Some subclasses of meromorphic functions involving the Hurwitz-Lerch Zeta function

Zhi-Gang Wang*† and Lei Shi‡

Abstract
The main purpose of this paper is to investigate some subclasses of meromorphic functions involving the meromorphic modified version of the familiar Srivastava-Attiya operator. Such results as inclusion relationships, convolution properties, coefficient inequalities, integral-preserving properties, subordination and superordination properties are proved.

Keywords: Analytic function; Meromorphic function; Hurwitz-Lerch Zeta function; Srivastava-Attiya operator; Differential subordination.

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1. Introduction
Let \( \Sigma \) denote the class of functions of the form
\[
(1.1) \quad f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k,
\]
which are analytic in the punctured open unit disk
\[
\mathbb{U}^* := \{ z : z \in \mathbb{C} \text{ and } 0 < |z| < 1 \} =: \mathbb{U} \setminus \{0\}.
\]
Let \( f, g \in \Sigma \), where \( f \) is given by (1.1) and \( g \) is defined by
\[
g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k.
\]

*Corresponding Author.
†School of Mathematics and Computing Science, Hunan First Normal University, Changsha 410205, Hunan, People’s Republic of China. Email: wangmath@163.com
‡School of Mathematics and Statistics, Anyang Normal University, Anyang 455000, Henan, People’s Republic of China. Email: shimath@163.com
Then the Hadamard product (or convolution) $f * g$ of the functions $f$ and $g$ is defined by

$$(f * g)(z) := \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k =: (g * f)(z).$$

Let $\mathcal{P}$ denote the class of functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

which are analytic and convex in $U$, and satisfy the condition

$$\Re(p(z)) > 0 \quad (z \in U).$$

For two functions $f$ and $g$, analytic in $U$, the function $f$ is said to be subordinate to $g$ in $U$, or the function $g$ is said to be superordinate to $f$ in $U$, and write

$$f(z) \prec g(z) \quad (z \in U),$$

if there exists a Schwarz function $\omega$, which is analytic in $U$ with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \quad (z \in U)$$

such that

$$f(z) = g(\omega(z)) \quad (z \in U).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in U) \implies f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Furthermore, if the function $g$ is univalent in $U$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in U) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).$$

The following we recall a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (cf., e. g., [20, p. 121 et sep.])

$$\Phi(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k + a)^s}$$

$$a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1,$$

where, as usual,

$$\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N} \quad (\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}; \mathbb{N} := \{1, 2, 3, \ldots\}).$$

Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ can be found in the recent investigations by (for example) Choi and Srivastava [1], Ferreira and López [4], Garg et al. [5], Lin et al. [7], Luo and Srivastava [10], Srivastava et al. [21], Ghanim [6] and others.

By making use of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$, Srivastava and Attiya [19] (see also [8, 9, 14, 17, 22, 23, 24, 27, 28, 29, 30]) recently introduced and investigated the integral operator

$$J_{s,b}(z) = z + \sum_{k=2}^{\infty} \left(1 + \frac{b}{k + b}\right)^s c_k z^k \quad (b \in \mathbb{C} \setminus \mathbb{Z}^-; s \in \mathbb{C}; z \in U).$$

Motivated essentially by the above-mentioned Srivastava-Attiya operator $J_{s,b}$, we now introduce the linear operator

$$W_{s,b} : \Sigma \longrightarrow \Sigma$$

defined, in terms of the Hadamard product (or convolution), by

$$(1.3) \quad W_{s,b} f(z) := \Theta_{s,b}(z) * f(z) \quad (b \in \mathbb{C} \setminus \mathbb{Z}_0^- \cup \{1\}; s \in \mathbb{C}; f \in \Sigma; z \in U^*).$$
where, for convenience,

\[(1.4) \quad \Theta_{s,b}(z) := (b - 1)^s \left[ \Phi(z, s, b) - b^{-s} + \frac{1}{z (b - 1)^s} \right] \quad (z \in \mathbb{U}^*). \]

It can easily be seen from (1.1) to (1.4) that

\[(1.5) \quad W_{s,b}f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left( \frac{b - 1}{b + k} \right)^s a_k z^k. \]

Indeed, the operator \( W_{s,b} \) can be defined for \( b \in \mathbb{C} \setminus \{ \mathbb{Z}^- \cup \{ 1 \} \} \), where

\[ W_{s,0}f(z) := \lim_{b \to 0} \{ W_{s,b}f(z) \}. \]

