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Some subclasses of meromorphic functions involving the Hurwitz-Lerch Zeta function

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Abstract

The main purpose of this paper is to investigate some subclasses of meromorphic functions involving the meromorphic modified version of the familiar Srivastava-Attiya operator. Such results as inclusion relationships, convolution properties, coefficient inequalities, integral-preserving properties, subordination and superordination properties are proved.

Keywords: Analytic function; Meromorphic function; Hurwitz-Lerch Zeta function; Srivastava-Attiya opertor; Differential subordination.

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1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \tag{1.1}$$

which are *analytic* in the *punctured* open unit disk

$$\mathbb{U}^* := \{z : \ z \in \mathbb{C} \ \text{ and } \ 0 < |z| < 1\} =: \mathbb{U}\backslash\{0\}.$$

Let $f, \ g \in \Sigma$, where f is given by (1.1) and g is defined by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k.$$

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Then the Hadamard product (or convolution) $f*g$ of the functions f and g is defined by

$$(f*g)(z) := \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k =: (g*f)(z).$$

Let $\mathcal{P}$ denote the class of functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

which are analytic and convex in $\mathbb{U}$, and satisfy the condition

$$\Re(p(z)) > 0 \quad (z \in \mathbb{U}).$$

For two functions f and g, analytic in $\mathbb{U}$, the function f is said to be subordinate to g in $\mathbb{U}$, or the function g is said to be superordinate to f in $\mathbb{U}$, and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \ (z \in \mathbb{U})$$

such that

$$f(z) = g\big(\omega(z)\big) \quad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \ (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \ (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

The following we recall a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (*cf., e.g.,* [20, p. 121 *et sep.*])

$$\Phi(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s} \tag{1.2}$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \ s \in \mathbb{C} \text{ when } |z| < 1; \ \Re(s) > 1 \text{ when } |z| = 1),$$

where, as usual,

$$\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N} \quad (\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}; \ \mathbb{N} := \{1, 2, 3, \ldots\}).$$

Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ can be found in the recent investigations by (for example) Choi and Srivastava [1], Ferreira and López [4], Garg *et al.* [5], Lin *et al.* [7], Luo and Srivastava [10], Srivastava *et al.* [21], Ghanim [6] and others.

By making use of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$, Srivastava and Attiya [19] (see also [8, 9, 14, 17, 22, 23, 24, 27, 28, 29, 30]) recently introduced and investigated the integral operator

$$\mathcal{J}_{s,b}\mathfrak{f}(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b}\right)^s c_k z^k \quad (b \in \mathbb{C} \setminus \mathbb{Z}^-; \ s \in \mathbb{C}; \ z \in \mathbb{U}).$$

Motivated essentially by the above-mentioned Srivastava-Attiya operator $\mathcal{J}_{s,b}$, we now introduce the linear operator

$$\mathcal{W}_{s,b} : \ \Sigma \longrightarrow \Sigma$$

defined, in terms of the Hadamard product (or convolution), by

$$\mathcal{W}_{s,b} f(z) := \Theta_{s,b}(z) * f(z) \quad \left(b \in \mathbb{C} \setminus \{\mathbb{Z}_0^- \cup \{1\}\}; \ s \in \mathbb{C}; \ f \in \Sigma; \ z \in \mathbb{U}^*\right), \tag{1.3}$$

where, for convenience,

$$\Theta_{s,\,b}(z) := (b-1)^s \left[\Phi(z,s,b) - b^{-s} + \frac{1}{z\,(b-1)^s}\right] \quad (z \in \mathbb{U}^*). \tag{1.4}$$

It can easily be seen from (1.1) to (1.4) that

$$\mathcal{W}_{s,\,b} f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{b-1}{b+k}\right)^s a_k z^k. \tag{1.5}$$

Indeed, the operator $\mathcal{W}_{s,\,b}$ can be defined for $b \in \mathbb{C} \setminus \{\mathbb{Z}^- \cup \{1\}\}$, where

$$\mathcal{W}_{s,\,0} f(z) := \lim_{b \to 0} \{\mathcal{W}_{s,\,b} f(z)\}.$$

We observe that

$$\mathcal{W}_{0,\,b} f(z) = f(z), \tag{1.6}$$

and

$$\mathcal{W}_{1,\,\gamma} f(z) = \frac{\gamma - 1}{z^\gamma} \int_0^z t^{\gamma-1} f(t)dt \quad (\Re(\gamma) > 1). \tag{1.7}$$

Furthermore, from the definition (1.5), we find that

$$\mathcal{W}_{s+1,\,b} f(z) = \frac{b-1}{z^b} \int_0^z t^{\,b-1} \mathcal{W}_{s,\,b} f(t)dt \quad (\Re(b) > 1). \tag{1.8}$$

Differentiating both sides of (1.8) with respect to z, we get the following useful relationship:

$$z\,(\mathcal{W}_{s+1,\,b} f)'\,(z) = (b-1)\,\mathcal{W}_{s,\,b} f(z) - b\,\mathcal{W}_{s+1,\,b} f(z). \tag{1.9}$$

By using the integral operator (1.5), we now introduce the following subclasses of the class Σ of meromorphic functions.

