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Dividend moments for two classes of risk processes with phase-type interclaim times

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Abstract

In this paper, we consider the distribution of discounted dividend payments until ruin under a risk model with two independent classes of claims in which both of the two interclaim times have phase-type distributions and a constant dividend barrier. We obtain the integro-differential equations with boundary conditions for the moment-generating function of the sum of the discounted dividend payments until ruin. Explicit expressions for arbitrary moments of the discounted dividend payments are derived if the distribution of the two classes claim amount both belong to the rational family. Finally, numerical illustrations are presented to show how the results are applied.

Keywords: Two classes of risk processes, Dividend payments, Moment-generating function, Dividend barrier, Phase-type distribution.

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1. Introduction

The ruin problems for a risk model involving two independent classes of risks have been considered by many researchers, see, for example, [9], [10], [15], and among others. As an extension of these papers, [5] investigated the risk model with two classes of renewal risk processes by assuming that both of the two claim number processes have phase-type interclaim times. The topics of these literatures are concentrated on the Gerber-Shiu discounted penalty function, which is an important tool to quantify the riskiness of the risk model.

In recent years, particular attention has been devoted to the risk models with dividend strategies. We refer the readers to, e.g. [6], [12], [13], [14] for details. The distribution of

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the discounted sum of dividend payments until ruin which is an important quantity in assessing the quality of dividend strategies has been studied by [7], [11], and the references therein. In particular, [1] presented some results on the distribution of dividend payments until ruin in a Sparre Andersen risk model with generalized Erlang(n)-distributed interclaim times and a constant dividend barrier which complemented the results of [8]. [16] considered dividend payments with a threshold strategy in the compound Poisson risk model perturbed by diffusion. [4] extended the results of [16] via assuming that the interclaim times follow a generalized Erlang(n) distribution. As a more general framework, [3] considered surplus processes of which the claim number is a Markovian arrival process perturbed by diffusion with dividend barrier strategies.

The main purpose of the current paper is to investigate the distribution of the discounted sum of dividend payments until ruin for two classes of risk processes in the presence of a constant dividend barrier, where both of the two claim number processes have phase-type interclaim. This paper is a natural extension of [1] and enriches the results for two classes of renewal risk processes. The rest of the paper is structured as follows. Section 2 describes the risk model. In Section 3, we derive systems of integro-differential equations for the moment-generating function of the sum of discounted dividend payments until ruin. Section 4 presents the results for arbitrary moments of the discounted dividend payments and derives explicit expressions when the two classes claim amount distributions both belong to the rational family. In Section 5, a numerical example is given.

2. Model setup

The surplus process $R(t)$ of an insurance portfolio is given by

$$R(t) = u + ct - S(t), \quad t \geq 0, \tag{2.1}$$

where $u \geq 0$ is the initial surplus, c denotes the insurer's premium income per unit time, and the aggregate-claim process $\{S(t) : t \geq 0\}$ is defined by

$$S(t) = \sum_{i=1}^{N_1(t)} X_i + \sum_{i=1}^{N_2(t)} Y_i, \quad t \geq 0,$$

where $\{X_1, X_2, \cdots\}$ and $\{Y_1, Y_2, \cdots\}$ are independent and identically distributed (i.i.d.) positive random variables representing the successive individual claim amounts from the first and the second class, respectively. The random variables $\{X_1, X_2, \cdots\}$ are assumed to have common cumulative distribution function $F(x) = 1 - \bar{F}(x), x \geq 0$, with probability density function $f(x) = F'(x)$, of which the Laplace transform is $\tilde{f}(s) = \int_0^\infty e^{-sx} f(x)dx$, $s \in \mathbb{C}$, $\mathbb{C}$ denotes the complex space. Similarly, common cumulative distribution function, density function and the Laplace transform of the density function of $\{Y_1, Y_2, \cdots\}$ are given by $G(x) = 1 - \bar{G}(x), x \geq 0$, $g(x) = G'(x)$ and $\tilde{g}(s) = \int_0^\infty e^{-sx} g(x)dx$. The renewal processes $\{N_1(t); t \geq 0\}$ and $\{N_2(t); t \geq 0\}$ denote the number of claims up to time t caused by the first and the second class of claim respectively, and are defined as follows:

$$N_1(t) = \sup\{n : T_1 + T_2 + \cdots + T_n \leq t\},$$

$$N_2(t) = \sup\{n : V_1 + V_2 + \cdots + V_n \leq t\},$$

where the i.i.d. interclaim times $\{T_1, T_2, \cdots\}$ have common cumulative distribution function $K_1(t), t \geq 0$ and density function $k_1(x) = K_1'(x)$, and $\{V_1, V_2, \cdots\}$ have common cumulative distribution function $K_2(t), t \geq 0$ and density function $k_2(x) = K_2'(x)$.

In addition, we assume that $\{X_1, X_2, \cdots\}$, $\{Y_1, Y_2, \cdots\}$, $\{N_1(t); t \geq 0\}$ and $\{N_2(t); t \geq 0\}$ are mutually independent. The net profit condition is given by $c > E(X_1)/E(T_1) + E(Y_1)/E(V_1)$.

In the present paper, we consider the risk model (2.1) with a constant dividend barrier $d(\geq 0)$. For such a dividend strategy, it is assumed that whenever the surplus process reaches the level d, the premium income is paid out as dividends to policyholders; otherwise, no dividend is paid. Let $R_d(t)$ be the surplus of an insurance company at time t under a constant dividend barrier d, then

$$dR_d(t) = \begin{cases} cdt - dS(t), & R_d(t) < b, \\ -dS(t), & R_d(t) \geq b. \end{cases}$$

The time of (ultimate) ruin is $T = \inf\{t | R(t) < 0\}$, where $T = \infty$ if $R(t) \geq 0$ for all $t \geq 0$. The probability of ruin is $\psi(u) = Pr(T < \infty)$.