We observe that

\[(1.6) \quad W_{0,b}f(z) = f(z), \]

and

\[(1.7) \quad W_{1,\gamma}f(z) = \frac{\gamma - 1}{z^\gamma} \int_0^z \tau^{\gamma-1} f(t)dt \quad (\Re(\gamma) > 1). \]

Furthermore, from the definition (1.5), we find that

\[(1.8) \quad W_{s+1,b}f(z) = \frac{b - 1}{z^b} \int_0^z t^{b-1} W_{s,b}f(t)dt \quad (\Re(b) > 1). \]

Differentiating both sides of (1.8) with respect to \( z \), we get the following useful relationship:

\[(1.9) \quad z \left( W_{s+1,b}f \right)'(z) = (b - 1) W_{s,b}f(z) - b W_{s+1,b}f(z). \]

By using the integral operator (1.5), we now introduce the following subclasses of the class \( \Sigma \) of meromorphic functions.

1.1. Definition. A function \( f \in \Sigma \) is said to be in the class \( \mathcal{MS}_{s,b}(\eta; \phi) \) if it satisfies the subordination

\[(1.10) \quad \frac{1}{1 - \eta} \left( - \frac{z \left( W_{s,b}f \right)'(z)}{W_{s,b}f(z)} - \eta \right) < \phi(z) \quad (s \in \mathbb{C}; \Re(b) > 1; \eta \in [0,1); \phi \in \mathcal{P}; z \in \mathbb{U}). \]

1.2. Definition. A function \( f \in \Sigma \) is said to be in the class \( \mathcal{MC}_{s,b}(\lambda; \phi) \) if it satisfies the condition

\[(1.11) \quad (1 - \lambda)z W_{s+1,b}f(z) + \lambda z W_{s,b}f(z) < \phi(z) \quad (s, \lambda \in \mathbb{C}; \Re(b) > 1; \phi \in \mathcal{P}; z \in \mathbb{U}). \]

For some recent investigations on meromorphic functions, see (for example) the earlier works [2, 3, 15, 16, 25, 26, 31] and the references cited therein. In this paper, we aim at deriving the inclusion relationships, convolution properties, coefficient inequalities, integral-preserving properties, subordination and superordination properties for the function classes \( \mathcal{MS}_{s,b}(\eta; \phi) \) and \( \mathcal{MC}_{s,b}(\lambda; \phi) \).
2. Preliminary results

The following lemmas will be required in the proof of our main results.

2.1. Lemma. ([11]) Let \( \vartheta, \gamma \in \mathbb{C} \). Suppose that \( \psi \) is convex and univalent in \( \mathbb{U} \) with \( \psi(0) = 1 \) and \( \Re(\partial \psi + \gamma) > 0 \) \((z \in \mathbb{U})\).

If \( \psi \) is analytic in \( \Omega(0) = 1 \) and \( \Theta(z) \), then the following subordination

\[
p(z) + \frac{z \psi'(z)}{\psi(z) + \gamma} < \psi(z) \quad (z \in \mathbb{U})
\]

implies that

\[
p(z) < \psi(z) \quad (z \in \mathbb{U}).
\]

2.2. Lemma. Let \( 0 \leq \alpha < 1, s \in \mathbb{C} \) and \( \Re(b) > 1 \). Suppose also that the sequence \( \{A_k\}_{k=1}^{\infty} \) is defined by

\[
A_1 = (1 - \alpha) \left| \frac{b + 1}{b - 1} \right|^s, \quad A_{k+1} = 2(1 - \alpha) \left( 1 + \sum_{m=1}^{k} \left| \frac{b - 1}{b + m} \right|^s A_m \right) \quad (k \in \mathbb{N}).
\]

Then

\[
A_k = (1 - \alpha) \left| \frac{b + 1}{b - 1} \right|^s \prod_{j=1}^{k-1} \left| \frac{j - 2\alpha + 3}{j + 2} \right| \left| \frac{b + j + 1}{b + j} \right|^s.
\]

Proof. From (2.1), we find that

\[
(k + 2) \left| \frac{b - 1}{b + k + 1} \right|^s A_{k+1} = 2(1 - \alpha) \left( 1 + \sum_{m=1}^{k} \left| \frac{b - 1}{b + m} \right|^s A_m \right),
\]

and

\[
(k + 1) \left| \frac{b - 1}{b + k} \right|^s A_k = 2(1 - \alpha) \left( 1 + \sum_{m=1}^{k-1} \left| \frac{b - 1}{b + m} \right|^s A_m \right).
\]