1.1. Definition. A function $f \in \Sigma$ is said to be in the class $\mathcal{MS}_{s,\,b}(\eta;\phi)$ if it satisfies the subordination

$$\frac{1}{1-\eta}\left(-\frac{z\,(\mathcal{W}_{s,\,b} f)'\,(z)}{\mathcal{W}_{s,\,b} f(z)} - \eta\right) \prec \phi(z) \tag{1.10}$$

$$(s \in \mathbb{C};\ \Re(b) > 1;\ \eta \in [0,1);\ \phi \in \mathcal{P};\ z \in \mathbb{U}).$$

1.2. Definition. A function $f \in \Sigma$ is said to be in the class $\mathcal{MC}_{s,\,b}(\lambda;\phi)$ if it satisfies the condition

$$(1-\lambda) z\,\mathcal{W}_{s+1,\,b} f(z) + \lambda z\,\mathcal{W}_{s,\,b} f(z) \prec \phi(z) \quad (s,\,\lambda \in \mathbb{C};\ \Re(b) > 1;\ \phi \in \mathcal{P};\ z \in \mathbb{U}). \tag{1.11}$$

For some recent investigations on meromorphic functions, see (for example) the earlier works [2, 3, 15, 16, 25, 26, 31] and the references cited therein. In this paper, we aim at deriving the inclusion relationships, convolution properties, coefficient inequalities, integral-preserving properties, subordination and superordination properties for the function classes $\mathcal{MS}_{s,\,b}(\eta;\phi)$ and $\mathcal{MC}_{s,\,b}(\lambda;\phi)$.

2. Preliminary results

The following lemmas will be required in the proof of our main results.

2.1. Lemma. ([11]) *Let ϑ, $\gamma \in \mathbb{C}$. Suppose that ψ is convex and univalent in $\mathbb{U}$ with*

$$\psi(0) = 1 \text{ and } \Re(\vartheta\psi(z) + \gamma) > 0 \ (z \in \mathbb{U}).$$

If $\mathfrak{p}$ is analytic in $\mathbb{U}$ with $\mathfrak{p}(0) = 1$, then the following subordination

$$\mathfrak{p}(z) + \frac{z\mathfrak{p}'(z)}{\vartheta\mathfrak{p}(z) + \gamma} \prec \psi(z) \quad (z \in \mathbb{U})$$

implies that

$$\mathfrak{p}(z) \prec \psi(z) \quad (z \in \mathbb{U}).$$

2.2. Lemma. *Let $0 \leq \alpha < 1$, $s \in \mathbb{C}$ and $\Re(b) > 1$. Suppose also that the sequence $\{A_k\}_{k=1}^{\infty}$ is defined by*

(2.1)

$$A_1 = (1-\alpha)\left|\frac{b+1}{b-1}\right|^s, \ A_{k+1} = \frac{2(1-\alpha)}{k+2}\left|\frac{b+k+1}{b-1}\right|^s\left(1 + \sum_{m=1}^{k}\left|\frac{b-1}{b+m}\right|^s A_m\right) \ (k \in \mathbb{N}).$$

Then

$$A_k = (1-\alpha)\left|\frac{b+1}{b-1}\right|^s \prod_{j=1}^{k-1}\frac{j-2\alpha+3}{j+2}\left|\frac{b+j+1}{b+j}\right|^s. \tag{2.2}$$

Proof. From (2.1), we find that

$$(k+2)\left|\frac{b-1}{b+k+1}\right|^s A_{k+1} = 2(1-\alpha)\left(1 + \sum_{m=1}^{k}\left|\frac{b-1}{b+m}\right|^s A_m\right), \tag{2.3}$$

and

$$(k+1)\left|\frac{b-1}{b+k}\right|^s A_k = 2(1-\alpha)\left(1 + \sum_{m=1}^{k-1}\left|\frac{b-1}{b+m}\right|^s A_m\right). \tag{2.4}$$