Denote by $D(t)$ the cumulative amount of dividends paid out up to time t and $\delta > 0$ the force of interest, then $\mathbb{D} = \int_0^T e^{-\delta t} dD(t)$ is the present value of all dividends until ruin time T. In the following text, we turn to the moment generating function of $\mathbb{D}$,

$$M(u, y, d) = E[e^{y\mathbb{D}} | R(0) = u]$$

(for those values of y where it exists) and the rth moment

$$W(u, r, d) = E[\mathbb{D}^r | R(0) = u], \quad r \in \mathbb{N}.$$

Note that $W(u, 0, d) \equiv 1$. We will always assume that $0 \leq u \leq d$ (otherwise the overflow is immediately paid out as dividends) and that $M(u, y, d)$ and $W(u, r, d)$ are sufficiently smooth functions in u and y, respectively.

Throughout the text of the paper, all bold-faced letters represent either vectors or matrices and all vectors are column vectors. We assume that the distribution $K_1(t)$ of the interclaim time random variable T_1 is phase-type with representation $(\boldsymbol{\alpha}^\top, \mathbf{A}, \mathbf{a})$, where $\boldsymbol{\alpha}^\top = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, with $\alpha_i \geq 0$, $\sum_{i=1}^n \alpha_i = 1$, $\mathbf{A} = (a_{ij})_{i,j=1}^n$ is an $n \times n$ matrix with $a_{ii} < 0, a_{ij} \geq 0$, for $i \neq j$, $\sum_{j=1}^n a_{ij} \leq 0$, for any $i = 1, 2, \cdots, n$, and $\mathbf{a} = (a_1, a_2, \cdots, a_n)^\top$ with $\mathbf{a} = -\mathbf{A}\mathbf{e}_n$, where $\mathbf{x}^\top$ denotes the transpose of $\mathbf{x}$ and $\mathbf{e}_n$ denotes a n-dimensional column vector with all elements being one. Following [2], we have

$$K_1(t) = 1 - \boldsymbol{\alpha}^\top e^{\mathbf{A}t}\mathbf{e}_n, \quad k_1(t) = \boldsymbol{\alpha}^\top e^{\mathbf{A}t}\mathbf{a}, \quad t \geq 0,$$

and

$$\tilde{k}_1(s) = \int_0^\infty e^{-st} k_1(t) dt = \boldsymbol{\alpha}^\top (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{a}. \tag{2.2}$$

By the definition of phase-type distributions, each of the interclaim times $T_i, i = 1, 2, \cdots$, corresponds to the time to absorption in a terminating continuous-time Markov Chain, say, $I_t^{(i)}$ with n transient states $\{E_1, E_2, \cdots, E_n\}$ and one absorbing state E_0.

Correspondingly, the distribution $K_2(t)$ of the interclaim time random variable V_1 is phase-type with representation $(\boldsymbol{\beta}^\top, \mathbf{B}, \mathbf{b})$, where $\boldsymbol{\beta}^\top = (\beta_1, \beta_2, \cdots, \beta_m)$, $\mathbf{B} = (b_{ij})_{i,j=1}^m$ is an $m \times m$ matrix, $\mathbf{b} = (b_1, b_2, \cdots, b_m)^\top$ with $\mathbf{b} = -\mathbf{B}\mathbf{e}_m$. Then we have

$$K_2(t) = 1 - \boldsymbol{\beta}^\top e^{\mathbf{B}t}\mathbf{e}_m, k_2(t) = \boldsymbol{\beta}^\top e^{\mathbf{B}t}\mathbf{b}, t \geq 0,$$

and

$$\tilde{k}_2(s) = \int_0^\infty e^{-st} k_2(t) dt = \boldsymbol{\beta}^\top (s\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}. \tag{2.3}$$

Similarly, $J_t^{(i)}$ denotes the terminating continuous-time Markov Chain of $V_i, i = 1, 2, \cdots$, with m transient states $\{F_1, F_2, \cdots, F_m\}$ and one absorbing state F_0.

Now, we construct a two-dimensional Markov process $\{(I(t), J(t)); t \geq 0\}$ by piecing the $\{I_t^{(i)}; i = 1, 2, \cdots\}$ and $\{J_t^{(i)}; i = 1, 2, \cdots\}$ together,

$$I(t) = \{I_t^{(1)}\}, 0 \leq t < T_1, \quad I(t) = \{I_{t-T_1}^{(2)}\}, T_1 \leq t < T_1 + T_2, \cdots,$$

$$J(t) = \{J_t^{(1)}\}, 0 \leq t < V_1, \quad J(t) = \{J_{t-V_1}^{(2)}\}, V_1 \leq t < V_1 + V_2, \cdots.$$

So $\{(I(t), J(t)); t \geq 0\}$ is the underlying state process with states $\{(E_1, F_1), (E_2, F_1), \cdots, (E_n, F_1), (E_1, F_2), (E_2, F_2), \cdots, (E_n, F_2), \cdots, (E_1, F_m), (E_2, F_m), \cdots, (E_n, F_m)\}$, initial distribution $\boldsymbol{\gamma} = \boldsymbol{\beta} \otimes \boldsymbol{\alpha}$, where $\otimes$ denotes the Kronecker product of two matrices.