Combining (2.3) and (2.4), we get

\[
\frac{A_{k+1}}{A_k} = \frac{k - 2\alpha + 3}{k + 2} \left| \frac{b + k + 1}{b + k} \right|^s.
\]

Thus, for \( k \geq 2 \), we deduce from (2.5) that

\[
A_k = \frac{A_k}{A_{k-1}} \cdots \frac{A_3}{A_2} A_1 = (1 - \alpha) \left| \frac{b + 1}{b - 1} \right|^s \prod_{j=1}^{k-1} \left| \frac{j - 2\alpha + 3}{j + 2} \right| \left| \frac{b + j + 1}{b + j} \right|^s.
\]

The proof of Lemma 2.2 is completed. \( \square \)

2.3. Lemma. ([12]) Let the function \( \Omega \) be analytic and convex (univalent) in \( \mathbb{U} \) with \( \Omega(0) = 1 \). Suppose also that the function \( \Theta \) given by

\[
\Theta(z) = 1 + d_n z^n + d_{n+1} z^{n+1} + \cdots
\]

is analytic in \( \mathbb{U} \). If

\[
\Theta(z) + \frac{z \Theta'(z)}{\zeta} < \Omega(z) \quad (\Re(\zeta) > 0; \; \zeta \neq 0; \; z \in \mathbb{U}),
\]

then

\[
\Theta(z) < \varpi(z) = \frac{\zeta}{n} z^{-\frac{n}{\zeta}} \int_0^{\varpi} t^{n-1} \Omega(t) dt \quad (z \in \mathbb{U}),
\]

\[
\Theta(z) < \varpi(z) = \frac{\zeta}{n} z^{-\frac{n}{\zeta}} \int_0^{\varpi} t^{n-1} \Omega(t) dt \quad (z \in \mathbb{U}),
\]

\[
\Theta(z) < \varpi(z) = \frac{\zeta}{n} z^{-\frac{n}{\zeta}} \int_0^{\varpi} t^{n-1} \Omega(t) dt \quad (z \in \mathbb{U}),
\]
and \(\omega\) is the best dominant of (2.6).

2.4. Lemma. ([18]) Let \(q\) be a convex univalent function in \(U\) and let \(\sigma, \eta \in \mathbb{C}\) with

\[
\Re\left(1 + \frac{q''(z)}{q'(z)}\right) > \max\left\{0, -\frac{\sigma}{\eta}\right\}.
\]

If \(p\) is analytic in \(U\) and

\[\sigma p(z) + \eta z p'(z) \prec \sigma q(z) + \eta z q'(z),\]

then \(p \prec q\) and \(q\) is the best dominant.

Denote by \(Q\) the set of all functions \(f\) that are analytic and injective on \(U - E(f)\), where

\[E(f) = \left\{\varepsilon \in \partial U : \lim_{z \to \varepsilon} f(z) = \infty\right\},\]

and such that \(f'(\varepsilon) \neq 0\) for \(\varepsilon \in \partial U - E(f)\). Let \(\mathcal{H}(U)\) denote the class of analytic functions in \(U\) and let \(\mathcal{H}[a, p]\) denote the subclass of the functions \(f \in \mathcal{H}(U)\) of the form:

\[f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots \quad (a \in \mathbb{C}; \ p \in \mathbb{N}).\]

2.5. Lemma. ([13]) Let \(q\) be convex univalent in \(U\) and \(\kappa \in \mathbb{C}\). Further assume that \(\Re(\kappa) > 0\). If

\[p \in \mathcal{H}[q(0), 1] \cap Q,\]

and \(p + \kappa z p'\) is univalent in \(U\), then

\[q(z) + \kappa z q'(z) \prec p(z) + \kappa z p'(z)\]

implies \(q \prec p\) and \(q\) is the best subdominant.

3. Main results

Firstly, we derive the following inclusion relationship for the function class \(\mathcal{M}S_{s, b}(\eta; \phi)\).