Combining (2.3) and (2.4), we get

$$\frac{A_{k+1}}{A_k} = \frac{k-2\alpha+3}{k+2}\left|\frac{b+k+1}{b+k}\right|^s. \tag{2.5}$$

Thus, for $k \geq 2$, we deduce from (2.5) that

$$A_k = \frac{A_k}{A_{k-1}} \cdot \cdots \cdot \frac{A_3}{A_2} \cdot \frac{A_2}{A_1} \cdot A_1 = (1-\alpha)\left|\frac{b+1}{b-1}\right|^s \prod_{j=1}^{k-1}\frac{j-2\alpha+3}{j+2}\left|\frac{b+j+1}{b+j}\right|^s.$$

The proof of Lemma 2.2 is completed. □

2.3. Lemma. ([12]) *Let the function Ω be analytic and convex (univalent) in $\mathbb{U}$ with $\Omega(0) = 1$. Suppose also that the function Θ given by*

$$\Theta(z) = 1 + d_n z^n + d_{n+1} z^{n+1} + \cdots$$

is analytic in $\mathbb{U}$. If

$$\Theta(z) + \frac{z\Theta'(z)}{\zeta} \prec \Omega(z) \quad (\Re(\zeta) > 0;\ \zeta \neq 0;\ z \in \mathbb{U}), \tag{2.6}$$

then

$$\Theta(z) \prec \varpi(z) = \frac{\zeta}{n} z^{-\frac{\zeta}{n}} \int_0^z t^{\frac{\zeta}{n}-1}\Omega(t)dt \prec \Omega(z) \quad (z \in \mathbb{U}),$$

and ϖ is the best dominant of (2.6).

2.4. Lemma. ([18]) *Let q be a convex univalent function in $\mathbb{U}$ and let $\sigma,\ \eta \in \mathbb{C}$ with*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0,\ -\Re\left(\frac{\sigma}{\eta}\right)\right\}.$$

If p is analytic in $\mathbb{U}$ and

$$\sigma p(z) + \eta z p'(z) \prec \sigma q(z) + \eta z q'(z),$$

then $p \prec q$ and q is the best dominant.

Denote by Q the set of all functions f that are analytic and injective on $\overline{\mathbb{U}} - E(f)$, where

$$E(f) = \left\{\varepsilon \in \partial\mathbb{U} :\ \lim_{z\to\varepsilon} f(z) = \infty\right\},$$

and such that $f'(\varepsilon) \neq 0$ for $\varepsilon \in \partial\mathbb{U} - E(f)$. Let $\mathcal{H}(\mathbb{U})$ denote the class of analytic functions in $\mathbb{U}$ and let $\mathcal{H}[a,p]$ denote the subclass of the functions $f \in \mathcal{H}(\mathbb{U})$ of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots \quad (a \in \mathbb{C};\ p \in \mathbb{N}).$$

2.5. Lemma. ([13]) *Let q be convex univalent in $\mathbb{U}$ and $\kappa \in \mathbb{C}$. Further assume that $\Re(\kappa) > 0$. If*

$$p \in \mathcal{H}[q(0),1] \cap Q,$$

and $p + \kappa z p'$ is univalent in $\mathbb{U}$, then

$$q(z) + \kappa z q'(z) \prec p(z) + \kappa z p'(z)$$

implies $q \prec p$ and q is the best subordinant.

3. Main results

Firstly, we derive the following inclusion relationship for the function class $\mathcal{MS}_{s,\,b}(\eta;\phi)$.

3.1. Theorem. *Let $0 \le \eta < 1$ and $\phi \in \mathcal{P}$ with*

$$\Re\left((1-\eta)\phi(z) + \eta - b\right) < 0 \quad (z \in \mathbb{U}). \tag{3.1}$$

Then

$$\mathcal{MS}_{s,\,b}(\eta;\phi) \subset \mathcal{MS}_{s+1,\,b}(\eta;\phi). \tag{3.2}$$

Proof. Let $f \in \mathcal{MS}_{s,\,b}(\eta;\phi)$ and suppose that

$$\varphi(z) := \frac{1}{1-\eta}\left(-\frac{z\left(\mathcal{W}_{s+1,\,b}f\right)'(z)}{\mathcal{W}_{s+1,\,b}f(z)} - \eta\right) \quad (z \in \mathbb{U}). \tag{3.3}$$

Then φ is analytic in $\mathbb{U}$ with $\varphi(0) = 1$. By virtue of (1.9) and (3.3), we get

$$(b-1)\frac{\mathcal{W}_{s,\,b}f(z)}{\mathcal{W}_{s+1,\,b}f(z)} = -(1-\eta)\varphi(z) - \eta + b. \tag{3.4}$$

