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Abelian model structures and Ding homological dimensions

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Abstract

Let R be an n-FC ring. For $0 < t \leq n$, we construct a new abelian model structure on R-Mod, called the Ding t-projective (t-injective) model structure. Based on this, we establish a bijective correspondence between dg-t-projective (dg-t-injective) R-complexes and Ding t-projective (t-injective) A-modules under some additional conditions, where $A = R[x]/(x^2)$. This gives a generalized version of the bijective correspondence established in [[14]] between dg-projective (dg-injective) R-complexes and Gorenstein projective (injective) A-modules. Finally, we show that the embedding functors $K(\mathcal{DP}) \longrightarrow K(R\text{-Mod})$ and $K(\mathcal{DI}) \longrightarrow K(R\text{-Mod})$ have right and left adjoints respectively, where $K(\mathcal{DP})$ ($K(\mathcal{DI})$) is the homotopy category of complexes of Ding projective (injective) modules, and $K(R\text{-Mod})$ denotes the homotopy category.

Keywords: model structures; Ding t-projective (injective) modules; dg-t-projective (injective) complexes; adjoint functors.

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1. Introduction and preliminaries. We study model structures on the categories R-Mod and $\mathrm{Ch}(R)$, where R is an n-FC ring. In [[15]], M. Hovey constructed an abelian model structure on R-Mod where the class of cofibrant objects is given by the class of Gorenstein projective modules, the class of fibrant objects is given by the category R-Mod, and the trivial objects are the left R-modules of finite projective dimension. Dually, there was a model structure on R-Mod with the same trivial objects, the class of cofibrant objects being R-Mod, and the class of fibrant objects being the class of the Gorenstein injective modules. Later in [[13]], J. Gillespie constructed another abelian model structure on R-Mod where the class of cofibrant objects is given by the class of Ding projective modules. Dually, there was a model structure on R-Mod where the class of fibrant objects is given by the class of Ding injective modules.

We construct two new abelian model structures on R-Mod, called the Ding t-projective and Ding t-injective model structures. In the first structure, the class of cofibrant objects is formed by the objects with Ding projective dimension at most t. In the second structure, the class of fibrant objects is given by the class of objects with Ding injective dimension at most t. In order to construct these structures, we use a result known by some authors as the Hovey's Criterion, which allows us to get abelian model structures from compatible and complete cotorsion pairs. In this sense, we prove the completeness of the cotorsion pair cogenerated by the class of Ding t-projective modules. Dually, the cotorsion pair generated by the class of Ding t-injective modules is also complete. These structures have their analogues in the category of chain complexes.

For any ring R, there exists an invertible functor from $\mathrm{Ch}(R)$ to the category of graded $R[x]/(x^2)$-modules. In [[14]], the authors proved that this functor gives rise to a bijective correspondence between the dg-projective complexes over R and the Gorenstein projective $R[x]/(x^2)$-modules. The same also occurred between dg-injective complexes over R and Gorenstein injective $R[x]/(x^2)$-modules. We prove the Ding version of these results.

In the end of this paper, we show that the embedding functors $K(\mathcal{DP}) \longrightarrow K(R\text{-Mod})$ and $K(\mathcal{DI}) \longrightarrow K(R\text{-Mod})$ have right and left adjoints respectively, where $K(\mathcal{DP})$ $(K(\mathcal{DI}))$ is the homotopy category with each complex constructed by Ding projective (injective) modules, and $K(R\text{-Mod})$ is the homotopy category.

We next recall some known notions and facts needed in the sequel.

In this paper, R denotes a ring with unity, R-Mod the category of left R-modules, and $\mathrm{Ch}(R)$ the category of complexes of left R-modules. A complex

$$\cdots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \xrightarrow{\delta_{-1}} \cdots$$

of left R-modules will be denoted (C, δ) or C. Given a left R-module M, we will denote by $D^m(M)$ the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\mathrm{id}} M \longrightarrow 0 \longrightarrow \cdots$$

with the M in the m and $(m-1)$-th position. Given a complex C, ΣC denotes the complex such that $(\Sigma C)_n = C_{n-1}$ and whose boundary operators are $-\delta^C_{n-1}$.

A homomorphism $\varphi : C \longrightarrow D$ of degree n is a family $(\varphi_i)_{i\in\mathbb{Z}}$ of homomorphisms of R-modules $\varphi_i : C_i \longrightarrow D_{n+i}$. All such homomorphisms form an abelian group, denoted $\mathcal{H}\mathrm{om}_R(C, D)_n$, it is clearly isomorphic to $\prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(C_i, D_{n+i})$. We let $\mathcal{H}\mathrm{om}_R(C, D)$ denote the complex of abelian groups with n-th component $\mathcal{H}\mathrm{om}_R(C, D)_n$ and boundary operator

$$\delta_n((\varphi_i)_{i\in\mathbb{Z}}) = (\delta^D_{n+i}\varphi_i - (-1)^n \varphi_{i-1}\delta^C_i)_{i\in\mathbb{Z}}.$$

A homomorphism $\varphi \in \mathcal{H}\mathrm{om}_R(C, D)_n$ is called a chain map if $\delta(\varphi) = 0$, that is, if $\delta^D_{n+i}\varphi_i = (-1)^n \varphi_{i-1}\delta^C_i$ for all $i \in \mathbb{Z}$. A chain map of degree 0 is called a morphism.

To every complex C we associate the numbers

$$\sup C = \sup\{i \mid C_i \neq 0\}, \quad \inf C = \inf\{i \mid C_i \neq 0\}.$$

The complex C is called bounded above when $\sup C < \infty$, bounded below when $\inf C > -\infty$ and bounded when it is bounded below and above.

For objects C and D of $\mathrm{Ch}(R)$ (R-Mod), $\mathrm{Hom}(C, D)$ ($\mathrm{Hom}_R(C, D)$) is the abelian group of morphisms from C to D in $\mathrm{Ch}(R)$ (R-Mod) and $\mathrm{Ext}^i(C, D)$ ($\mathrm{Ext}^i_R(C, D)$) for $i \geq 1$ will denote the groups we get from the right derived functor of $\mathrm{Hom}(C, D)$ ($\mathrm{Hom}_R(C, D)$).

Let $\mathcal{A}, \mathcal{B}$ be two classes of R-modules. The pair $(\mathcal{A}, \mathcal{B})$ is called a cotorsion pair (also called a cotorsion theory) if $\mathcal{A}^{\perp} = \mathcal{B}$ and $\mathcal{A} = {}^{\perp}\mathcal{B}$. Here $\mathcal{A}^{\perp}$ is the class of R-modules C such that $\mathrm{Ext}^1(A, C) = 0$ for all $A \in \mathcal{A}$, and similarly ${}^{\perp}\mathcal{B}$ is the class of R-modules C such that $\mathrm{Ext}^1(C, B) = 0$ for all $B \in \mathcal{B}$. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be hereditary, if whenever $0 \to \widetilde{A} \to A \to \widehat{A} \to 0$ is exact with $A, \widehat{A} \in \mathcal{A}$ then $\widetilde{A}$ is also in $\mathcal{A}$, or equivalently, if $0 \to \widetilde{B} \to B \to \widehat{B} \to 0$ is exact with $\widetilde{B}, B \in \mathcal{B}$ then $\widehat{B}$ is also in $\mathcal{B}$. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is cogenerated by a set $\mathcal{S} \subseteq \mathcal{A}$ if $\mathcal{B} = \mathcal{S}^{\perp}$. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to have enough injectives (projectives) [[10]] if for any object M there exists an exact sequence $0 \to M \to B \to A \to 0$ ($0 \to B \to A \to M \to 0$) with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. By [[10], Proposition 1.1.5], a cotorsion pair $(\mathcal{A}, \mathcal{B})$ has enough projectives if and only if it has enough injectives. The cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called complete if it has enough projectives and injectives.