3.1. Theorem. Let \(0 \leq \eta < 1\) and \(\phi \in \mathcal{P}\) with

\[(1 - \eta)\phi(z) + \eta - b < 0 \quad (z \in U).\]

Then

\[\mathcal{M}S_{s, b}(\eta; \phi) \subset \mathcal{M}S_{s+1, b}(\eta; \phi).\]

Proof. Let \(f \in \mathcal{M}S_{s, b}(\eta; \phi)\) and suppose that

\[(1 - \eta)\phi(z) + \eta - b < 0 \quad (z \in U).\]

Then \(\phi\) is analytic in \(U\) with \(\phi(0) = 1\). By virtue of (1.9) and (3.3), we get

\[(b - 1)\left(z W_{s, b}(f(z)) W_{s+1, b}(f(z))\right) = -(1 - \eta)\phi(z) - \eta + b.\]

Differentiating both sides of (3.4) with respect to \(z\) logarithmically and using (3.3), we have

\[
\frac{1}{1 - \eta} \left(-z \frac{W_{s+1, b}(f(z))'}{W_{s, b}(f(z))} - \eta\right) = \phi(z) + \frac{z \phi'(z)}{(1 - \eta)\phi(z) - \eta + b} < \phi(z).
\]

By means of (3.1), an application of Lemma 2.1 to (3.5) yields

\[
\phi(z) = \frac{1}{1 - \eta} \left(-z \frac{W_{s+1, b}(f(z))'}{W_{s, b}(f(z))} - \eta\right) < \phi(z),
\]

that is \(f \in \mathcal{M}S_{s+1, b}(\eta; \phi)\), which implies that the assertion (3.2) of Theorem 3.1 holds. \(\square\)
Next, we derive some convolution properties of the class \( MS_{s,b}(\eta;\phi) \).

### 3.2. Theorem

Let \( f \in MS_{s,b}(\eta;\phi) \). Then

\[
(3.6) \quad f(z) = \left[ z^{-1} \cdot \exp \left( (\eta - 1) \int_0^z \phi(\omega(\xi)) - 1 \frac{d\xi}{\xi} \right) \right] \ast \left( \frac{1}{z} + \sum_{k=1}^{\infty} \left( \frac{b+k}{b+1} \right)^s z^k \right),
\]

where \( \omega \) is analytic in \( U \) with
\[
\omega(0) = 0 \text{ and } |\omega(z)| < 1 \quad (z \in \mathbb{U}).
\]

**Proof.** Suppose that \( f \in MS_{s,b}(\eta;\phi) \). We find from (1.10) that

\[
(3.7) \quad \frac{z (W_{s,b}f)'(z)}{W_{s,b}f(z)} = (\eta - 1)\phi(\omega(z)) - \eta,
\]

where \( \omega \) is analytic in \( U \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) \((z \in \mathbb{U})\). From (3.7), we get

\[
(3.8) \quad \frac{(W_{s,b}f)'(z)}{W_{s,b}f(z)} + \frac{1}{z} = (\eta - 1)\phi(\omega(z)) - 1,
\]

which, upon integration, yields

\[
(3.9) \quad \log(z W_{s,b}f(z)) = (\eta - 1) \int_0^z \phi(\omega(\xi)) - 1 \frac{d\xi}{\xi}.
\]

It follows from (3.9) that

\[
(3.10) \quad W_{s,b}f(z) = z^{-1} \cdot \exp \left( (\eta - 1) \int_0^z \phi(\omega(\xi)) - 1 \frac{d\xi}{\xi} \right).
\]

The assertion (3.6) of Theorem 3.2 can directly be derived from (1.5) and (3.10). \( \square \)

### 3.3. Theorem

Let \( f \in \Sigma \) and \( \phi \in \mathcal{P} \). Then \( f \in MS_{s,b}(\eta;\phi) \) if and only if

\[
(3.11) \quad \frac{1}{z} \left\{ f \ast \left\{ -\frac{1}{z} + \sum_{k=1}^{\infty} k \left( \frac{b-1}{b+k} \right)^s z^k - [ (\eta - 1)\phi(e^{i\theta}) - \eta] \left( \frac{1}{z} + \sum_{k=1}^{\infty} \left( \frac{b-1}{b+k} \right)^s z^k \right) \right\} \right\} \neq 0
\]

\((z \in \mathbb{U}^*; \ 0 \leq \theta < 2\pi)\).