Differentiating both sides of (3.4) with respect to z logarithmically and using (3.3), we have

$$\frac{1}{1-\eta}\left(-\frac{z\left(\mathcal{W}_{s,\,b}f\right)'(z)}{\mathcal{W}_{s,\,b}f(z)} - \eta\right) = \varphi(z) + \frac{z\varphi'(z)}{-(1-\eta)\varphi(z) - \eta + b} \prec \phi(z). \tag{3.5}$$

By means of (3.1), an application of Lemma 2.1 to (3.5) yields

$$\varphi(z) = \frac{1}{1-\eta}\left(-\frac{z\left(\mathcal{W}_{s+1,\,b}f\right)'(z)}{\mathcal{W}_{s+1,\,b}f(z)} - \eta\right) \prec \phi(z),$$

that is $f \in \mathcal{MS}_{s+1,\,b}(\eta;\phi)$, which implies that the assertion (3.2) of Theorem 3.1 holds. $\square$

Next, we derive some convolution properties of the class $\mathcal{MS}_{s,b}(\eta;\phi)$.

3.2. Theorem. *Let $f \in \mathcal{MS}_{s,b}(\eta;\phi)$. Then*

$$f(z) = \left[z^{-1} \cdot \exp\left((\eta-1)\int_0^z \frac{\phi(\omega(\xi))-1}{\xi} d\xi\right)\right] * \left(\frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{b+k}{b-1}\right)^s z^k\right), \tag{3.6}$$

where ω is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \ (z \in \mathbb{U}).$$

Proof. Suppose that $f \in \mathcal{MS}_{s,b}(\eta;\phi)$. We find from (1.10) that

$$\frac{z\left(\mathcal{W}_{s,b}f\right)'(z)}{\mathcal{W}_{s,b}f(z)} = (\eta-1)\phi(\omega(z)) - \eta, \tag{3.7}$$

where ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$. From (3.7), we get

$$\frac{\left(\mathcal{W}_{s,b}f\right)'(z)}{\mathcal{W}_{s,b}f(z)} + \frac{1}{z} = (\eta-1)\frac{\phi(\omega(z))-1}{z}, \tag{3.8}$$

which, upon integration, yields

$$\log\left(z\,\mathcal{W}_{s,b}f(z)\right) = (\eta-1)\int_0^z \frac{\phi(\omega(\xi))-1}{\xi} d\xi. \tag{3.9}$$

It follows from (3.9) that

$$\mathcal{W}_{s,b}f(z) = z^{-1} \cdot \exp\left((\eta-1)\int_0^z \frac{\phi(\omega(\xi))-1}{\xi} d\xi\right). \tag{3.10}$$

The assertion (3.6) of Theorem 3.2 can directly be derived from (1.5) and (3.10). □

3.3. Theorem. *Let $f \in \Sigma$ and $\phi \in \mathcal{P}$. Then $f \in \mathcal{MS}_{s,b}(\eta;\phi)$ if and only if*

(3.11)

$$\frac{1}{z}\left\{f * \left\{-\frac{1}{z} + \sum_{k=1}^{\infty} k\left(\frac{b-1}{b+k}\right)^s z^k - \left[(\eta-1)\phi\left(e^{i\theta}\right) - \eta\right]\left(\frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{b-1}{b+k}\right)^s z^k\right)\right\}\right\} \neq 0$$
$$(z \in \mathbb{U}^*;\ 0 \leq \theta < 2\pi).$$

Proof. Suppose that $f \in \mathcal{MS}_{s,b}(\eta;\phi)$. We know that (1.6) is equivalent to

$$\frac{1}{1-\eta}\left(-\frac{z\left(\mathcal{W}_{s,b}f\right)'(z)}{\mathcal{W}_{s,b}f(z)} - \eta\right) \neq \phi\left(e^{i\theta}\right) \quad (z \in \mathbb{U};\ 0 \leq \theta < 2\pi). \tag{3.12}$$

It is easy to see that the condition (3.12) can be written as follows:

$$\frac{1}{z}\left\{z\left(\mathcal{W}_{s,b}f\right)'(z) - \left[(\eta-1)\phi\left(e^{i\theta}\right) - \eta\right]\mathcal{W}_{s,b}f(z)\right\} \neq 0 \quad (z \in \mathbb{U}^*;\ 0 \leq \theta < 2\pi). \tag{3.13}$$