Given a class $\mathcal{B}$ of objects of $\mathrm{Ch}(R)$, a morphism $\phi : X \to B$ is called a $\mathcal{B}$-preenvelope ([[6]]) if $B \in \mathcal{B}$ and $\mathrm{Hom}(B, B') \to \mathrm{Hom}(X, B') \to 0$ is exact for all $B' \in \mathcal{B}$. If, moreover, any $f : B \to B$ such that $f\phi = \phi$ is an automorphism of B then $\phi : X \to B$ is called a $\mathcal{B}$-envelope of X. A complex X is said to have a special $\mathcal{B}$-preenvelope [[9]] if there is an exact sequence $0 \to X \to B \to L \to 0$ with $B \in \mathcal{B}$ and $L \in {}^{\perp}\mathcal{B}$. (Special) precovers and covers of X are defined dually.

2. Ding t-projective and Ding t-injective model structures. Ding and Chen extended FC rings to n-FC rings [[2], [3]], which are seen to have many properties similar to those of n-Gorenstein rings. Just as a ring is called Gorenstein when it is n-Gorenstein for some nonnegative integer n (a ring R is called n-Gorenstein if it is a left and right Noetherian ring with self injective dimension at most n on both sides for some nonnegative integer n), Gillespie first called a ring Ding-Chen when it is n-FC for some n [[13], Definition 4.1]. An R-module M is called Ding projective if there exists an exact sequence of projective R-modules $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots$ with $M = \mathrm{Ker}(P_0 \longrightarrow P_{-1})$ and which remains exact after applying $\mathrm{Hom}(-, F)$ for any flat R-module F [[5]]. The class of Ding projective R-modules is denoted by $\mathcal{DP}$. An R-module N is called Ding injective if there exists an exact sequence of injective R-modules $\cdots \longrightarrow I_1 \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow I_{-2} \longrightarrow \cdots$ with $N = \mathrm{Ker}(I_0 \longrightarrow I_{-1})$ and which remains exact after applying $\mathrm{Hom}(E, -)$ for any FP-injective R-module E [[17]]. The class of Ding injective R-modules is denoted by $\mathcal{DI}$. Note that every Ding injective (respectively, Ding projective) R-module N is Gorenstein injective (respectively, Gorenstein projective), and if R is Gorenstein, then every Gorenstein injective R-module is Ding injective (respectively, Gorenstein projective)[[13]].

From [[13], Theorem 4.2], we know that for a Ding-Chen ring R, the class of all modules with finite flat dimension and the class of all modules with finite FP-injective dimension are the same, and we use $\mathcal{W}_R$ to denote this class throughout this section.

Ding and Mao proved that $({}^{\perp}\mathcal{W}_R, \mathcal{W}_R)$ forms a complete cotorsion pair when R is a Ding-Chen ring [[4], Theorem 3.8]. Also, $(\mathcal{W}_R, \mathcal{W}_R^{\perp})$ forms a complete cotorsion pair when R is a Ding-Chen ring [[16], Theorem 3.4]. Moreover, Gillespie proved that an R-module M is Ding projective if and only if $M \in^{\perp} \mathcal{W}_R$, an R-module N is Ding injective if and only if $N \in \mathcal{W}_R^{\perp}$ [[13], Corollaries 4.5 and 4.6]. So $(\mathcal{DP}, \mathcal{W}_R)$ and $(\mathcal{W}_R, \mathcal{DI})$ are complete hereditary cotorsion pairs (each cogenerated by a set). Hence for every $M \in R$-Mod there exists an epimorphism $D_0 \longrightarrow M$ where D_0 is a Ding projective module. This allows us to construct an exact sequence

$$\cdots \longrightarrow D_k \longrightarrow \cdots \longrightarrow D_1 \longrightarrow D_0 \longrightarrow M \longrightarrow 0,$$

where D_k is a Ding projective module, for every $k \geq 0$. We shall say that this sequence is a *left Ding projective resolution* of M. An R-module M is said to be *Ding t-projective*, if M admits a *left Ding projective resolution* of length at most t (that is, M has Ding projective dimension at most t), where t is a nonnegative integer. Let $\mathcal{DP}_t$ denote the class of Ding t-projective modules. We shall denote by $\mathrm{Dpd}(M)$ the (left) Ding projective dimension of M. Note that $\mathcal{DP}_t = \{M \in R\text{-Mod} : \mathrm{Dpd}(M) \leq t\}$ and that $\mathcal{DP}_0 = \mathcal{DP}$, similarly, we let $\mathcal{P}_t$ denote the class of t-projective R-modules.

Similarly, we can define Ding t-injective modules, and we let $\mathcal{DI}_t$ denote the class of Ding t-injective modules and $\mathcal{I}_t$ the class of t-injective R-modules.

Let

$$\cdots \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0, \quad (1)$$

be a projective resolution of X. We shall say that $\mathrm{Im}(f_i)$ is the i-th projective syzygy of X in (1). We shall use the notation $\Omega^i(X)$ for the class of all i-th projective syzygies of X. Dually, given an injective coresolution of X, say

$$0 \longrightarrow X \longrightarrow I^0 \xrightarrow{f^0} I^1 \xrightarrow{f^1} \cdots \longrightarrow I^{n-1} \xrightarrow{f^{n-1}} I^n \longrightarrow \cdots, \quad (2)$$

we shall say that $\mathrm{Ker}(f^i)$ is the i-th injective cosyzygy of X in (2), and we shall use the notation $\Omega^{-i}(X)$ for the class of all i-th injective cosyzygies of X.

We begin with the following result.

1.1. Lemma ([[5], Lemma 3.4]). *Let R be a Ding-Chen ring. Then the following are equivalent:*

(1) *M is Ding t-projective.*
(2) $\mathrm{Ext}_R^i(M, W) = 0$ *for all $i > t$ and for all $W \in \mathcal{W}_R$.*
(3) $\mathrm{Ext}_R^{t+1}(M, W) = 0$ *for all $W \in \mathcal{W}_R$.*
(4) *Every tth Ding projective syzygy of M is Ding projective.*
(5) *Every tth projective syzygy of M is Ding projective.*

1.2. Corollary. *Let R be an n-FC ring. Then for every $0 \leq t \leq n, \mathcal{DP}_t \cap \mathcal{W}_R = \mathcal{P}_t$.*

Proof The inclusion $\mathcal{P}_t \subseteq \mathcal{DP}_t \cap \mathcal{W}_R$ is clear. Now let $M \in \mathcal{DP}_t \cap \mathcal{W}_R$. Then every $G \in \Omega^t(M)$ is in $\mathcal{DP}$ by Lemma 1.1. Since $M \in \mathcal{W}_R$, we have $G \in \mathcal{W}_R$. Then $G \in \mathcal{DP} \cap \mathcal{W}_R = \mathcal{P}_0$ by [[5], Lemma 2.4]. It follows $M \in \mathcal{P}_t$. □

The following results show that $(\mathcal{DP}_t, (\mathcal{DP}_t)^{\perp})$ is a complete cotorsion pair for every $1 \leq t \leq n$.