**Proof.** Suppose that \( f \in MS_{s,b}(\eta;\phi) \). We know that (1.6) is equivalent to

\[
(3.12) \quad \frac{1}{1 - \eta} \left( -\frac{z (W_{s,b}f)'(z)}{W_{s,b}f(z)} - \eta \right) \neq \phi(e^{i\theta}) \quad (z \in \mathbb{U}; \ 0 \leq \theta < 2\pi).
\]

It is easy to see that the condition (3.12) can be written as follows:

\[
(3.13) \quad \frac{1}{z} \left\{ z (W_{s,b}f)'(z) - [ (\eta - 1)\phi(e^{i\theta}) - \eta] W_{s,b}f(z) \right\} \neq 0 \quad (z \in \mathbb{U}^*; \ 0 \leq \theta < 2\pi).
\]

On the other hand, we find from (1.5) that

\[
(3.14) \quad z (W_{s,b}f)'(z) = -\frac{1}{z} + \sum_{k=1}^{\infty} k \left( \frac{b-1}{b+k} \right)^s a_k z^k.
\]

Combining (1.5), (3.13) and (3.14), we get the assertion (3.11) of Theorem 3.3. \( \square \)
3.4. Theorem. If \( f \in \mathcal{M}S_{a,b}(0; [1 + (1 - 2\alpha)z]/(1 - z)) \), then

\[
|a_1| \leq (1 - \alpha) \left| \frac{b + 1}{b - 1} \right|^s,
\]

and

\[
|a_k| \leq (1 - \alpha) \left| \frac{b + 1}{b - 1} \right|^s \prod_{j=1}^{k-1} \left| \frac{j - 2\alpha + 3}{j + 2} \right| \left| \frac{b + j + 1}{b + j} \right|^s \quad (k \in \mathbb{N}\setminus\{1\}).
\]

Proof. Suppose that

\[
(3.15) \quad h(z) := -\frac{s(W_{a,b})'(z)}{W_{a,b}(z)} = 1 + c_1z + c_2z^2 + \cdots.
\]

It follows from \( f \in \mathcal{M}S_{a,b}(0; [1 + (1 - 2\alpha)z]/(1 - z)) \) that \( h \in \mathcal{P} \), and subsequently one has \( |c_k| \leq 2 \) for \( k \in \mathbb{N} \).

By virtue of (3.15), we know that

\[
(3.16) \quad z(W_{a,b})'(z) = [(\alpha - 1)h(z) - \alpha]}W_{a,b}(z).
\]

It now follows from (1.5), (3.15) and (3.16) that

\[
(3.17) \quad \frac{1}{z} \sum_{k=1}^{\infty} k \left( \frac{b - 1}{b + k} \right)^s a_k z^k = -1 + (\alpha - 1) (c_1z + c_2z^2 + \cdots) \left[ \frac{1}{z} \sum_{k=1}^{\infty} \left( \frac{b - 1}{b + k} \right)^s a_k z^k \right].
\]

By evaluating the coefficients of \( z^k \) in both sides of (3.17), we get

\[
(3.18) \quad k \left( \frac{b - 1}{b + k} \right)^s a_k = -\left( \frac{b - 1}{b + k} \right)^s a_k + (\alpha - 1) \left[ c_{k+1} + \sum_{l=1}^{k-1} c_l \left( \frac{b - 1}{b + k - l} \right)^s a_{k-l} \right].
\]

By observing the fact that \( |c_k| \leq 2 \) for \( k \in \mathbb{N} \), we find from (3.18) that

\[
(3.19) \quad |a_k| \leq \frac{2(1 - \alpha)}{k + 1} \left| \frac{b + k}{b - 1} \right|^s \left( 1 + \sum_{m=1}^{k-1} \left| \frac{b - 1}{b + m} \right|^s |a_m| \right).
\]

Now, we define the sequence \( \{A_k\}_{k=1}^{\infty} \) as follows:

\[
(3.20) \quad A_1 = (1 - \alpha) \left| \frac{b + 1}{b - 1} \right|^s, \quad A_{k+1} = \frac{2(1 - \alpha)}{k + 2} \left| \frac{b + k + 1}{b - 1} \right|^s \left( 1 + \sum_{m=1}^{k} \left| \frac{b - 1}{b + m} \right|^s A_m \right) \quad (k \in \mathbb{N}).
\]

In order to prove that

\[
|a_k| \leq A_k \quad (k \in \mathbb{N}),
\]

we make use of the principle of mathematical induction. By noting that

\[
|a_1| \leq A_1 = (1 - \alpha) \left| \frac{b + 1}{b - 1} \right|^s.
\]