For $k = 1, 2; i = 1, 2, \cdots, n; j = 1, 2, \cdots, m$, let $M^{(k)}(u, y, d)$ denote the moment generating function of $\mathbb{D}$ if the ruin is caused by a claim from class k and $R(0) = u$. $M_{ij}^{(k)}(u, y, d)$ denotes the moment generating function of $\mathbb{D}$ when the ruin is caused by a claim from class k and initial state $(I_0^{(1)}, J_0^{(1)}) = (E_i, F_j)$, then the moment generating function can be written as

$$M^{(k)}(u, y, d) = \boldsymbol{\gamma}^\top \mathbf{M}^{(k)}(u, y, d), \tag{2.4}$$

where $\mathbf{M}^{(k)}(u, y, d) \equiv \Big(M_{11}^{(k)}(u, y, d), M_{21}^{(k)}(u, y, d), \cdots, M_{n1}^{(k)}(u, y, d),\ M_{12}^{(k)}(u, y, d), M_{22}^{(k)}(u, y, d), \cdots, M_{n2}^{(k)}(u, y, d), \cdots,\ M_{1m}^{(k)}(u, y, d), M_{2m}^{(k)}(u, y, d), \cdots, M_{nm}^{(k)}(u, y, d)\Big)^\top$. Thus

$$M(u, y, d) = \boldsymbol{\gamma}^\top \mathbf{M}(u, y, d) = \boldsymbol{\gamma}^\top [\mathbf{M}^{(1)}(u, y, d) + \mathbf{M}^{(2)}(u, y, d)]. \tag{2.5}$$

Let $W_{ij}(u, r, d)$ denote the rth moment of $\mathbb{D}$ if $(I_0^{(1)}, J_0^{(1)}) = (E_i, F_j)$. Then the moment can be computed by

$$W(u, r, d) = \boldsymbol{\gamma}^\top \mathbf{W}(u, r, d), \tag{2.6}$$

where $\mathbf{W}(u, r, d) \equiv (W_{11}(u, r, d), W_{21}(u, r, d), \cdots, W_{n1}(u, r, d),\ W_{12}(u, r, d), W_{22}(u, r, d), \cdots, W_{n2}(u, r, d), \cdots,\ W_{1m}(u, r, d),\ W_{2m}(u, r, d), \cdots,\ W_{nm}(u, r, d))^\top$.

3. Integro-differential system for $\mathbf{M}^{(k)}(u, y, d)$

Let $\frac{\partial \cdot}{\partial u}$ and $\frac{\partial \cdot}{\partial y}$ denote the differentiation operators with respect to u and y, respectively.

3.1. Theorem. *The vectors* $\mathbf{M}^{(k)}(u, y, d), 0 \leq u \leq d, k = 1, 2$ *satisfy the following partial integro-differential system, respectively,*

$$\begin{aligned}&\left(c\frac{\partial}{\partial u} - y\delta\frac{\partial}{\partial y}\right)\mathbf{M}^{(1)}(u, y, d) + I_{m\times m} \otimes \mathbf{A}\mathbf{M}^{(1)}(u, y, d) + \\ &\mathbf{B} \otimes I_{n\times n}\mathbf{M}^{(1)}(u, y, d) + I_{m\times m} \otimes (\mathbf{a}\boldsymbol{\alpha}^\top) \textstyle\int_0^u \mathbf{M}^{(1)}(u - x, y, d) f(x) dx + \\ &(\mathbf{b}\boldsymbol{\beta}^\top) \otimes I_{n\times n} \textstyle\int_0^u \mathbf{M}^{(1)}(u - x, y, d) g(x) dx + (\mathbf{e}_m \otimes \mathbf{a})\bar{F}(u) = \mathbf{0},\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}&\left(c\frac{\partial}{\partial u} - y\delta\frac{\partial}{\partial y}\right)\mathbf{M}^{(2)}(u, y, d) + I_{m\times m} \otimes \mathbf{A}\mathbf{M}^{(2)}(u, y, d) + \\ &\mathbf{B} \otimes I_{n\times n}\mathbf{M}^{(2)}(u, y, d) + I_{m\times m} \otimes (\mathbf{a}\boldsymbol{\alpha}^\top) \textstyle\int_0^u \mathbf{M}^{(2)}(u - x, y, d) f(x) dx + \\ &(\mathbf{b}\boldsymbol{\beta}^\top) \otimes I_{n\times n} \textstyle\int_0^u \mathbf{M}^{(2)}(u - x, y, d) g(x) dx + (\mathbf{b} \otimes \mathbf{e}_n)\bar{G}(u) = \mathbf{0},\end{aligned} \tag{3.2}$$

with boundary conditions

$$\left.\frac{\partial \mathbf{M}^{(k)}(u, y, d)}{\partial u}\right|_{u=d} = y\mathbf{M}^{(k)}(d, y, d), \quad k = 1, 2, \tag{3.3}$$

where $I_{n\times n}$ *denotes the* $n \times n$ *identity matrix,* $\mathbf{0}$ *denotes a column vector of length* mn *with all elements being 0.*