1.3. Theorem. *Let R be an n-FC ring. $(\mathcal{DP}_t, (\mathcal{DP}_t)^{\perp})$ is a cotorsion pair cogenerated by a set, and so it is complete for every $1 \leq t \leq n$.*

Proof First we prove that $(\mathcal{DP}_t, (\mathcal{DP}_t)^{\perp})$ is a cotorsion pair.

It suffices to show that ${}^{\perp}((\mathcal{DP}_t)^{\perp}) \subseteq \mathcal{DP}_t$. Let $M \in^{\perp} ((\mathcal{DP}_t)^{\perp})$. Consider a left partial projective resolution of M, say $0 \longrightarrow G \longrightarrow P_{t-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$.

By Lemma 1.1, it suffices to show that G is a Ding projective module. Suppose $t = 1$ and let $W \in \mathcal{W}_R$. We have the exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^1_R(P_0, W) \longrightarrow \mathrm{Ext}^1_R(G, W) \longrightarrow \mathrm{Ext}^2_R(M, W) \longrightarrow \cdots,$$

where $\mathrm{Ext}^1_R(P_0, W) = 0$, since P_0 is projective. On the other hand, $\mathrm{Ext}^2_R(M, W) = \mathrm{Ext}^1_R(M, L)$, where $L \in \Omega^{-1}(W)$. We show $L \in (\mathcal{DP}_1)^\perp$. Let $K \in \mathcal{DP}_1$ and consider the short exact sequence $0 \longrightarrow W \longrightarrow I \longrightarrow L \longrightarrow 0$, where I is injective. Then we have an exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^1_R(K, I) \longrightarrow \mathrm{Ext}^1_R(K, L) \longrightarrow \mathrm{Ext}^2_R(K, W) \longrightarrow \cdots,$$

where $\mathrm{Ext}^1_R(K, I) = 0$, since I is injective, and $\mathrm{Ext}^2_R(K, W) = 0$ since $K \in \mathcal{DP}_1$ and $W \in \mathcal{W}_R$. Then $\mathrm{Ext}^1_R(K, L) = 0$ for every $K \in \mathcal{DP}_1$, i.e. $L \in (\mathcal{DP}_1)^\perp$. It follows $\mathrm{Ext}^2_R(M, W) = 0$. Hence $\mathrm{Ext}^1_R(G, W) = 0$ for every $W \in \mathcal{W}_R$, i.e. $G \in \mathcal{DP}$.

Suppose the result is true for every $1 \leq j \leq t-1$. We have an exact sequence

$$0 \longrightarrow G \longrightarrow P_{t-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow L \longrightarrow 0,$$

where $L \in \Omega^1(M)$, and a short exact sequence $0 \longrightarrow L \longrightarrow P_0 \longrightarrow M \longrightarrow 0$. Let $K \in (\mathcal{DP}_{t-1})^\perp$. We have $\mathrm{Ext}^1_R(L, K) \cong \mathrm{Ext}^1_R(M, K')$, where $K' \in \Omega^{-1}(K)$. Let $N \in \mathcal{DP}_t$. Then $N' \in \mathcal{DP}_{t-1}$, for every $N' \in \Omega^1(N)$. We have $\mathrm{Ext}^1_R(N, K') \cong \mathrm{Ext}^1_R(N', K) = 0$. So $K' \in (\mathcal{DP}_t)^\perp$. It follows $\mathrm{Ext}^1_R(L, K) \cong \mathrm{Ext}^1_R(M, K') = 0$, for every $K \in (\mathcal{DP}_{t-1})^\perp$. Hence $L \in {}^\perp((\mathcal{DP}_{t-1})^\perp) = \mathcal{DP}_{t-1}$. It follows $M \in \mathcal{DP}_t$.

Now we prove that $(\mathcal{DP}_t, (\mathcal{DP}_t)^\perp)$ is a cotorsion pair cogenerated by a set.

Consider the cogenerating set $\mathcal{U}$ of $(\mathcal{DP}, \mathcal{W}_R)$. On the other hand, it is known that $(\mathcal{P}_t, (\mathcal{P}_t)^\perp)$ is cogenerated by the set $\mathcal{P}_t^{\leq\kappa} := \{M \in \mathcal{P}_t : \mathrm{Card}(M) \leq \kappa\}$, where $\kappa \geq \mathrm{Card}(R)$ is a fixed infinite cardinal number. Set $\mathcal{G}_t := \mathcal{U} \cup \mathcal{P}_t^{\leq\kappa}$. We prove $(\mathcal{DP}_t)^\perp = (\mathcal{G}_t)^\perp$. Since $\mathcal{G}_t \subseteq \mathcal{DP}_t$, we have $(\mathcal{DP}_t)^\perp \subseteq (\mathcal{G}_t)^\perp$. Now let $N \in (\mathcal{G}_t)^\perp$, and consider $M \in \mathcal{DP}_t$. Since $(\mathcal{DP}, \mathcal{W}_R)$ is a complete cotorsion pair, there exists a short exact sequence $0 \longrightarrow M \longrightarrow W \longrightarrow G \longrightarrow 0$, where $W \in \mathcal{W}_R$ and $G \in \mathcal{DP}$. Then $W \in \mathcal{DP}_t \cap \mathcal{W}_R = \mathcal{P}_t$ by Corollary 1.2. We apply the contravariant functor $\mathrm{Ext}(-, N)$ and obtain a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^1_R(W, N) \longrightarrow \mathrm{Ext}^1_R(M, N) \longrightarrow \mathrm{Ext}^2_R(G, N) \longrightarrow \cdots.$$

Note that $\mathrm{Ext}^2_R(G, N) = 0$, since $N \in (\mathcal{G}_t)^\perp \subseteq \mathcal{U}^\perp = \mathcal{W}_R$ and $(\mathcal{DP}, \mathcal{W}_R)$ is hereditary. On the other hand, $N \in (\mathcal{P}_t^{\leq\kappa})^\perp = (\mathcal{P}_t)^\perp$ and $W \in \mathcal{P}_t$, so $\mathrm{Ext}^1_R(W, N) = 0$. Hence $\mathrm{Ext}^1_R(M, N) = 0$ for every $M \in \mathcal{DP}_t$, i.e. $N \in (\mathcal{DP}_t)^\perp$. □

This gives the following result.

1.4. Corollary. *Let R be an n-FC ring. Then $(\mathcal{DP}_t)^\perp = \mathcal{W}_R \cap (\mathcal{P}_t)^\perp$ for every $1 \leq t \leq n$.*

Proof Since $\mathcal{DP} \subseteq \mathcal{DP}_t$ and $\mathcal{P}_t \subseteq \mathcal{DP}_t$, we get $(\mathcal{DP}_t)^\perp \subseteq (\mathcal{DP})^\perp = \mathcal{W}_R$ and $(\mathcal{DP}_t)^\perp \subseteq (\mathcal{P}_t)^\perp$, and so $(\mathcal{DP}_t)^\perp \subseteq \mathcal{W}_R \cap (\mathcal{P}_t)^\perp$. Let $N \in \mathcal{W}_R \cap (\mathcal{P}_t)^\perp$. Since $(\mathcal{DP}_t, (\mathcal{DP}_t)^\perp)$ is complete by Theorem 1.3, there exists a short exact sequence $0 \longrightarrow N \longrightarrow K \longrightarrow C \longrightarrow 0$ where $K \in (\mathcal{DP}_t)^\perp$ and $C \in \mathcal{DP}_t$. Since $N, K \in \mathcal{W}_R$, $C \in \mathcal{W}_R$. Then $C \in \mathcal{DP}_t \cap \mathcal{W}_R = \mathcal{P}_t$ by Corollary 1.2 and hence $\mathrm{Ext}^1_R(C, N) = 0$. It follows $K \cong N \oplus C$. Since $(\mathcal{DP}_t)^\perp$ is closed under direct summands and $K \in (\mathcal{DP}_t)^\perp$, we get $N \in (\mathcal{DP}_t)^\perp$. □