Therefore, assuming that

\[
|a_m| \leq A_m \quad (m = 1, 2, 3, \ldots, k; \ k \in \mathbb{N}).
\]
Combining (3.19) and (3.20), we get
\[ |a_{k+1}| \leq \frac{2(1-\alpha)}{k+2} \left| \frac{b+k+1}{b-1} \right|^\kappa \left( 1 + \sum_{m=1}^{k} \left| \frac{b-1}{b+m} \right|^\kappa |a_m| \right) \]
\[ \leq \frac{2(1-\alpha)}{k+2} \left| \frac{b+k+1}{b-1} \right|^\kappa \left( 1 + \sum_{m=1}^{k} \left| \frac{b-1}{b+m} \right|^\kappa A_m \right) \]
\[ = A_{k+1}. \]

Hence, by the principle of mathematical induction, we have
\[ (3.21) \quad |a_k| \leq A_k \quad (k \in \mathbb{N}) \]
as desired.

By virtue of Lemma 2.2 and (3.20), we know that (2.2) holds. Combining (3.21) and (2.2), we readily get the coefficient estimates asserted by Theorem 3.4. □

In what follows, we derive some integral-preserving properties for the class \( MS_{s,b}(\eta; \phi) \).

3.5. Theorem. Let \( f \in MS_{s,b}(\eta; \phi) \) with
\[ \Re((1-\eta)\phi(z) + \eta - \mu) < 0 \quad (z \in U; \quad \Re(\mu) > 1). \]
Then the integral operator \( F \) defined by
\[ \begin{align*}
F(z) &:= \frac{\mu - 1}{z^\mu} \int_0^z \frac{t^{\mu-1} f(t)}{W_{s,b}(t)} dt \quad (z \in U^*; \quad \Re(\mu) > 1) \\
\end{align*} \]
belongs to the class \( MS_{s,b}(\eta; \phi) \).

Proof. Let \( f \in MS_{s,b}(\eta; \phi) \). We then find from (3.22) that
\[ (3.23) \quad z (W_{s,b}F)'(z) + \mu W_{s,b}F(z) = (\mu - 1)W_{s,b}f(z). \]
By setting
\[ (3.24) \quad q(z) := \frac{1}{1-\eta} \left( -z \frac{(W_{s,b}F)'(z)}{W_{s,b}F(z)} - \eta \right), \]
we observe that \( q \) is analytic in \( U \) with \( q(0) = 1 \). It follows from (3.23) and (3.24) that
\[ (3.25) \quad -(1-\eta)\eta q(z) - \eta + \mu = (\mu - 1) \frac{W_{s,b}f(z)}{W_{s,b}F(z)}. \]
Differentiating both sides of (3.25) with respect to \( z \) logarithmically and using (3.24), we get
\[ (3.26) \quad q(z) + \frac{zq'(z)}{-(1-\eta)\eta q(z) - \eta + \mu} = \frac{1}{1-\eta} \left( -z \frac{(W_{s,b}F)'(z)}{W_{s,b}F(z)} - \eta \right) \prec \phi(z). \]
Since
\[ \Re(-(1-\eta)\phi(z) - \eta + \mu) > 0 \quad (z \in U), \]
by virtue of Lemma 2.1 and (3.26), we obtain
\[ \frac{1}{1-\eta} \left( -z \frac{(W_{s,b}F)'(z)}{W_{s,b}F(z)} - \eta \right) \prec \phi(z), \]
which implies that the assertion of Theorem 3.5 holds. □
3.6. Theorem. Let $f \in \text{MS}_s, b(\eta; \phi)$ with
\[
\Re((1 - \eta)\delta \phi(z) + \eta \delta - \mu) < 0 \quad (z \in \mathbb{U}; \delta \neq 0; \mu \in \mathbb{C}).
\]
Then the function $K \in \Sigma$ defined by
\[
W_{s, b}K(z) := \left( \frac{\mu - \delta}{z^\mu} \int_0^z t^{\mu - 1} (W_{s, b}f(t))^{\delta} dt \right)^{1/\delta} \quad (z \in \mathbb{U}^*; \delta \neq 0)
\]
belongs to the class $\text{MS}_s, b(\eta; \phi)$.