Proof. Considering an infinitesimal time interval $(0, dt)$ for $0 \le u \le d$, there are four possible events regarding to the occurrence of the claim and change of the environment: (i) no claim arrival and no change of state; (ii) a claim arrival but no change of state; (iii) a change of state but no claim arrival; (iv) two or more events occur. Taking into account the above four events in $(0, dt)$ and using the total expectation formula, it follows that

$$\begin{aligned}
&M_{ij}^{(1)}(u,y,d)\\
&= (1+a_{ii}dt)(1+b_{jj}dt)M_{ij}^{(1)}(u+cdt, ye^{-\delta dt}, d)\\
&+(1+b_{jj}dt)\sum_{k=1,k\neq i}^{n}(a_{ik}dt)M_{kj}^{(1)}(u+cdt, ye^{-\delta dt}, d)\\
&+(1+a_{ii}dt)\sum_{h=1,h\neq j}^{m}(b_{jh}dt)M_{ih}^{(1)}(u+cdt, ye^{-\delta dt}, d)\\
&+(1+b_{jj}dt)(a_i dt)\left[\sum_{s=1}^{n}\alpha_s\int_0^{u+cdt}M_{sj}^{(1)}(u+cdt-x, ye^{-\delta dt}, d)f(x)dx\right.\\
&\left.+\int_{u+cdt}^{\infty}f(x)dx\right]\\
&+(1+a_{ii}dt)(b_j dt)\sum_{r=1}^{m}\beta_r\int_0^{u+cdt}M_{ir}^{(1)}(u+cdt-x, ye^{-\delta dt}, d)g(x)dx + o(dt).
\end{aligned} \tag{3.4}$$

By Taylor expansion,

$$\begin{aligned}
& M_{ij}^{(1)}(u+cdt, ye^{-\delta dt}, d)\\
= \;& M_{ij}^{(1)}(u,y,d) + cdt\frac{\partial M_{ij}^{(1)}(u,y,d)}{\partial u} + y(e^{-\delta dt}-1)\frac{\partial M_{ij}^{(1)}(u,y,d)}{\partial y} + o(dt).
\end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.4), dividing by dt and then letting $dt \to 0$, it yields that

$$\begin{aligned}
& c\frac{\partial M_{ij}^{(1)}(u,y,d)}{\partial u} - y\delta\frac{\partial M_{ij}^{(1)}(u,y,d)}{\partial y} + \sum_{k=1}^{n}a_{ik}M_{kj}^{(1)}(u,y,d) + \sum_{h=1}^{m}b_{jh}M_{ih}^{(1)}(u,y,d)\\
& + a_i\left(\sum_{s=1}^{n}\alpha_s\int_0^u M_{sj}^{(1)}(u-x,y,d)f(x)dx + \int_u^\infty f(x)dx\right)\\
& + b_j\sum_{r=1}^{m}\beta_r\int_0^u M_{ir}^{(1)}(u-x,y,d)g(x)dx = 0.
\end{aligned} \tag{3.6}$$

Rewriting (3.6) in matrix form and rearranging it, we have (3.1). By similar derivation to (3.4)-(3.6), we get (3.2).

When $u = d$, we have

$$\begin{aligned}
&M_{ij}^{(1)}(d,y,d)\\
&= (1+a_{ii}dt)(1+b_{jj}dt)e^{ycdt}M_{ij}^{(1)}(d, ye^{-\delta dt}, d)\\
&+(1+b_{jj}dt)e^{ycdt}\sum_{k=1,k\neq i}^{n}(a_{ik}dt)M_{kj}^{(1)}(d, ye^{-\delta dt}, d)\\
&+(1+a_{ii}dt)e^{ycdt}\sum_{h=1,h\neq j}^{m}(b_{jh}dt)M_{ih}^{(1)}(d, ye^{-\delta dt}, d)\\
&+(1+b_{jj}dt)(a_i dt)e^{ycdt}\left[\sum_{s=1}^{n}\alpha_s\int_0^{d}M_{sj}^{(1)}(d-x, ye^{-\delta dt}, d)f(x)dx\right.\\
&\left.+\int_{d}^{\infty}f(x)dx\right]\\
&+(1+a_{ii}dt)(b_j dt)e^{ycdt}\sum_{r=1}^{m}\beta_r\int_0^{d}M_{ir}^{(1)}(d-x, ye^{-\delta dt}, d)g(x)dx + o(dt).
\end{aligned} \tag{3.7}$$

It follows from Taylor expansion that

$$
\begin{aligned}
&ycM_{ij}^{(1)}(d,y,d) - y\delta\frac{\partial M_{ij}^{(1)}(d,y,d)}{\partial y} + \sum_{k=1}^{n} a_{ik}M_{kj}^{(1)}(d,y,d) + \sum_{h=1}^{m} b_{jh}M_{ih}^{(1)}(d,y,d)\\
&+a_i\left(\sum_{s=1}^{n}\alpha_s\int_0^d M_{sj}^{(1)}(d-x,y,d)f(x)dx + \int_d^\infty f(x)dx\right)\\
&+b_j\sum_{r=1}^{m}\beta_r\int_0^d M_{ir}^{(1)}(d-x,y,d)g(x)dx = 0.
\end{aligned}
\tag{3.8}
$$

Comparing the above equations with the corresponding equations in (3.6) and utilizing the continuity of $M_{ij}^{(1)}(u,y,d)$ at $u=d$, then

$$
\left.\frac{\partial \mathbf{M}^{(1)}(u,y,d)}{\partial u}\right|_{u=d} = y\mathbf{M}^{(1)}(d,y,d).
$$

By the same approach, we can obtain the boundary conditions (3.3) for $k=2$. $\square$

3.2. Remark. When $m=1$ and $G(0)=1$, from Eq.(3.1), we have

$$
\begin{aligned}
&\left(c\frac{\partial}{\partial u} - y\delta\frac{\partial}{\partial y}\right)\mathbf{M}^{(1)}(u,y,d) + \mathbf{A}\mathbf{M}^{(1)}(u,y,d)\\
&+\left[\int_0^u \boldsymbol{\alpha}^\top\mathbf{M}^{(1)}(u-x,y,d)f(x)dx + \bar{F}(u)\right]\mathbf{a} = \mathbf{0}.
\end{aligned}
\tag{3.9}
$$