On the other hand, we find from (1.5) that

$$z\left(\mathcal{W}_{s,b}f\right)'(z) = -\frac{1}{z} + \sum_{k=1}^{\infty} k\left(\frac{b-1}{b+k}\right)^s a_k z^k. \tag{3.14}$$

Combining (1.5), (3.13) and (3.14), we get the assertion (3.11) of Theorem 3.3. □

3.4. Theorem. *If* $f \in \mathcal{MS}_{s,\,b}(0; [1+(1-2\alpha)z]/(1-z))$, *then*

$$|a_1| \leq (1-\alpha)\left|\frac{b+1}{b-1}\right|^s,$$

and

$$|a_k| \leq (1-\alpha)\left|\frac{b+1}{b-1}\right|^s \prod_{j=1}^{k-1} \frac{j-2\alpha+3}{j+2}\left|\frac{b+j+1}{b+j}\right|^s \quad (k \in \mathbb{N}\backslash\{1\}).$$

Proof. Suppose that

$$h(z) := \frac{-\frac{z(\mathcal{W}_{s,\,b}f)'(z)}{\mathcal{W}_{s,\,b}f(z)} - \alpha}{1-\alpha} = 1 + c_1 z + c_2 z^2 + \cdots. \tag{3.15}$$

It follows from $f \in \mathcal{MS}_{s,\,b}(0; [1+(1-2\alpha)z]/(1-z))$ that $h \in \mathcal{P}$, and subsequently one has $|c_k| \leq 2$ for $k \in \mathbb{N}$.

By virtue of (3.15), we know that

$$z\left(\mathcal{W}_{s,\,b}f\right)'(z) = [(\alpha-1)h(z) - \alpha]\mathcal{W}_{s,\,b}f(z). \tag{3.16}$$

It now follows from (1.5), (3.15) and (3.16) that

(3.17)

$$\frac{1}{z} + \sum_{k=1}^{\infty} k\left(\frac{b-1}{b+k}\right)^s a_k z^k = \left[-1 + (\alpha-1)\left(c_1 z + c_2 z^2 + \cdots\right)\right]\left[\frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{b-1}{b+k}\right)^s a_k z^k\right].$$

By evaluating the coefficients of z^k in both sides of (3.17), we get

$$k\left(\frac{b-1}{b+k}\right)^s a_k = -\left(\frac{b-1}{b+k}\right)^s a_k + (\alpha-1)\left[c_{k+1} + \sum_{l=1}^{k-1} c_l\left(\frac{b-1}{b+k-l}\right)^s a_{k-l}\right]. \tag{3.18}$$

By observing the fact that $|c_k| \leq 2$ for $k \in \mathbb{N}$, we find from (3.18) that

$$|a_k| \leq \frac{2(1-\alpha)}{k+1}\left|\frac{b+k}{b-1}\right|^s\left(1 + \sum_{m=1}^{k-1}\left|\frac{b-1}{b+m}\right|^s |a_m|\right). \tag{3.19}$$

Now, we define the sequence $\{A_k\}_{k=1}^{\infty}$ as follows:

(3.20)

$$A_1 = (1-\alpha)\left|\frac{b+1}{b-1}\right|^s, \;\; A_{k+1} = \frac{2(1-\alpha)}{k+2}\left|\frac{b+k+1}{b-1}\right|^s\left(1 + \sum_{m=1}^{k}\left|\frac{b-1}{b+m}\right|^s A_m\right) \; (k \in \mathbb{N}).$$

In order to prove that

$$|a_k| \leq A_k \quad (k \in \mathbb{N}),$$

we make use of the principle of mathematical induction. By noting that

$$|a_1| \leq A_1 = (1-\alpha)\left|\frac{b+1}{b-1}\right|^s.$$

Therefore, assuming that

$$|a_m| \leq A_m \quad (m = 1, 2, 3, \cdots, k;\; k \in \mathbb{N}).$$

Combining (3.19) and (3.20), we get

$$\begin{aligned}|a_{k+1}| &\leq \frac{2(1-\alpha)}{k+2}\left|\frac{b+k+1}{b-1}\right|^s\left(1+\sum_{m=1}^{k}\left|\frac{b-1}{b+m}\right|^s|a_m|\right)\\ &\leq \frac{2(1-\alpha)}{k+2}\left|\frac{b+k+1}{b-1}\right|^s\left(1+\sum_{m=1}^{k}\left|\frac{b-1}{b+m}\right|^s A_m\right)\\ &= A_{k+1}.\end{aligned}$$

Hence, by the principle of mathematical induction, we have

$$|a_k| \leq A_k \quad (k \in \mathbb{N}) \tag{3.21}$$

as desired.