Proof. Let $f \in \text{MS}_s, b(\eta; \phi)$ and suppose that
\[
\rho(z) := \frac{1}{1 - \eta} \left( -\frac{zW_{s, b}K'(z)}{W_{s, b}K(z)} - \eta \right) \quad (z \in \mathbb{U}).
\]
In view of (3.27) and (3.28), we have
\[
\lambda - \eta \delta - (1 - \eta)\delta \rho(z) = (\lambda - \delta) \left( \frac{W_{s, b}f(z)}{W_{s, b}K(z)} \right)^\delta.
\]
Now, by means of (3.27), (3.28) and (3.29), we obtain
\[
\rho(z) + \frac{z \rho'(z)}{\mu - \eta \delta - (1 - \eta)\delta \rho(z)} = \frac{1}{1 - \eta} \left( -z \left( W_{s, b}f(z) \right)^\delta - \eta \right) \prec \phi(z).
\]
Since
\[
\Re(\lambda - \eta \delta - (1 - \eta)\delta \rho(z)) > 0 \quad (z \in \mathbb{U}),
\]
it follows from (3.30) and Lemma 2.1 that $\rho(z) \prec \phi(z)$, that is $K \in \text{MS}_s, b(\eta; \phi)$. We thus complete the proof of Theorem 3.6.

Now, we derive the following subordination property for the class $\text{MC}_s, b(\lambda; \phi)$.

3.7. Theorem. Let $f \in \text{MC}_s, b(\lambda; \phi)$ with $\Re(\lambda/(b - 1)) > 0$. Then
\[
z W_{s+1, b}f(z) \prec \frac{b - 1}{2 \lambda} z^{-\frac{b - 1}{2 \lambda}} \int_0^z t^{\frac{b - 1}{2 \lambda} - 1} \phi(t) dt \prec \phi(z).
\]

Proof. Let $f \in \text{MC}_s, b(\lambda; \phi)$ and suppose that
\[
b(z) := z W_{s+1, b}f(z) \quad (z \in \mathbb{U}).
\]
Then $b$ is analytic in $\mathbb{U}$. By virtue of (1.5), (1.11) and (3.32), we find that
\[
b(z) + \frac{\lambda}{b - 1} z b'(z) = (1 - \lambda)z W_{s+1, b}f(z) + \lambda z W_{s, b}f(z) \prec \phi(z).
\]
Thus, an application of Lemma 2.3 to (3.33) yields the desired assertion (3.31) of Theorem 3.7.

3.8. Theorem. Let $\lambda_2 > \lambda_1 \geq 0$. Then $\text{MC}_s, b(\lambda_2; \phi) \subset \text{MC}_s, b(\lambda_1; \phi)$.

Proof. Suppose that $f \in \text{MC}_s, b(\lambda_2; \phi)$. It follows that
\[
(1 - \lambda_2)z W_{s+1, b}f(z) + \lambda_2 z W_{s, b}f(z) \prec \phi(z) \quad (z \in \mathbb{U}).
\]
Since
\[
0 \leq \frac{\lambda_1}{\lambda_2} < 1
\]
and the function $\phi$ is convex and univalent in $\mathbb{U}$, we deduce from (3.31) and (3.34) that
\[
(1 - \lambda_1)z W_{s+1, b}f(z) + \lambda_1 z W_{s, b}f(z)
\]
\[
= \frac{\lambda_1}{\lambda_2} [(1 - \lambda_2)z W_{s+1, b}f(z) + \lambda_2 z W_{s, b}f(z)] + \left( 1 - \frac{\lambda_1}{\lambda_2} \right) z W_{s+1, b}f(z) \prec \phi(z),
\]
which implies that \( f \in \mathcal{MC}_{s, b}(\lambda; \phi) \). The proof of Theorem 3.8 is thus completed. \( \square \)

**3.9. Theorem.** Let \( f \in \mathcal{MC}_{s, b}(\lambda; \phi) \). If the function \( F \in \Sigma \) is defined by (3.22), then
\[
(3.35) \quad z W_{s+1, b} F(z) \prec \phi(z) \quad (z \in U).
\]