1.5. Definition. *Given two cotorsion pairs $(\mathcal{A}, \mathcal{B}')$ and $(\mathcal{A}', \mathcal{B})$ in an abelian category, we shall say that they are compatible if there exists a class of objects $\mathcal{W}$ such that $\mathcal{A}' = \mathcal{A} \cap \mathcal{W}$ and $\mathcal{B}' = \mathcal{B} \cap \mathcal{W}$.*

1.6. Lemma. ***(Hovey's criterion)*** *Let $(\mathcal{A}, \mathcal{B} \cap \mathcal{W})$ and $(\mathcal{A} \cap \mathcal{W}, \mathcal{B})$ be two compatible cotorsion pairs in a bicomplete abelian category $\mathcal{C}$ with enough projective and injective* *objects, where the class $\mathcal{W}$ is thick. Then there exists a unique abelian model structure on $\mathcal{C}$ such that $\mathcal{A}$ is the class of cofibrant objects, $\mathcal{B}$ is the class of fibrant objects, and $\mathcal{W}$ is the class of trivial objects.*

From the above results, there exists a unique abelian model strucutre on R-Mod where $\mathcal{DP}_t$ is the class of cofibrant objects, $(\mathcal{P}_t)^{\perp}$ is the class of fibrant objects, and $\mathcal{W}_R$ is the class of trivial objects. We call this structure the Ding t-projective model structure on R-Mod. Similarly, there is a unique abelian model structure on R-Mod such that ${}^{\perp}(\mathcal{I}_t)$ is the class of cofibrant objects, $\mathcal{DI}_t$ is the class of fibrant objects, and $\mathcal{W}_R$ is the class of trivial objects. We call this structures the Ding t-injective model structure on R-Mod.

We also have the following result.

1.7. Proposition. *Let X be a chain complex bounded below. Then X is Ding t-projective if and only if X_m is a Ding t-projective module for every $m \in \mathbb{Z}$.*

Proof Let X be a Ding t-projective chain complex. Then there exists an exact sequence in $\mathrm{Ch}(R)$

$$0 \longrightarrow D^t \longrightarrow D^{t-1} \longrightarrow \cdots \longrightarrow D^1 \longrightarrow D^0 \longrightarrow X \longrightarrow 0,$$

such that D^i is a Ding projective complex for every $0 \leq i \leq t$. For each $m \in \mathbb{Z}$, we have an exact sequence in R-Mod

$$0 \longrightarrow D^t_m \longrightarrow D^{t-1}_m \longrightarrow \cdots \longrightarrow D^1_m \longrightarrow D^0_m \longrightarrow X_m \longrightarrow 0.$$

Since each D^i is a Ding projective complex, we have that D^i_m is a Ding projective module. So the previous exact sequence turns out to be a right Ding projective resolution of X_m of length t, i.e. $X_m \in \mathcal{DP}_t$.

Now suppose that X_m is a Ding t-projective module for every $m \in \mathbb{Z}$. Consider a partial left projective resolution

$$0 \longrightarrow D^t \longrightarrow P^{t-1} \longrightarrow \cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow X \longrightarrow 0.$$

It suffices to show that D^t is a Ding projective chain complex. For each $m \in \mathrm{Z}$, we have a exact sequence

$$0 \longrightarrow D^t_m \longrightarrow P^{t-1}_m \longrightarrow \cdots \longrightarrow P^1_m \longrightarrow P^0_m \longrightarrow X_m \longrightarrow 0.$$

Note that each P^i_m is a projective module. Since $X_m \in \mathcal{DP}_t$, we have $D^t_m \in \Omega^t(X_m) \in \mathcal{DP}$. Hence D^t is a Ding projective complex by [[20], Proposition 3.14]. $\square$

1.8. Definition ([[12], Definition 3.3]). *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in R-Mod and X an R-complex.*

(1) *X is called an $\mathcal{A}$ complex if it is exact and $\mathrm{Z}_n X \in \mathcal{A}$ for all $n \in \mathbb{Z}$.*

(2) *X is called a $\mathcal{B}$ complex if it is exact and $\mathrm{Z}_n X \in \mathcal{B}$ for all $n \in \mathbb{Z}$.*

(3) *X is called a dg-$\mathcal{A}$ complex if $X_n \in \mathcal{A}$ for each $n \in \mathbb{Z}$, and $\mathcal{H}\mathrm{om}_R(X, B)$ is exact whenever B is a $\mathcal{B}$ complex.*

(4) *X is called a dg-$\mathcal{B}$ complex if $X_n \in \mathcal{B}$ for each $n \in \mathbb{Z}$, and $\mathcal{H}\mathrm{om}_R(A, X)$ is exact whenever A is an $\mathcal{A}$ complex.*

We denote the class of $\mathcal{A}$ complexes by $\widetilde{\mathcal{A}}$ and the class of dg-$\mathcal{A}$ complexes by $dg\widetilde{\mathcal{A}}$. Similarly, the class of $\mathcal{B}$ complexes is denoted by $\widetilde{\mathcal{B}}$ and the class of dg-$\mathcal{B}$ complexes is denoted by $dg\widetilde{\mathcal{B}}$.

As we did in the category R-Mod, we can prove that $(\widetilde{\mathcal{DP}_t}, (\widetilde{\mathcal{DP}_t})^{\perp})$ and $({}^{\perp}(\widetilde{\mathcal{DI}_t}), \widetilde{\mathcal{DI}_t})$ are complete cotorsion pairs. Moreover, we can see that $(\widetilde{\mathcal{DP}_t}, (\widetilde{\mathcal{DP}_t})^{\perp})$ and $(\widetilde{\mathcal{P}}_t, (\widetilde{\mathcal{P}}_t)^{\perp})$ are compatible. So there exists a unique abelian model structure on $\mathrm{Ch}(R)$ such that $\widetilde{\mathcal{DP}_t}$ is the class of cofibrant objects, $(\widetilde{\mathcal{P}}_t)^{\perp}$ is the class of fibrant objects, and $\widetilde{\mathcal{W}}_R$ is the class of trivial objects. Similarly, there is a unique abelian model structure on $\mathrm{Ch}(R)$ such that ${}^{\perp}(\widetilde{\mathfrak{I}}_t)$ is the class of cofibrant objects, $\widetilde{\mathcal{D}\mathfrak{I}}_t$ is the class of fibrant objects, and $\widetilde{\mathcal{W}}_R$ is the class of trivial objects. We call these structures the Ding t-projective model structure and the Ding t-injective model structure on $\mathrm{Ch}(R)$, respectively.

3. Ding homological dimensions over graded rings. A $\mathbb{Z}$-graded ring A is a ring that has a direct sum decomposition into (abelian) additive groups $A = \oplus_{n\in\mathbb{Z}}A_n = \cdots \oplus A_{-1} \oplus A_0 \oplus A_1 \oplus \cdots$ such that the ring multiplication $\cdot$ satisfies $A_m \cdot A_n \subseteq A_{m+n}$, for every $m, n \in \mathbb{Z}$. A graded module is left module over a $\mathbb{Z}$-graded ring A with a direct sum decomposition $M = \oplus_{n\in\mathbb{Z}}M_n$ such that the product $\cdot : A \cdot M \to M$ satisfies $A_m \cdot M_n \subseteq M_{m+n}$, for every $m,\ n \in \mathbb{Z}$.