By virtue of Lemma 2.2 and (3.20), we know that (2.2) holds. Combining (3.21) and (2.2), we readily get the coefficient estimates asserted by Theorem 3.4. □

In what follows, we derive some integral-preserving properties for the class $\mathcal{MS}_{s,\,b}(\eta;\phi)$.

3.5. Theorem. *Let $f \in \mathcal{MS}_{s,\,b}(\eta;\phi)$ with*

$$\Re((1-\eta)\phi(z)+\eta-\mu) < 0 \quad (z \in \mathbb{U};\ \Re(\mu) > 1).$$

Then the integral operator F defined by

$$F(z) := \frac{\mu-1}{z^{\mu}}\int_0^z t^{\mu-1}f(t)dt \quad (z \in \mathbb{U}^*;\ \Re(\mu) > 1) \tag{3.22}$$

belongs to the class $\mathcal{MS}_{s,\,b}(\eta;\phi)$.

Proof. Let $f \in \mathcal{MS}_{s,\,b}(\eta;\phi)$. We then find from (3.22) that

$$z\left(\mathcal{W}_{s,\,b}F\right)'(z)+\mu\mathcal{W}_{s,\,b}F(z) = (\mu-1)\mathcal{W}_{s,\,b}f(z). \tag{3.23}$$

By setting

$$q(z) := \frac{1}{1-\eta}\left(-\frac{z\left(\mathcal{W}_{s,\,b}F\right)'(z)}{\mathcal{W}_{s,\,b}F(z)}-\eta\right), \tag{3.24}$$

we observe that q is analytic in $\mathbb{U}$ with $q(0) = 1$. It follows from (3.23) and (3.24) that

$$-(1-\eta)q(z)-\eta+\mu = (\mu-1)\frac{\mathcal{W}_{s,\,b}f(z)}{\mathcal{W}_{s,\,b}F(z)}. \tag{3.25}$$

Differentiating both sides of (3.25) with respect to z logarithmically and using (3.24), we get

$$q(z)+\frac{zq'(z)}{-(1-\eta)q(z)-\eta+\mu} = \frac{1}{1-\eta}\left(-\frac{z\left(\mathcal{W}_{s,\,b}f\right)'(z)}{\mathcal{W}_{s,\,b}f(z)}-\eta\right) \prec \phi(z). \tag{3.26}$$

Since

$$\Re(-(1-\eta)\phi(z)-\eta+\mu) > 0 \quad (z \in \mathbb{U}),$$

by virtue of Lemma 2.1 and (3.26), we obtain

$$\frac{1}{1-\eta}\left(-\frac{z\left(\mathcal{W}_{s,\,b}F\right)'(z)}{\mathcal{W}_{s,\,b}F(z)}-\eta\right) \prec \phi(z),$$

which implies that the assertion of Theorem 3.5 holds. □

3.6. Theorem. *Let* $f \in \mathcal{MS}_{s,\ b}(\eta;\phi)$ *with*

$$\Re((1-\eta)\delta\,\phi(z)+\eta\,\delta-\mu)<0 \quad (z\in\mathbb{U};\ \delta\neq 0;\ \mu\in\mathbb{C}).$$

Then the function $K\in\Sigma$ *defined by*

$$\text{(3.27)}\qquad \mathcal{W}_{s,\ b}K(z) := \left(\frac{\mu-\delta}{z^{\mu}}\int_0^z t^{\mu-1}\left(\mathcal{W}_{s,\ b}f(t)\right)^{\delta}dt\right)^{1/\delta} \qquad (z\in\mathbb{U}^*;\ \delta\neq 0)$$

belongs to the class $\mathcal{MS}_{s,\ b}(\eta;\phi)$.

Proof. Let $f\in\mathcal{MS}_{s,\ b}(\eta;\phi)$ and suppose that

$$\text{(3.28)}\qquad \varrho(z) := \frac{1}{1-\eta}\left(-\frac{z\left(\mathcal{W}_{s,\ b}K\right)'(z)}{\mathcal{W}_{s,\ b}K(z)}-\eta\right) \qquad (z\in\mathbb{U}).$$

In view of (3.27) and (3.28), we have

$$\text{(3.29)}\qquad \mu-\eta\,\delta-(1-\eta)\delta\,\varrho(z) = (\mu-\delta)\left(\frac{\mathcal{W}_{s,\ b}f(z)}{\mathcal{W}_{s,\ b}K(z)}\right)^{\delta}.$$