In this case, $\mathbf{M}^{(2)}(u,y,d)$ need not be considered. Specially, when the distribution $K_1(t)$ of the interclaim time is a generalized Erlang(n) distribution, i.e.,

$$
\boldsymbol{\alpha}^\top = (1,0,\ldots,0), \mathbf{A} = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & 0 & 0\\ 0 & -\lambda_2 & \lambda_2 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & 0 & 0 & -\lambda_n \end{pmatrix}, \mathbf{a} = \begin{pmatrix} 0\\ 0\\ \vdots\\ \lambda_n\end{pmatrix}.
$$

Then (3.9) can be expressed as

$$
\left(\prod_{i=1}^{n}\frac{y\delta\frac{\partial}{\partial y} - c\frac{\partial}{\partial u} + \lambda_i}{\lambda_i}\right)M^{(1)}(u,y,d) - \int_0^u M^{(1)}(u,y,d)f(x)dx - \bar{F}(u) = 0,
$$

which is identical to (2) in [1].

4. The moments of the discounted dividend payments

4.1. Integro-differential system. Adding (3.1) to (3.2), by virtue of (2.5) leads to

$$
\begin{aligned}
&\left(c\frac{\partial}{\partial u} - y\delta\frac{\partial}{\partial y}\right)\mathbf{M}(u,y,d) + I_{m\times m}\otimes\mathbf{A}\mathbf{M}(u,y,d)+\\
&\mathbf{B}\otimes I_{n\times n}\mathbf{M}(u,y,d) + I_{m\times m}\otimes(\mathbf{a}\boldsymbol{\alpha}^\top)\int_0^u\mathbf{M}(u-x,y,d)f(x)dx+\\
&(\mathbf{b}\boldsymbol{\beta}^\top)\otimes I_{n\times n}\int_0^u\mathbf{M}(u-x,y,d)g(x)dx + (\mathbf{e}_m\otimes\mathbf{a})\bar{F}(u)\\
&+(\mathbf{b}\otimes\mathbf{e}_n)\bar{G}(u) = \mathbf{0}.
\end{aligned}
\tag{4.1}
$$

Note that $W(u,r,d) = E[\mathbb{D}^r|R(0)=u]$. With the help of the representation

$$
\mathbf{M}(u,y,d) = \mathbf{e}_{mn} + \sum_{r=1}^{\infty}\frac{y^r}{r!}\mathbf{W}(u,r,d),
$$

by equating the coefficients of $y^r(r\in\mathbb{N})$ in (4.1), using $\mathbf{a} = -\mathbf{A}\mathbf{e}_n$, $\mathbf{b} = -\mathbf{B}\mathbf{e}_m$, $I_{m\times m}\otimes\mathbf{A}\mathbf{e}_{mn} = -I_{m\times m}\otimes(\mathbf{a}\boldsymbol{\alpha}^\top)\mathbf{e}_{mn} = -\mathbf{e}_m\otimes\mathbf{a}$ and $\mathbf{B}\otimes I_{n\times n}\mathbf{e}_{mn} = -(\mathbf{b}\boldsymbol{\beta}^\top)\otimes I_{n\times n}\mathbf{e}_{mn} = -\mathbf{b}\otimes\mathbf{e}_n$, we have the following result.

4.1. Theorem. *The vector* $\mathbf{W}(u,r,d), 0 \le u \le d$, *satisfies the following integro-differential system,*

$$\begin{aligned} &c\tfrac{d\mathbf{W}(u,r,d)}{du} - r\delta\mathbf{W}(u,r,d) + I_{m\times m}\otimes \mathbf{A}\mathbf{W}(u,r,d) + \\ &\mathbf{B}\otimes I_{n\times n}\mathbf{W}(u,r,d) + I_{m\times m}\otimes(\mathbf{a}\boldsymbol{\alpha}^\top)\textstyle\int_0^u \mathbf{W}(u-x,r,d)f(x)dx + \\ &(\mathbf{b}\boldsymbol{\beta}^\top)\otimes I_{n\times n}\textstyle\int_0^u \mathbf{W}(u-x,r,d)g(x)dx = \mathbf{0}, \end{aligned} \tag{4.2}$$

with boundary conditions

$$\left.\frac{\partial \mathbf{W}(u,r,d)}{\partial u}\right|_{u=d} = r\mathbf{W}(d,r-1,d). \tag{4.3}$$

4.2. Remark. When $m=1$ and $G(0)=1$, from Eq.(4.2), we get

$$c\frac{d\mathbf{W}(u,r,d)}{du} - r\delta\mathbf{W}(u,r,d) + \mathbf{A}\mathbf{W}(u,r,d) + (\mathbf{a}\boldsymbol{\alpha}^\top)\int_0^u \mathbf{W}(u-x,r,d)f(x)dx = \mathbf{0}. \tag{4.4}$$

Furthermore, when the distribution $K_1(t)$ of the interclaim time is a generalized Erlang(n) distribution, see Remark 3.1 for the representation $(\boldsymbol{\alpha}^\top, \mathbf{A}, \mathbf{a})$. Under this scenario, we recover (9) in [1] from (4.4) as follows:

$$\left(\prod_{i=1}^{n} \frac{r\delta - c\frac{\partial}{\partial u} + \lambda_i}{\lambda_i}\right) W(u,r,d) - \int_0^u W(u-x,r,d)f(x)dx = 0.$$

4.2. Explicit results for claim-size with rational family distributions. Now define the Laplace transforms $\tilde{\mathbf{W}}(s,r,d) = \int_0^\infty e^{-su}\mathbf{W}(u,r,d)du$ by ignoring for a moment that $\mathbf{W}(u,r,d)$ is only defined for $0 \le u \le d$.