Given any associative ring with unit R, consider the ring of polynomials $R[x]$ and the ideal (x^2). It is easy to see that the quotient $A := R[x]/(x^2)$ is a $\mathbb{Z}$-graded ring with a direct sum decomposition given by $R[x]/(x^2) = \cdots \oplus 0 \oplus (x) \oplus R \oplus 0 \oplus \cdots$, where the scalars $r \in R$ are the elements of degree 0, and the elements in the ideal (x) form the terms of degree -1. The following we will check that the category A-Mod is isomorphic to the category $\mathrm{Ch}(R)$ of unbounded R-chain complexes. Through this isomorphism, the A-module A corresponds to $D^0(R)$. In particular, we have $\mathrm{Ext}^i_A(-, A) \cong \mathrm{Ext}^i_{\mathrm{Ch}(R)}(-, D^0(R))$.

Now we prove that every A-module can be viewed as a chain complex over R, and vice versa.

Let $\Phi : A\text{-Mod} \longrightarrow \mathrm{Ch}(R)$ be the application defined as follows:
(1) Given a graded A-module $M = \oplus_{n\in\mathbb{Z}}M_n$, note that if $y \in M_n$ then $x \cdot y \in M_{n-1}$, since x has degree -1. Denote by $\Phi(M)_n$ the set M_n endowed with the structure of R-module provided by the graded multiplication. Let $\partial_n : \Phi(M)_n \longrightarrow \Phi(M)_{n-1}$ be the map $y \mapsto x \cdot y$. It is clear that ∂_n is an R-homomorphism. Moreover, $\partial_{n-1} \circ \partial_n(y) = x \cdot (x \cdot y) = x^2 \cdot y = 0 \cdot y = 0$. Then, $\Phi(M) = (\Phi(M)_n, \partial_n)_{n\in\mathrm{Z}}$ is a chain complex over R.
(2) Let $f : M \longrightarrow N$ be a homomorphism of graded A-modules. Then $f(M_n) \subseteq N_n$, for every $n \in \mathrm{Z}$. It follows that $f|_{M_n}$ is an R-homomorphism. Let $\Phi(f)_n := f|_{M_n} : \Phi(M)_n \longrightarrow \Phi(N)_n$. We have $\Phi(f)_{n-1} \circ \partial^M_n(y) = f|_{M_{n-1}}(x \cdot y) = x \cdot f|_{M_n}(y) = \partial^N_n \circ \Phi(f)_n(y)$. So $\Phi(f) = (\Phi(f)_n)_{n\in\mathbb{Z}}$ is a chain map.

Note that $\Phi : A\text{-Mod} \longrightarrow \mathrm{Ch}(R)$ defines a covariant functor. We show that this functor is an isomorphism, by giving an inverse functor $\Psi : \mathrm{Ch}(R) \longrightarrow A\text{-Mod}$.
(1) Let $M = (M_n, \partial_n)_{n\in\mathbb{Z}}$ be a chain complex over R. Let $y \in M_n$ and define the product $r \cdot y = ry \in M_n$ for every $r \in R$, and $x \cdot y = \partial_n(y) \in M_{n-1}$. This gives rise to a graded A-module, that we denote by $\Psi(M) = (\Psi(M)_n)_{n\in\mathbb{Z}}$, where $\Psi(M)_n = M_n$ as sets.
(2) Given a chain map $f : M \longrightarrow N$, we have $x \cdot f(y) = \partial \circ f(y) = f \circ \partial(y) = f(x \cdot y)$. Then f gives rise to a graded A-module homomorphism denoted by $\Phi(f)$.

It is easy to show that $\Psi \circ \Phi = \mathrm{Id}_{A\text{-Mod}}$ and $\Phi \circ \Psi = \mathrm{Id}_{\mathrm{Ch}(R\text{-Mod})}$. It follows that Ψ and Φ map projective (resp., injective, flat) objects into projective (resp., injective, flat) objects. It is also easy to check that both Ψ and Φ preserves exact sequences.

2.1. Definition ([[8]]). *A complex C is called finitely generated if, in case $C = \sum_{i\in I} D^i$, with $D^i \in \mathrm{Ch}(R)$ subcomplexes of C, then there exists a finite subset $J \subset I$ such that $C = \sum_{i\in J} D^i$; A complex C is called finitely presented if C is finitely generated and for every exact sequence of complexes $0 \to K \to L \to C \to 0$ with L finitely generated, K is also finitely generated.*

2.2. Lemma ([[8]]). *An R-complex C is finitely generated if and only if C is bounded and C_n is finitely generated in R-*Mod *for all $n \in \mathbb{Z}$. A complex C is finitely presented if and only if C is bounded and C_n is finitely presented in R-*Mod *for all $n \in \mathbb{Z}$.*

It is obvious that Ψ and Φ map finitely presented objects into finitely presented objects by Lemma 2.2. Next, we prove Ψ and Φ map FP-injective objects into FP-injective objects.

2.3. Lemma *Let E be an FP-injective A-module, and Y be an FP-injective R-complex. Then $\Phi(E)$ is an FP-injective R-complex, and $\Psi(Y)$ is an FP-injective A-module.*

Proof We prove the first assertion, the second one can be proven similarly. Let F be a finitely presented A-module. We first prove that $\mathrm{Ext}^i(\Phi(F),\Phi(E)) \cong \mathrm{Ext}^i_A(F,E) = 0$ for every $i \geq 1$. Given a class $[0 \longrightarrow E \longrightarrow Q \longrightarrow F \longrightarrow 0] \in \mathrm{Ext}^1_A(F,E)$, map its representative to the sequence

$$0 \longrightarrow \Phi(E) \longrightarrow \Phi(Q) \longrightarrow \Phi(F) \longrightarrow 0.$$

This sequence is exact since Φ is an exact functor. Also, Φ preserves pullbacks, and hence it preserves Baer sums. It follows

$$[0 \longrightarrow \Phi(E) \longrightarrow \Phi(Q) \longrightarrow \Phi(F) \longrightarrow 0] \in \mathrm{Ext}^1(\Phi(F),\Phi(E)).$$

It is clear that this mapping defines a group isomorphism from $\mathrm{Ext}^1_A(F,E)$ to $\mathrm{Ext}^1(\Phi(F),\Phi(E))$. The same argument works for any $i > 1$. Since $\Phi(F)$ is a finitely presented R-complex, $\Phi(E)$ is an FP-injective R-complex. □

2.4. Proposition *If R is an n-FC ring, then the graded ring $A := R[x]/(x^2)$ is n-FC with weak global dimension ∞.*

Proof Any homogeneous left (resp. right) ideal of A is of the form $I_0 + I_1x$, where I_0 and I_1 are left (resp. right) ideals of R. Let $I_0 + I_1x$ be finitely generated. So I_0, I_1 is finitely generated. Since R is left (right) coherent, I_0, I_1 is finitely presented. Hence $I_0 + I_1x$ is finitely presented. Hence A is left and right coherent. If $M = \oplus_{n\in\mathbb{Z}} M_n$ is a finitely presented A-module, then M_n is a finitely presented R-module for every $n \in \mathbb{Z}$. Since

$$\mathrm{Ext}^i_A(-,A) \cong \mathrm{Ext}^i_{\mathrm{Ch}(R)}(-,D^0(R)),$$

and

$$\mathrm{Hom}_{\mathrm{Ch}(R)}(X,D^0(R)) \cong \mathrm{Hom}_R(X_{-1},R),$$

where X_{-1} is the degree -1 part of X. Since this functor $(-)_{-1}$ is exact and preserves projectives, we see that

$$\mathrm{Ext}^i_{\mathrm{Ch}(R)}(-,D^0(R)) \cong \mathrm{Ext}^i_R((-)_{-1},R).$$

In particular, if R is n-FC, so is A.