Now, by means of (3.27), (3.28) and (3.29), we obtain

$$\text{(3.30)}\qquad \varrho(z)+\frac{z\varrho'(z)}{\mu-\eta\,\delta-(1-\eta)\delta\,\varrho(z)} = \frac{1}{1-\eta}\left(-\frac{z\left(\mathcal{W}_{s,\ b}f\right)'(z)}{\mathcal{W}_{s,\ b}f(z)}-\eta\right) \prec \phi(z).$$

Since

$$\Re(\mu-\eta\,\delta-(1-\eta)\delta\,\phi(z))>0 \quad (z\in\mathbb{U}),$$

it follows from (3.30) and Lemma 2.1 that $\varrho(z)\prec\phi(z)$, that is $K\in\mathcal{MS}_{s,\ b}(\eta;\phi)$. We thus complete the proof of Theorem 3.6. □

Now, we derive the following subordination property for the class $\mathcal{MC}_{s,\ b}(\lambda;\phi)$.

3.7. Theorem. *Let* $f\in\mathcal{MC}_{s,\ b}(\lambda;\phi)$ *with* $\Re(\lambda/(b-1))>0$. *Then*

$$\text{(3.31)}\qquad z\,\mathcal{W}_{s+1,\ b}f(z) \prec \frac{b-1}{2\lambda}z^{-\frac{b-1}{2\lambda}}\int_0^z t^{\frac{b-1}{2\lambda}-1}\phi(t)dt \prec \phi(z).$$

Proof. Let $f\in\mathcal{MC}_{s,\ b}(\lambda;\phi)$ and suppose that

$$\text{(3.32)}\qquad \mathfrak{h}(z) := z\,\mathcal{W}_{s+1,\ b}f(z) \quad (z\in\mathbb{U}).$$

Then $\mathfrak{h}$ is analytic in $\mathbb{U}$. By virtue of (1.5), (1.11) and (3.32), we find that

$$\text{(3.33)}\qquad \mathfrak{h}(z)+\frac{\lambda}{b-1}z\mathfrak{h}'(z) = (1-\lambda)z\,\mathcal{W}_{s+1,\ b}f(z)+\lambda z\,\mathcal{W}_{s,\ b}f(z)\prec\phi(z).$$

Thus, an application of Lemma 2.3 to (3.33) yields the desired assertion (3.31) of Theorem 3.7. □

3.8. Theorem. *Let* $\lambda_2>\lambda_1\geq 0$. *Then* $\mathcal{MC}_{s,\ b}(\lambda_2;\phi)\subset\mathcal{MC}_{s,\ b}(\lambda_1;\phi)$.

Proof. Suppose that $f\in\mathcal{MC}_{s,\ b}(\lambda_2;\phi)$. It follows that

$$\text{(3.34)}\qquad (1-\lambda_2)z\,\mathcal{W}_{s+1,\ b}f(z)+\lambda_2 z\,\mathcal{W}_{s,\ b}f(z)\prec\phi(z) \quad (z\in\mathbb{U}).$$

Since

$$0\leq\frac{\lambda_1}{\lambda_2}<1$$

and the function ϕ is convex and univalent in $\mathbb{U}$, we deduce from (3.31) and (3.34) that

$$\begin{aligned}&(1-\lambda_1)z\,\mathcal{W}_{s+1,\ b}f(z)+\lambda_1 z\,\mathcal{W}_{s,\ b}f(z)\\ &=\frac{\lambda_1}{\lambda_2}\left[(1-\lambda_2)z\,\mathcal{W}_{s+1,\ b}f(z)+\lambda_2 z\,\mathcal{W}_{s,\ b}f(z)\right]+\left(1-\frac{\lambda_1}{\lambda_2}\right)z\,\mathcal{W}_{s+1,\ b}f(z)\prec\phi(z),\end{aligned}$$

which implies that $f \in \mathcal{MC}_{s,\ b}(\lambda_1;\phi)$. The proof of Theorem 3.8 is thus completed. □

3.9. Theorem. *Let $f \in \mathcal{MC}_{s,\ b}(\lambda;\phi)$. If the function $F \in \Sigma$ is defined by (3.22), then*

(3.35) $\quad z\,\mathcal{W}_{s+1,\ b}F(z) \prec \phi(z) \quad (z \in \mathbb{U}).$

Proof. Let $f \in \mathcal{MC}_{s,\ b}(\lambda;\phi)$ and suppose that

(3.36) $\quad \chi(z) := z\,\mathcal{W}_{s+1,\ b}F(z) \quad (z \in \mathbb{U}).$

From (3.22), we find that

(3.37) $\quad z\left(\mathcal{W}_{s+1,\ b}F\right)'(z) + \mu\,\mathcal{W}_{s+1,\ b}F(z) = (\mu-1)\,\mathcal{W}_{s+1,\ b}f(z).$

