$

$$W_3 = \binom{B}{j}\frac{(-1)^j}{D^{C-A-j}}\int_0^{\infty} s^{A+j}e^{-s}$$

$$\times \Big(e^{-Ds}\sum_{i=1}^{A-C+j}(-1)^{i+1}\frac{(Ds)^{C-A-j+i-1}}{(A-C+j)!}(A-C+j-i)!$$

$$+ \quad \frac{(-1)^{A-C+j}}{(A-C+j)!}\Big(-\gamma - \ln(Ds) + \sum_{i=1}^{\infty}(-1)^{i+1}\frac{(Ds)^i}{i \cdot i!}\Big)\Big)ds$$

$$= \binom{B}{j}\sum_{i=1}^{A-C+j}\frac{(-1)^{i+j+1}D^{i-1}(A-C+j-i)!}{(A-C+j)!}$$

$$\times \int_0^{\infty} s^{C+i-1}e^{-(D+1)s}ds + \binom{B}{j}\frac{(-1)^{A-C+1}}{(A-C+j)!}\frac{1}{D^{C-A-j}}$$

$$\times \Big((\gamma + \ln D)\int_0^{\infty} s^{A+j}e^{-s}ds + \int_0^{\infty}\ln s\; s^{A+j}e^{-s}ds$$

$$+ \quad \sum_{i=1}^{\infty}(-1)^i\frac{D^i}{i \cdot i!}\int_0^{\infty}s^{A+j+i}e^{-s}ds\Big)$$

$$= \binom{B}{j}\sum_{i=1}^{A-C+j}\frac{(-1)^{i+j+1}D^{i-1}(A-C+j-i)!}{(A-C+j)!}\frac{(C+i-1)!}{(D+1)^{C+i}}$$

$$+ \quad \binom{B}{j}\frac{(-1)^{A-C+1}}{(A-C+j)!}\frac{1}{D^{C-A-j}}\Big((\gamma + \ln D)(A+j)!$$

$$+ \quad \psi(A+j+1)(A+j)! + \sum_{i=1}^{\infty}(-1)^i\frac{D^i}{i \cdot i!}(A+j+i)!\Big)$$

$$= \binom{B}{j}(C-1)!\sum_{i=1}^{A-C+j}(-1)^{i+j+1}\frac{D^{i-1}\binom{C+i-1}{i}}{(D+1)^{C+i}\binom{A-C+j}{i}} + \frac{(-1)^{A-C+1}}{D^{C-A-j}}$$

$$\times \binom{B}{j}\binom{A+j}{C}C!\left[\ln D + \sum_{i=1}^{\infty}\frac{(-1)^i}{i}\left(D^i\binom{A+j+i}{i} - \binom{A+j}{i}\right)\right].$$

This type of summand appears in (.1) whenever $C - A < B$.

Expressing (.1) via these summands we get (4.1).

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