Since flat chain complexes are exact, any chain complex that is not exact must have infinite flat dimension, so the weak global dimension of A is ∞. □

We say a chain complex X is projective (resp., injective, flat, FP-injective) if it is exact and each cycle Z_nX is projective (resp., injective, flat, FP-injective). We denote these classes of chain complexes by $\widetilde{\mathcal{P}}, \widetilde{\mathcal{I}}, \widetilde{\mathcal{F}}$, and $\widetilde{\mathcal{FI}}$ respectively.

2.5. Lemma *If R is an n-FC ring, then the class of chain complexes with finite FP-injective dimension and the class of chain complexes with finite flat dimension coincide and every exact complex E with cycles of finite flat (FP-injective) dimension has $\mathrm{fd}(E) \leq n$ (FP-$\mathrm{id}(E) \leq n$).*

Proof From [[19], Theorem 2.26], we know that the class of chain complexes with finite FP-injective (flat) dimension is the class of exact complexes with cycles of bounded FP-injective (flat) dimension. If R is n-FC, then these classes coincide. □

By Proposition 2.4, for an n-FC ring R and $A := R[x]/(x^2)$, the class $\mathcal{W}_A$ must correspond to some collection of chain complexes. Next we will characterize these chain complexes.

2.6. Corollary. *Let R be left and right coherent with finite weak global dimension. Then $\mathcal{W}_A$ corresponds the class of all exact complexes.*

Proof By [[18], Proposition 3.5], [[13], Theorem 4.2] and Lemma 2.5 the conclusion is obvious. □

Recall from [[7]] that a complex P is said to be dg-projective if each P_m is projective and $\mathcal{H}om_R(P, E)$ is exact for any exact complex E. A dg-injective complex is defined dually.

Now we get the following result.

2.7. Proposition. *Suppose R is a ring and let A be the graded ring $R[x]/(x^2)$. Then every dg-projective chain complex over R is a Ding projective A-module. The converse holds if R is left and right coherent and of finite weak global dimension.*

Proof Suppose X is a dg-projective chain complex. We want to show that it is a Ding projective A-module. We first take a projective resolution of X

$$\cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow X \longrightarrow 0.$$

Note that since X is dg-projective, the kernel at any spot in the sequence is also dg-projective. Next we use the fact that $(dg\widetilde{\mathcal{P}}, \mathcal{E})$ is complete to find a short exact sequence $0 \longrightarrow X \longrightarrow P_0 \longrightarrow K \longrightarrow 0$ where P_0 is exact and K is dg-projective. But P_0 must also be dg-projective since it is an extension of two dg-projective complexes. Therefore P_0 is a projective complex. Continuing with the same procedure on K we can build a projective coresolution of X as below:

$$0 \longrightarrow X \longrightarrow P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots.$$

Again the kernel at each spot is dg-projective. Pasting this "right" coresolution together with the "left" resolution above we get an exact sequence

$$\cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow P_0 \longrightarrow P_1 \longrightarrow \cdots$$

of projective complexes which satisfies the definition of X being a Ding projective A-module. Indeed since X dg-projective implies $\operatorname{Ext}^1(X, E) = 0$ for any exact chain complex E, we certainly have $\operatorname{Ext}^1(X, F) = 0$ for any flat chain complex F. Therefore applying $\operatorname{Hom}_A(-, F)$ will leave the sequence exact.

Next we let X be a Ding projective A-module and argue that it is a dg-projective R-chain complex, when R is both left and right Coherent and $\operatorname{wD.dim}(R) = n$. Note that by the definition of Ding projective we have $\operatorname{Ext}^i(X, F) = 0$ for all $i > 0$ and flat complexes F. We will be done if we can show that $\operatorname{Ext}^1(X, E) = 0$ for any exact complex E. By Corollary 2.6 $\operatorname{fd}(E) \leq n$, so there exists a finite flat resolution

$$0 \longrightarrow F^n \longrightarrow \cdots \longrightarrow F^1 \longrightarrow F^0 \longrightarrow E \longrightarrow 0.$$

By a dimension shifting argument we see that $\operatorname{Ext}^1(X, E) \cong \operatorname{Ext}^{n+1}(X, F^n) = 0$. □

With a dual proof we get the following.

2.8. Proposition. *Suppose R is a ring and let A be the graded ring $R[x]/(x^2)$. Then every dg-injective chain complex over R is a Ding injective A-module. The converse holds if R is left and right coherent and of finite weak global dimension.*

Now we extend Proposition 2.7 as follows.

2.9. Theorem. *The functor $\Psi : \mathrm{Ch}(R) \longrightarrow A$-Mod maps dg-t-projective complexes into Ding t-projective A-modules. If R is a left and right coherent ring of finite weak global dimension, then the inverse functor $\Phi : A$-Mod $\longrightarrow \mathrm{Ch}(R)$ maps Ding t-projective A-modules into dg-t-projective complexes.*

Proof Let $X \in dg\widetilde{\mathcal{P}_t}$. Consider $\Psi(X)$ and a partial left projective resolution

$$0 \longrightarrow G \longrightarrow P_{t-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \Psi(X) \longrightarrow 0.$$

We show that G is a Ding projective A-module. Consider the complex $\Phi(G)$ and let E be an exact complex. By the proof of Lemma 2.3 we have $\mathrm{Ext}^1(\Phi(G), E) \cong \mathrm{Ext}^1(X, E')$, where $E' \in \Omega^{-t}(E)$. Note that $E' \in (\widetilde{\mathcal{P}_t})^{\perp}$. In fact, if $Z \in \widetilde{\mathcal{P}_t}$ then $\mathrm{Ext}^1(Z, E') \cong \mathrm{Ext}^{t+1}(Z, E) = 0$. Also, it is easy to check that $E' \in \mathcal{E}$. So $E' \in (\widetilde{\mathcal{P}_t})^{\perp} \cap \mathcal{E} = (dg\widetilde{\mathcal{P}_t})^{\perp}$. It follows $\mathrm{Ext}^1(\Phi(G), E) \cong \mathrm{Ext}^1(X, E') = 0$, for every $E \in \mathcal{E}$. In other words, $\Phi(G)$ is dg-projective, and by Proposition 2.7 we have $G = \Psi(\Phi(G))$ is a Ding projective A-module.