**Proof.** Let \( f \in \mathcal{MC}_{s, b}(\lambda; \phi) \) and suppose that
\[
(3.36) \quad \chi(z) := z W_{s+1, b} F(z) \quad (z \in U).
\]
From (3.22), we find that
\[
(3.37) \quad z (W_{s+1, b} F)'(z) + \mu W_{s+1, b} F(z) = (\mu - 1) W_{s+1, b} F(z).
\]
By virtue of (3.31), (3.36) and (3.37), we have
\[
(3.38) \quad \chi(z) + \frac{1}{\mu - 1} z \chi'(z) = z W_{s+1, b} f(z) \prec \phi(z).
\]
Thus, an application of Lemma 2.3 to (3.38), we get the assertion of Theorem 3.9. \( \square \)

**3.10. Theorem.** Let \( q_1 \) be univalent in \( U \). Suppose also that \( q_1 \) satisfies the condition
\[
(3.39) \quad \mathbb{R} \left( 1 + \frac{z q_1''(z)}{q_1'(z)} \right) > \max \left\{ 0, -\frac{b - 1}{\lambda} \right\}.
\]
If \( f \in \Sigma \) satisfies the following subordination
\[
(3.40) \quad (1 - \lambda) z W_{s+1, b} f(z) + \lambda z W_{s, b} f(z) \prec q_1(z) + \frac{\lambda}{b - 1} z q_1'(z),
\]
then
\[
z W_{s+1, b} f(z) \prec q_1(z),
\]
and \( q_1 \) is the best dominant.

**Proof.** Let the function \( b \) be defined by (3.32). We know that (3.33) holds. Combining (3.33) and (3.40), we find that
\[
(3.41) \quad b(z) + \frac{\lambda}{b - 1} z b'(z) \prec q_1(z) + \frac{\lambda}{b - 1} z q_1'(z).
\]
By Lemma 2.4 and (3.41), we obtain the assertion of Theorem 3.10. \( \square \)

We now derive the following superordination result for the class \( \mathcal{MC}_{s, b}(\lambda; \phi) \).

**3.11. Theorem.** Let \( q_2 \) be convex univalent in \( U \), \( \lambda \in \mathbb{C} \) with \( \mathbb{R}(\lambda) > 0 \). Also let \( z W_{s+1, b} f(z) \in \mathcal{H}(q_2(0), 1] \cap Q \) and \( (1 - \lambda) z W_{s+1, b} f(z) + \lambda z W_{s, b} f(z) \) be univalent in \( U \). If
\[
q_2(z) + \frac{\lambda}{b - 1} z q_2'(z) \prec (1 - \lambda) z W_{s+1, b} f(z) + \lambda z W_{s, b} f(z),
\]
then
\[
q_2(z) \prec z W_{s+1, b} f(z),
\]
and \( q_2 \) is the best subordinant.

**Proof.** Let the function \( b \) be defined by (3.32). Then
\[
q_2(z) + \frac{\lambda}{b - 1} z q_2'(z) \prec (1 - \lambda) z W_{s+1, b} f(z) + \lambda z W_{s, b} f(z) = b(z) + \frac{\lambda}{b - 1} z b'(z).
\]
Thus, an application of Lemma 2.5, yields the assertion of Theorem 3.11. \( \square \)

Finally, combining the above-mentioned subordination and superordination results, we obtain the following sandwich type result.
3.12. Corollary. Let \( q_3 \) be convex univalent and let \( q_4 \) be univalent in \( U, \lambda \in \mathbb{C} \) with \( \Re(\lambda) > 0 \). Suppose also that \( q_4 \) satisfies the condition
\[
\Re \left( 1 + \frac{zq_4''(z)}{q_4'(z)} \right) > \max \left\{ 0, -\Re \left( \frac{b-1}{\lambda} \right) \right\}.
\]
If \( 0 \neq z \mathcal{W}_{s+1}, b f(z) \in \mathbb{H}[q_3(0), 1] \cap Q \) and \((1-\lambda)z \mathcal{W}_{s+1}, b f(z)+\lambda z \mathcal{W}_s, b f(z) \) is univalent in \( U \), also
\[
q_3(z) + \frac{\lambda}{b-1} zq_3'(z) \prec (1-\lambda)z \mathcal{W}_{s+1}, b f(z)+\lambda z \mathcal{W}_s, b f(z) \prec q_4(z) + \frac{\lambda}{b-1} zq_4'(z),
\]
then
\[
q_3(z) \prec z \mathcal{W}_{s+1}, b f(z) \prec q_4(z),
\]
and \( q_3 \) and \( q_4 \) are, respectively, the best subordinant and the best dominant.

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