Taking Laplace transforms on both sides of (4.2) yields

$$\begin{aligned} &\Big[(cs - r\delta)\, I_{mn\times mn} + I_{m\times m}\otimes \mathbf{A} + \mathbf{B}\otimes I_{n\times n} + I_{m\times m}\otimes(\mathbf{a}\boldsymbol{\alpha}^\top)\tilde{f}(s) \\ &+ (\mathbf{b}\boldsymbol{\beta}^\top)\otimes I_{n\times n}\tilde{g}(s)\Big]\, \tilde{\mathbf{W}}(s,r,d) = c\mathbf{W}(0,r,d). \end{aligned} \tag{4.5}$$

Let $\mathbf{L}(s) = (cs - r\delta)\, I_{mn\times mn} + I_{m\times m}\otimes \mathbf{A} + \mathbf{B}\otimes I_{n\times n} + I_{m\times m}\otimes(\mathbf{a}\boldsymbol{\alpha}^\top)\tilde{f}(s) + (\mathbf{b}\boldsymbol{\beta}^\top)\otimes I_{n\times n}\tilde{g}(s)$, and $\mathbf{L}^*(s)$ be the adjoint of matrix $\mathbf{L}(s)$. In the following, we assume $det[\mathbf{L}(s)] \ne 0$. So, from (4.5), it holds that

$$\tilde{\mathbf{W}}(s,r,d) = \frac{\mathbf{L}^*(s)}{det[\mathbf{L}(s)]} c\mathbf{W}(0,r,d). \tag{4.6}$$

Thanks to [5], the generalized Lundberg's equation $det[\mathbf{L}(s)] = 0$ has exactly mn roots in the right half of the complex plane when $\delta > 0$. We denote them by $\rho_1, \rho_2, \cdots, \rho_{mn}$ respectively, and for simplicity, we assume that they are different from each other.

Next, we present some explicit results for the moments of the discounted dividend payments by assuming that the claim amount distributions F and G are both from the rational family distribution. That is, the Laplace transforms of the density functions are of the forms

$$\tilde{f}(s) = \frac{p_{m_1-1}(s)}{p_{m_1}(s)}, \quad \tilde{g}(s) = \frac{q_{m_2-1}(s)}{q_{m_2}(s)}, \quad m_1, m_2 \in \mathbb{N}^+,$$

where $p_{m_1-1}(s), q_{m_2-1}(s)$ are polynomials of degree m_1-1 and m_2-1 or less, respectively, while $p_{m_1}(s)$ and $q_{m_2}(s)$ are polynomials of degree m_1 and m_2 with only negative roots, and satisfy $p_{m_1-1}(0) = p_{m_1}(0)$, $q_{m_2-1}(0) = q_{m_2}(0)$. Without loss of generality, we assume that $p_{m_1}(s)$ and $q_{m_2}(s)$ have leading coefficient 1. This wide class of distributions includes the phase-type distributions, and in particular, it includes the Erlang, Coxian and exponential distribution and all the mixtures of them.

In what follows, let $h(s) = [p_{m_1}(s)q_{m_2}(s)]^{mn}$. Multiplying both numerator and denominator of (4.6) by $h(s)$ results in

$$(4.7) \qquad \tilde{\mathbf{W}}(s,r,d) = \frac{h(s)\mathbf{L}^*(s)}{h(s)det[\mathbf{L}(s)]} c\mathbf{W}(0,r,d).$$

Obviously, the factor $h(s)det[\mathbf{L}(s)]$ of the denominator is a polynomial of degree $mn(m_1+m_2+1)$ with leading coefficient c^{mn}. Therefore, the equation $h(s)det[\mathbf{L}(s)] = 0$ has $mn(m_1+m_2+1)$ roots on the complex plane. We can factorize $h(s)det[\mathbf{L}(s)]$ as follows

$$(4.8) \qquad h(s)det[\mathbf{L}(s)] = c^{mn}\prod_{i=1}^{mn}(s-\rho_i)\prod_{j=1}^{(m_1+m_2)mn}(s+R_j),$$

where R_j for each j has positive real part and we assume that all of them are distinct from each other.

Since the numerator $h(s)\mathbf{L}^*(s)$ in (4.7) is a polynomial with degree less than $mn(m_1+m_2+1)$. By the partial fraction decomposition, it follows that

$$(4.9) \qquad \tilde{\mathbf{W}}(s,r,d) = \sum_{j=1}^{mn}\frac{\mathbf{\Gamma}_j(d)}{s-\rho_j} + \sum_{j=1}^{(m_1+m_2)mn}\frac{\mathbf{\Lambda}_j(d)}{s+R_j},$$

where $\mathbf{\Gamma}_j(d)$, for $j = 1,2,\cdots,mn$, and $\mathbf{\Lambda}_j(d)$, for $j = 1,2,\cdots,(m_1+m_2)mn$, are the coefficient matrices defined respectively by

$$(4.10) \qquad \mathbf{\Gamma}_j(d) = -\frac{h(\rho_j)\mathbf{L}^*(\rho_j)\mathbf{W}(0,r,d)}{c^{mn-1}\left[\prod_{k=1}^{(m_1+m_2)mn}(R_k+\rho_j)\right]\left[\prod_{i=1,i\neq j}^{mn}(\rho_i-\rho_j)\right]},$$

and

$$(4.11) \qquad \mathbf{\Lambda}_j(d) = \frac{h(-R_j)\mathbf{L}^*(-R_j)\mathbf{W}(0,r,d)}{c^{mn-1}\left[\prod_{k=1}^{mn}(\rho_k+R_j)\right]\left[\prod_{i=1,i\neq j}^{(m_1+m_2)mn}(R_i-R_j)\right]}.$$