Now suppose that R is a left and right coherent ring of finite weak global dimension. Note that Ψ and Φ define an one-to-one correspondence between the projective objects of $\mathrm{Ch}(R)$ and A-Mod. It follows that Ψ and Φ also define an one-to-one correspondence between t-projective complexes over R and t-projective A-modules. Let $X \in (\widetilde{\mathcal{P}_t})^{\perp}$ and consider $\Psi(X)$. Let M be an t-projective A-module. Then $\Phi(M)$ is a t-projective complex. We have $\mathrm{Ext}^1_A(M, \Psi(X)) \cong \mathrm{Ext}^1(\Phi(M), X) = 0$. It follows $\Psi(X) \in (\mathcal{P}_t)^{\perp}$. Hence, Ψ and Φ give rise to a one-to-one correspondence between $(\widetilde{\mathcal{P}_t})^{\perp}$ and $(\mathcal{P}_t)^{\perp}$. Also, by Corollary 2.6, we have the same correspondence between $\mathcal{E}$ and $\mathcal{W}_A$. Since $(dg\widetilde{\mathcal{P}_t})^{\perp} = (\widetilde{\mathcal{P}_t})^{\perp} \cap \mathcal{E}$ and $\mathcal{W}_A \cap (\mathcal{P}_t)^{\perp} = (\mathcal{DP}_t)^{\perp}$ by Corollary 1.4, we have that a complex Y is in $(dg\widetilde{\mathcal{P}_t})^{\perp}$ if and only if $\Psi(Y)$ is in $(\mathcal{DP}_t)^{\perp}$. Since $dg\widetilde{\mathcal{P}_t} = {}^{\perp}((dg\widetilde{\mathcal{P}_t})^{\perp})$ and $\mathcal{DP}_t = {}^{\perp}((\mathcal{DP}_t)^{\perp})$, we have that Φ maps Ding t-projective A-modules into dg-t-projective complexes. □

The following result is the dual version of Theorem 2.9.

2.10. Theorem. *The functor $\Psi : \mathrm{Ch}(R) \longrightarrow A$-Mod maps dg-t-injective complexes into Ding t-injective A-modules. If R is a left and right coherent ring of finite weak global dimension, then the inverse functor $\Phi : A$-Mod $\longrightarrow \mathrm{Ch}(R)$ maps Ding t-injective A-modules into dg-t-injective complexes.*

4. Adjoint functors. In this section, we show that the embedding functors $K(\mathcal{DP}) \longrightarrow K(R\text{-Mod})$ and $K(\mathcal{DI}) \longrightarrow K(R\text{-Mod})$ have right and left adjoints respectively, where $K(\mathcal{DP})$ ($K(\mathcal{DI})$) is the homotopy category of complexes of Ding projective (injective) modules, and $K(R\text{-Mod})$ denotes the homotopy category. To this end, we will be concerned with the category $\mathrm{Ch}(R)$ and the category $K(R\text{-Mod})$ firstly. These categories have the same objects, and the morphisms in $K(R\text{-Mod})$ are homotopy equivalence classes of chain maps, that is, for objects C and D of $K(R\text{-Mod})$, $\mathrm{Hom}_{K(R\text{-Mod})}(C, D) = \mathrm{H}_0(\mathcal{H}om_R(C, D))$, where $\mathrm{Hom}_{K(R\text{-Mod})}(C, D)$ denotes the abelian group of morphisms from C to D in $K(R\text{-Mod})$. We recall that if $f : C \longrightarrow D$ is a morphism in $\mathrm{Ch}(R)$, then we have the mapping cone $con(f)$ of f. We have that $(con(f))_n = D_n \oplus C_{n-1}$ and the differential d is such that $d(y, x) = (d(y) + f(x), -d(x))$. We have the short exact sequence $0 \longrightarrow D \longrightarrow con(f) \longrightarrow \Sigma C \longrightarrow 0$ where the maps $D \longrightarrow con(f)$ and $con(f) \longrightarrow \Sigma C$ are given by $y \longmapsto (y, 0)$ and $(y, x) \longmapsto x$ respectively. Given $f, g \in \mathrm{Hom}(C, D)$ we will let $f \sim g$ mean that f and g are homotopic. The idea of the next lemma derives from Bravo et al. in [[1]].

3.1. Lemma *Let R be a Ding-Chen ring, X be an R-complex, and $0 \longrightarrow C \longrightarrow D \longrightarrow X \longrightarrow 0$ be exact where $D \in \widetilde{\mathcal{DP}}$, $C \in dg\widetilde{\mathcal{W}_R}$. If $D' \in \widetilde{\mathcal{DP}}$, $f_i \in \mathrm{Hom}(D', X)$ and $g_i \in \mathrm{Hom}(D', D)$ such that*

$$\begin{tikzcd} & D' \arrow[dl, "g_i"'] \arrow[d, "f_i"] \\ D \arrow[r] & X \end{tikzcd}$$

are commutative for $i = 1, 2$, then $f_1 \sim f_2$ if and only if $g_1 \sim g_2$.

Proof If $g_1 \sim g_2$ then easily $f_1 \sim f_2$. For the converse let $f = f_1 - f_2$ and $g = g_1 - g_2$ we see that we only need show that when $f \sim 0$ we have $g \sim 0$. With such f and g we get the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & D \arrow[r] \arrow[d] & con(g) \arrow[r] \arrow[d] & \Sigma D' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X \arrow[r] & con(f) \arrow[r] & \Sigma D' \arrow[r] & 0. \end{tikzcd}$$

Since $f \sim 0$, by [[11], Lemma 2.3.2] we get that the lower short exact sequence splits. A retraction $con(f) \longrightarrow X$ provides us with a commutative diagram

$$\begin{tikzcd} D \arrow[r] \arrow[d] & con(g) \arrow[dl] \\ X. & \end{tikzcd}$$

Since $\widetilde{\mathcal{DP}}$ is closed under extensions and suspensions we have $con(g) \in \widetilde{\mathcal{DP}}$. Since $D \longrightarrow X$ is a $\widetilde{\mathcal{DP}}$-precover we get a lifting $con(g) \longrightarrow D$. We now prove that $con(g) \longrightarrow D$ provides a retraction of $D \longrightarrow con(g)$ in $K(R\text{-Mod})$. For this note that the difference of the composition $D \longrightarrow con(g) \longrightarrow D$ and the identity map $\mathrm{id}D$ maps D into the kernel of $D \longrightarrow X$, that is into C. Since $(\widetilde{\mathcal{DP}}, dg\widetilde{\mathcal{W}_R})$ is a complete hereditary cotorsion pair, this difference (as a map into C) is homotopic to 0 by [[11], Lemma 2.3.2]. But then the difference as a map into D is homotopic to 0. So $con(g) \longrightarrow D$ provides a retraction of $D \longrightarrow con(g)$ in $K(R\text{-Mod})$. Next we prove that $con(g) \longrightarrow D$ provides a retraction of $D \longrightarrow con(g)$ in $\mathrm{Ch}(R)$. Let $s : con(g) \longrightarrow D$ (s a morphism in $\mathrm{Ch}(R)$) give a retraction of $D \longrightarrow con(g)$ in $K(R\text{-Mod})$. Let r be the corresponding homotopy, i.e. for $y \in D$ we have $(dr + rd)(y) = y - s(y, 0)$. Define $con(g) \longrightarrow D$ by $(y, x) \mapsto y + rg(x) + s(0, x)$ for $(y, x) \in con(g)$. We can easily prove that this map is a morphism of complexes and it gives the desired retraction. So we get that the short exact sequence $0 \longrightarrow D \longrightarrow con(g) \longrightarrow \Sigma D' \longrightarrow 0$ is split exact in $\mathrm{Ch}(R)$. So by [[11], Lemma 2.3.2] we get that $g \sim 0$. □

3.2. Corollary. *Let R be a Ding-Chen ring, X be an R-complex, and $0 \longrightarrow C \longrightarrow D \longrightarrow X \longrightarrow 0$ be exact where $D \in \widetilde{\mathcal{DP}}$ and $C \in dg\widetilde{\mathcal{W}_R}$. If $D' \in \widetilde{\mathcal{DP}}$, then $\mathrm{Hom}_{K(R\text{-Mod})}(D', D) \longrightarrow \mathrm{Hom}_{K(R\text{-Mod})}(D', X)$ is a bijection.*

Proof We first note that the exact sequence $0 \longrightarrow C \longrightarrow D \longrightarrow X \longrightarrow 0$ gives the exact sequence $\mathrm{Hom}(D', D) \longrightarrow \mathrm{Hom}(D', X) \longrightarrow \mathrm{Ext}^1(D', C) = 0$. So $\mathrm{Hom}(D', D) \longrightarrow \mathrm{Hom}(D', X)$ is surjective. This gives that $\mathrm{Hom}_{K(R\text{-Mod})}(D', D) \longrightarrow \mathrm{Hom}_{K(R\text{-Mod})}(D', X)$ is surjective. Lemma 3.1 guarantees that this function is injective and so bijective. □

This gives the following result.