By virtue of (3.31), (3.36) and (3.37), we have

(3.38) $$\chi(z) + \frac{1}{\mu-1}z\,\chi'(z) = z\,\mathcal{W}_{s+1,\ b}f(z) \prec \phi(z).$$

Thus, an application of Lemma 2.3 to (3.38), we get the assertion of Theorem 3.9. □

3.10. Theorem. *Let q_1 be univalent in $\mathbb{U}$. Suppose also that q_1 satisfies the condition*

(3.39) $$\Re\left(1 + \frac{zq_1''(z)}{q_1'(z)}\right) > \max\left\{0,\ -\Re\left(\frac{b-1}{\lambda}\right)\right\}.$$

If $f \in \Sigma$ satisfies the following subordination

(3.40) $$(1-\lambda)z\,\mathcal{W}_{s+1,\ b}f(z) + \lambda z\,\mathcal{W}_{s,\ b}f(z) \prec q_1(z) + \frac{\lambda}{b-1}zq_1'(z),$$

then

$$z\,\mathcal{W}_{s+1,\ b}f(z) \prec q_1(z),$$

and q_1 is the best dominant.

Proof. Let the function $\mathfrak{h}$ be defined by (3.32). We know that (3.33) holds. Combining (3.33) and (3.40), we find that

(3.41) $$\mathfrak{h}(z) + \frac{\lambda}{b-1}z\mathfrak{h}'(z) \prec q_1(z) + \frac{\lambda}{b-1}zq_1'(z).$$

By Lemma 2.4 and (3.41), we obtain the assertion of Theorem 3.10. □

We now derive the following superordination result for the class $\mathcal{MC}_{s,\ b}(\lambda;\phi)$.

3.11. Theorem. *Let q_2 be convex univalent in $\mathbb{U}$, $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$. Also let $z\,\mathcal{W}_{s+1,\ b}\ f(z) \in \mathcal{H}[q_2(0),1] \cap Q$ and $(1-\lambda)z\,\mathcal{W}_{s+1,\ b}\ f(z) + \lambda z\,\mathcal{W}_{s,\ b}f(z)$ be univalent in $\mathbb{U}$. If*

$$q_2(z) + \frac{\lambda}{b-1}zq_2'(z) \prec (1-\lambda)z\,\mathcal{W}_{s+1,\ b}\ f(z) + \lambda z\,\mathcal{W}_{s,\ b}f(z),$$

then

$$q_2(z) \prec z\,\mathcal{W}_{s+1,\ b}f(z),$$

and q_2 is the best subordinant.

Proof. Let the function $\mathfrak{h}$ be defined by (3.32). Then

$$q_2(z) + \frac{\lambda}{b-1}zq_2'(z) \prec (1-\lambda)z\,\mathcal{W}_{s+1,\ b}f(z) + \lambda z\,\mathcal{W}_{s,\ b}f(z) = \mathfrak{h}(z) + \frac{\lambda}{b-1}z\mathfrak{h}'(z).$$

Thus, an application of Lemma 2.5, yields the assertion of Theorem 3.11. □

Finally, combining the above-mentioned subordination and superordination results, we obtain the following sandwich type result.

3.12. Corollary. *Let q_3 be convex univalent and let q_4 be univalent in $\mathbb{U}$, $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$. Suppose also that q_4 satisfies the condition*

$$\Re\left(1 + \frac{zq_4''(z)}{q_4'(z)}\right) > \max\left\{0,\ -\Re\left(\frac{b-1}{\lambda}\right)\right\}.$$

If $0 \neq z\,\mathcal{W}_{s+1,\ b}f(z) \in \mathcal{H}[q_3(0), 1] \cap Q$ *and* $(1-\lambda)z\,\mathcal{W}_{s+1,\ b}f(z) + \lambda z\,\mathcal{W}_{s,\ b}f(z)$ *is univalent in* $\mathbb{U}$, *also*

$$q_3(z) + \frac{\lambda}{b-1} zq_3'(z) \prec (1-\lambda)z\,\mathcal{W}_{s+1,\ b}f(z) + \lambda z\,\mathcal{W}_{s,\ b}f(z) \prec q_4(z) + \frac{\lambda}{b-1} zq_4'(z),$$

then

$$q_3(z) \prec z\,\mathcal{W}_{s+1,\ b}f(z) \prec q_4(z),$$

and q_3 and q_4 are, respectively, the best subordinant and the best dominant.

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