Obviously, $\mathbf{\Gamma}_j(d)$ and $\mathbf{\Lambda}_j(d)$ depend on dividend barrier d. Inverting (4.9) leads to

$$(4.12) \qquad \mathbf{W}(u,r,d) = \sum_{j=1}^{mn}\mathbf{\Gamma}_j(d)e^{\rho_j u} + \sum_{j=1}^{(m_1+m_2)mn}\mathbf{\Lambda}_j(d)e^{-R_j u}.$$

Since we don't need to distinguish $\mathbf{\Gamma}_j(d)$ and $\mathbf{\Lambda}_j(d)$, for notational convenience, (4.12) can be reexpressed as

$$(4.13) \qquad \mathbf{W}(u,r,d) = \sum_{j=1}^{mn(m_1+m_2+1)}\mathbf{\Upsilon}_j(d)e^{\kappa_j u},$$

where $\kappa_j, j = 1,\ldots,mn(m_1+m_2+1)$ denote $mn(m_1+m_2+1)$ roots of $h(s)det[\mathbf{L}(s)] = 0$.

Now we announce that the explicit forms for arbitrary moments of the discounted dividend payments can be obtained from (4.13) if the two classes claim amount distributions both belong to the rational family. The coefficients $\mathbf{\Upsilon}_j(d)$ can be determined by boundary conditions (4.3), and we can obtain the other demand equations for determining these coefficients by substituting (4.13) into (4.2), and equating coefficients of the resulting exponential terms. At the same time, the asymptotic behavior $\lim_{d\to\infty}\mathbf{W}(u,r,d) = \mathbf{0}$ holds.

5. Numerical illustrations

In this section, we will illustrate numerically an application of the main results in this paper. We assume that the claim amounts from class 1 and class 2 both follow exponentially distributions with density functions, respectively,

$$f(x)=\mu_1 e^{-\mu_1 x}, \quad \mu_1>0, x>0, \quad g(y)=\mu_2 e^{-\mu_2 y}, \quad \mu_2>0, y>0.$$

We also assume $\mu_1 \neq \mu_2$ for simplicity. Thus, the Laplace transforms $\tilde{f}(s)=\frac{\mu_1}{s+\mu_1}$, $\tilde{g}(s)=\frac{\mu_2}{s+\mu_2}$. At the same time, we suppose that the interclaim times from class 1 occur following a Poisson process with parameter λ and interclaim times from class 2 occur following a phase-type distribution with the following parameters: $\boldsymbol{\beta}^\top=(1/2,1/2), \mathbf{B}=\begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, \mathbf{b}=\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$. So, we also have $\boldsymbol{\alpha}=(1), \mathbf{A}=(-\lambda), \mathbf{a}=(\lambda)$, and

$$\mathbf{L}(s)=\begin{pmatrix} cs-r\delta-\lambda-\lambda_1+\frac{\lambda\mu_1}{s+\mu_1}+\frac{\lambda_1\mu_2}{2(s+\mu_2)} & \frac{\lambda_1\mu_2}{2(s+\mu_2)} \\ \frac{\lambda_2\mu_2}{2(s+\mu_2)} & cs-r\delta-\lambda-\lambda_2+\frac{\lambda\mu_1}{s+\mu_1}+\frac{\lambda_2\mu_2}{2(s+\mu_2)} \end{pmatrix}.$$

From (4.2), we have

$$\begin{aligned} & c\tfrac{d\mathbf{W}(u,r,d)}{du}-r\delta\mathbf{W}(u,r,d)+\begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}\mathbf{W}(u,r,d)+ \\ & \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}\mathbf{W}(u,r,d)+\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\textstyle\int_0^u \mathbf{W}(u-x,r,d)f(x)dx+ \\ & \begin{pmatrix} \frac{\lambda_1}{2} & \frac{\lambda_1}{2} \\ \frac{\lambda_2}{2} & \frac{\lambda_2}{2} \end{pmatrix}\textstyle\int_0^u \mathbf{W}(u-x,r,d)g(x)dx=\mathbf{0}. \end{aligned} \tag{5.1}$$

Using (4.13), we obtain the representation

$$\mathbf{W}(u,r,d)=\sum_{j=1}^{6}\boldsymbol{\Upsilon}_j(d)e^{\kappa_j u}. \tag{5.2}$$

Obviously, $s=-\mu_2$ is one of the roots of $h(s)det[\mathbf{L}(s)]=0$. Hence, (5.2) can be rewritten as

$$\mathbf{W}(u,r,d)=\sum_{j=1}^{5}\boldsymbol{\Upsilon}_j(d)e^{\kappa_j u}+\boldsymbol{\Upsilon}_6(d)e^{-\mu_2 u}. \tag{5.3}$$