3.3. Theorem. *Let R be a Ding-Chen ring. Then the embedding $K(\mathcal{DP}) \longrightarrow K(R\text{-Mod})$ has a right adjoint.*

Proof For each $X \in \mathrm{Ch}(R)$, there exists an exact sequence $0 \longrightarrow C \longrightarrow D \longrightarrow X \longrightarrow 0$ in $\mathrm{Ch}(R)$ with $D \in \widetilde{\mathcal{DP}}$ and $C \in dg\widetilde{\mathcal{W}_R}$. We want to define a functor $T : K(R\text{-Mod}) \longrightarrow K(\mathcal{DP})$ so that $T(X) = D$. If $f : X \longrightarrow X'$ is a morphism in $\mathrm{Ch}(R)$ we let $[f]$ represent the corresponding morphism in $K(R\text{-Mod})$. So $[f]$ consists of all $f' : X \longrightarrow X'$ such that $f \sim f'$. We use the following procedure to define $T([f])$. We have the exact sequence $\mathrm{Hom}(D, D') \longrightarrow \mathrm{Hom}(D, X') \longrightarrow \mathrm{Ext}^1(D, C') = 0$. This means that there is a morphism $g \in \mathrm{Hom}(D, D')$ whose image in $\mathrm{Hom}(D, X')$, which is the composition $D \longrightarrow X \longrightarrow X'$. So we have the commutative diagram

$$\begin{array}{ccc} D & \longrightarrow & X \\ {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ D' & \longrightarrow & X'. \end{array}$$

For $f' \in [f]$ (so $f \sim f'$) we use the same argument and find a morphism $g' : D \longrightarrow D'$ so that the diagram

$$\begin{array}{ccc} D & \longrightarrow & X \\ {\scriptstyle g'}\downarrow & & {\scriptstyle f'}\downarrow \\ D' & \longrightarrow & X' \end{array}$$

is commutative. Then an application of Lemma 3.1 gives that $g \sim g'$. This means that we can define $T([f])$ to be $[g]$ with f and g as above. Then it can be quickly checked that T is an additive functor. Note that the maps $D \longrightarrow X$ then become maps $T(X) \longrightarrow X$ and give a natural transformation from T to the identity functor on $K(R\text{-Mod})$.

Now we appeal to Corollary 3.2. This Corollary says that $\mathrm{Hom}_{K(R\text{-Mod})}(D', D) \longrightarrow \mathrm{Hom}_{K(R\text{-Mod})}(D', X)$ is a bijection if $D' \in \widetilde{\mathcal{DP}}$ and $0 \longrightarrow C \longrightarrow D \longrightarrow X \longrightarrow 0$ is as above. But $T(X) = D$, so we have the bijection

$$\mathrm{Hom}_{K(R\text{-Mod})}(D', T(X)) \longrightarrow \mathrm{Hom}_{K(R\text{-Mod})}(D', X).$$

From the definition of this map we see that it is natural in D'. From the natural transformation above we see that it is natural in X. So this establishes that T is a right adjoint of the embedding functor $K(\mathcal{DP}) \longrightarrow K(R\text{-Mod})$. □

3.4. Remark *We also have the duals of Lemma 3.1 and the Corollary 3.2. The embedding $K(\mathcal{DI}) \longrightarrow K(R\text{-Mod})$ has a left adjoint.*

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References

[1] Bravo D., Enochs E.E., Iacob A.C., Jenda O.M.G., Rada J.: Cotorsion pairs in C(R-Mod), Rocky Mountain J. Math. 42, 1787–1802 (2012).

[2] Ding N.Q., Chen J.L.: The flat dimensions of injective modules, Manuscripta Math. 78, 165–177 (1993).

[3] Ding N.Q., Chen J.L.: Coherent rings with finite self-FP-injective dimension, Comm. Algebra 24, 2963–2980 (1996).

[4] Ding N.Q., Mao L.X.: Relative FP-projective modules, Comm. Algebra 33, 1587–1602 (2005).

[5] Ding N.Q., Li Y.L., Mao L.X.: Strongly Gorenstein flat modules, J. Aust. Math. Soc. 86, 323–338 (2009).

[6] Enochs, E.E.: Injective and flat covers, envelopes and resolvents, Israel J. Mathematics, 39, 189–209 (1981).

[7] Enochs, E.E., Jenda, O.M.G., Xu, J.: Orthogonality in the category of complexes, Math. J. Okayama Univ. 38, 25–46 (1996).

[8] Enochs E.E., García Rozas, J.R.: Flat covers of complexes, J. Algebra 210, 86–102, (1998).

[9] Enochs, E.E., Jenda, O.M.G.: Relative Homological Algebra, de Gruyter Exp. Math., vol. 30, de Gruyter, Berlin, 2000.

[10] Enochs, E.E., Oyonarte, L.: Covers, envelopes and cotorsion theories, Nova Science Publishers, Ins. Hauppauge, New York, 2002.

[11] García Rozas, J.R.: Covers and envelope in the category of complexes of modules. Boca Raton London New York Washington, D.C. 1999.

[12] Gillespie, J.: The flat model structure on Ch(R), Tran. Amer. Math. Soc. 356, 3369–3390 (2004).

[13] Gillespie J.: Model structures on modules over Ding-Chen rings, Homology, Homotopy Appl. 12, 61–73 (2010).

[14] Gillespie J., Hovey M.: Gorenstein model structures and generalized derived categories. Submitted to Proceedings of the Edinburgh Mathematical Society.

[15] Hovey M.: Cotorsion pairs, model category structures and representation theory. Mathematical Zeitschrift. 241, 553–592 (2002).

[16] Mao, L.X., Ding, N.Q.: Envelopes and covers by modules of finite FP-injective and flat dimensions, Comm. Algebra 35, 833–849 (2007).

[17] Mao, L.X., Ding N.Q.: Gorenstein FP-injective and Gorenstein flat modules, J. Algebra Appl. 7, 491–506 (2008).

[18] Stenström, B.: Coherent rings and FP-injective modules, J. London Math. Soc. 2(2), 323–329 (1970).

[19] Wang, Z.P., Liu, Z.K.: FP-injective complexes and FP-injective dimension for complexes, J. Aust. Math. Soc. 91, 163–187 (2011).

[20] Yang, G., Liu, Z.K., Liang, L.: Model structures on categories of complexes over Ding-Chen rings, Comm. Algebra 41, 50–69 (2013).