Substituting (5.3) into (5.1) results in

$$\begin{aligned} & \textstyle\sum_{j=1}^{5}\mathbf{L}(\kappa_j)\boldsymbol{\Upsilon}_j(d)e^{\kappa_j u}= \\ & \left[\textstyle\sum_{j=1}^{5}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\boldsymbol{\Upsilon}_j(d)\frac{\mu_1}{\kappa_j+\mu_1}+\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\boldsymbol{\Upsilon}_6(d)\frac{\mu_1}{-\mu_2+\mu_1}\right]e^{-\mu_1 u}+ \\ & \left\{\begin{pmatrix} c\mu_2+r\delta+\lambda+\lambda_1-\frac{\lambda\mu_1}{-\mu_2+\mu_1} & 0 \\ 0 & c\mu_2+r\delta+\lambda+\lambda_2-\frac{\lambda\mu_1}{-\mu_2+\mu_1} \end{pmatrix}\boldsymbol{\Upsilon}_6(d)+\right. \\ & \left.\textstyle\sum_{j=1}^{5}\begin{pmatrix} \frac{\lambda_1}{2} & \frac{\lambda_1}{2} \\ \frac{\lambda_2}{2} & \frac{\lambda_2}{2} \end{pmatrix}\boldsymbol{\Upsilon}_j(d)\frac{\mu_2}{\kappa_j+\mu_2}-\begin{pmatrix} \frac{\lambda_1}{2} & \frac{\lambda_1}{2} \\ \frac{\lambda_2}{2} & \frac{\lambda_2}{2} \end{pmatrix}\boldsymbol{\Upsilon}_6(d)\mu_2 u\right\}e^{-\mu_2 u}, \end{aligned} \tag{5.4}$$

from which we have the following conditions

$$\sum_{j=1}^{5}\mathbf{L}(\kappa_j)\boldsymbol{\Upsilon}_j(d)=\mathbf{0}, \tag{5.5}$$

$$\sum_{j=1}^{5}\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\boldsymbol{\Upsilon}_j(d)\frac{\mu_1}{\kappa_j+\mu_1}+\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\boldsymbol{\Upsilon}_6(d)\frac{\mu_1}{-\mu_2+\mu_1}=\mathbf{0}, \tag{5.6}$$

$$\begin{pmatrix} c\mu_2+r\delta+\lambda+\lambda_1-\frac{\lambda\mu_1}{-\mu_2+\mu_1} & 0\\ 0 & c\mu_2+r\delta+\lambda+\lambda_2-\frac{\lambda\mu_1}{-\mu_2+\mu_1}\end{pmatrix}\boldsymbol{\Upsilon}_6(d)+ \sum_{j=1}^{5}\begin{pmatrix}\frac{\lambda_1}{2} & \frac{\lambda_1}{2}\\ \frac{\lambda_2}{2} & \frac{\lambda_2}{2}\end{pmatrix}\boldsymbol{\Upsilon}_j(d)\frac{\mu_2}{\kappa_j+\mu_2}=\mathbf{0}, \tag{5.7}$$

and

$$\begin{pmatrix}\frac{\lambda_1}{2} & \frac{\lambda_1}{2}\\ \frac{\lambda_2}{2} & \frac{\lambda_2}{2}\end{pmatrix}\boldsymbol{\Upsilon}_6(d)=\mathbf{0}. \tag{5.8}$$

For $r=1$ we have from (4.3) $\left.\frac{\partial \mathbf{W}(u,1,d)}{\partial u}\right|_{u=d}=\mathbf{e}_{mn}$, which yields

$$\sum_{j=1}^{5}\boldsymbol{\Upsilon}_j(d)\kappa_j e^{\kappa_j d}-\boldsymbol{\Upsilon}_6(d)\mu_2 e^{-\mu_2 d}=\mathbf{e}_2. \tag{5.9}$$

By virtue of the asymptotic behavior $\lim_{d\to\infty}\mathbf{W}(u,r,d)=\mathbf{0}$, we have

$$\lim_{d\to\infty}\left[\sum_{j=1}^{5}\boldsymbol{\Upsilon}_j(d)e^{\kappa_j u}+\boldsymbol{\Upsilon}_6(d)e^{-\mu_2 u}\right]=\mathbf{0}. \tag{5.10}$$

Thus the coefficients $\boldsymbol{\Upsilon}_j(d), j=1,\dots,6$, can be determined from Eqs. (5.5)-(5.10), then we obtain $\mathbf{W}(u,1,d)$. By the same arguments and in view of the boundary conditions (4.3), we can derive $\mathbf{W}(u,r,d)$ for $r=2,3,\dots$.

For illustration purpose, we set $c=2.5, \delta=0.01, \lambda=1, \lambda_1=1, \lambda_2=2, \mu_1=1, \mu_2=2$. It is easy to check that the net profit condition holds. Now, we consider the expectation of discounted dividend payments, namely, $r=1$. In this case the solutions of $h(s)det[\mathbf{L}(s)]=0$ are $\kappa_1=0.8082, \kappa_2=0.0118, \kappa_3=-0.4017, \kappa_4=-0.7713, \kappa_5=-1.6390, \kappa_6=-\mu_2=-2.000$. In the following, Table 1 gives some numerical values of $W(u,1,d)=\boldsymbol{\gamma}^\top\mathbf{W}(u,1,d)$.

Table 1. Exact values for $W(u,1,d)$.

d\u	0	1	2	3	4	5	6	7	8
0	1.0372								
1	0.7157	1.4795							
2	0.3980	0.7842	1.4831						
3	0.2003	0.3858	0.7050	1.3935					
4	0.0621	0.1376	0.2784	0.5856	1.2718				
5	0.0449	0.0847	0.1498	0.2878	0.5942	1.2803			
6	0.0206	0.0388	0.0680	0.1296	0.2663	0.5722	1.2582		
7	0.0094	0.0176	0.0307	0.0582	0.1191	0.2555	0.5613	1.2472	
8	0.0042	0.0079	0.0138	0.0261	0.0532	0.1140	0.2503	0.5560	1.2419

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