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Intersection local time of subfractional Ornstein-Uhlenbeck processes

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Abstract

In this paper, we consider Ornstein-Uhlenbeck process

$$\mathrm{d}X_t^H = -X_t^H \mathrm{d}t + v\mathrm{d}S_t^H, \quad X_0^H = x,$$

driven by a subfractional Brownian motion S^H. We prove that the subfractional Ornstein-Uhlenbeck process X^H is local nondeterministic and give some properties of this process. As an application, assume $d \geq 2$, we prove that the intersection local time of two independent, $d-$dimensional subfractional Ornstein-Uhlenbeck process, X^H and $\widetilde{X}^H$, exists in L^2 if and only if $Hd < 2$.

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1. Introduction

The classical Ornstein-Uhlenbeck process (see Revuz and Yor[23]) has a remarkable history in physics. It was introduced to model the velocity of the particle diffusion process and later it has been heavily used in finance, and thus in econophysics. It can be constructed as the unique strong solution of Itô stochastic differential equation

$$\mathrm{d}X_t = -X_t \mathrm{d}t + v\mathrm{d}B_t, \quad X_0 = x, \tag{1.1}$$

where B is a standard Brownian motion starting at 0.

On the other hand, extensions of the classical Ornstein-Uhlenbeck process have been suggested mainly on demand of applications. The fractional Ornstein-Uhlenbeck process

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was an extension of the classical Ornstein-Uhlenbeck process, where fractional Brownian motion B^H was used as integrator

$$dX_t^H = -X_t^H dt + v dB_t^H, \quad X_0 = x. \tag{1.2}$$

The equation (1.2) has a unique solution $X_t^H = \{X_t^H, 0 \le t \le T\}$, which can be expressed as

$$X_t^H = e^{-t}(x + v\int_0^t e^s dB_s^H), \tag{1.3}$$

and the solution was called the fractional Ornstein-Uhlenbeck process. Recall that fractional Brownian motion B^H with Hurst index $H \in (0,1)$ is a central Gaussian process with $B_0^H = 0$ and the covariance function

$$\mathrm{E}[B_t^H B_s^H] = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}], \tag{1.4}$$

for all $t, s \geqslant 0$. This process was first introduced by Kolmogorov and studied by Mandelbrot and Van Ness [19]. Clearly, when $H = \frac{1}{2}$ the fractional Ornstein-Uhlenbeck process is the classical Ornstein-Uhlenbeck process X with parameter v starting at $x \in \mathbb{R}$. A class of superpositions of Ornstein-Uhlenbeck type processes is constructed in terms of integrals with respect to independently scattered random measures in Barndorff-Nielsen [3]. Barndorff-Nielsen and Shephard [4] construct continuous time stochastic volatility models for financial assets where the volatility processes are superpositions of positive Ornstein-Uhlenbeck processes, and they study these models in relation to financial data and theory. Recently, Habtemicael and SenGupta [12] shown that the Gamma-Ornstein-Uhlenbeck process is a possible candidate for earthquake data modeling. SenGupta [25] uses Ornstein-Uhlenbeck process in forming a partial integro differential equations in finance. More works for the fractional Ornstein-Uhlenbeck process can be found in Cheridito *et al.* [9], Hu and Nualart [15], Es-Sebaiy [11], Yan *et al.* [32, 33].

The intersection properties of Brownian motion paths have been investigated since the forties (see lévy [17]), and since then a large number of results on intersection local times of Brownian motion have been accumulated (see Albeverio *et al.* [1] and the references therein). The intersection local time of independent fractional Brownian motions has been studied by Chen and Yan [8], Jiang and Wang [16], Nualart and Ortiz-Latorre [22], Rosen [24], Wu-Xiao [30] and the references therein.

Motivated by all these results, in this paper, we will study the Ornstein-Uhlenbeck process

$$dX_t^H = -X_t^H dt + v dS_t^H, \quad X_0^H = x,$$

driven by a subfractional Brownian motion S^H(see section 2 for a precise definition). The solution

$$X_t^H = e^{-t}(x + v\int_0^t e^s dS_s^H), \tag{1.5}$$

is called the subfractional Ornstein-Uhlenbeck process (see Mendy [20]).

The rest of this paper is organized as follows. In section 2 we briefly recall the subfractional Brownian motion and the related Wiener-Itô integral. In section 3 we show that the subfractional Ornstein-Uhlenbeck process X^H is local nondeterministic and establish some estimates for the increments of the process, that is, there exist two constant $c_{H,T}, C_{H,T} > 0$ depending on H, T only which may not be the same in each occurrence such that the estimates

$$c_{H,T} v^2 (t-s)^{2H} \le E(X_t^H - X_s^H)^2 \le C_{H,T} v^2 (t-s)^{2H},$$

and

$$c_{H,T} v^2 G(t,s) \le E(X_t^H X_s^H) \le C_{H,T} v^2 G(t,s),$$

hold for all $0 < s < t < T$, where $G(t,s) = t^{2H} + s^{2H} - \frac{1}{2}[(t+s)^{2H} + (t-s)^{2H}]$. In section 4 we consider the intersection local time of two independent subfractional Ornstein-Uhlenbeck process $X^H = \{X_t^H, 0 \le t \le T\}$ and $\widetilde{X}^H = \{\widetilde{X}_t^H, 0 \le t \le T\}$ on $\mathbb{R}^d, d \ge 2$ with the same indices $H \in (0,1)$. The intersection local time is formally defined as

$$\ell_T = \int_0^T \int_0^T \delta(X_t^H - \widetilde{X}_s^H) \mathrm{d}s\mathrm{d}t, \tag{1.6}$$

where δ denotes the Dirac delta function. It is a measure of the amount of time that the trajectories of the two processes, X^H and $\widetilde{X}^H$, intersect on the time interval $[0,T]$. In order to give a rigorous meaning to ℓ_T we approximate the Dirac function by the heat kernel

$$p_\varepsilon(x) = (2\pi\varepsilon)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2\varepsilon}}, x \in \mathbb{R}^d.$$

Then, we can consider the following family of random variables indexed by $\varepsilon > 0$

$$\ell_{\varepsilon,T} = \int_0^T \int_0^T p_\varepsilon(X_t^H - \widetilde{X}_s^H) \mathrm{d}s\mathrm{d}t. \tag{1.7}$$

We get the convergence of $\ell_{\varepsilon,T}$ as ε tends to zero in the $L^2(\Omega)$.

2. Preliminaries

In this section, we briefly recall the definition and properties of the Wiener-Itô integer with respect to the subfractional Brownian motion. As an extension of Brownian motion, Bojdecki *et al.* [6] introduced and studied a rather special class of self-similar Gaussian processes which preserves many properties of the fractional Brownian motion, which is called the subfractional Brownian motion. This process arised from occupation time fluctuations of branching particle systems with Poisson initial condition, and it also appeared independently in a different context in Dzhaparidze and Van Zanten[10]. The so-called subfractional Brownian motion (subfBm in short) with index $H \in (0,1)$ is a mean zero Gaussian process $S^H = \{S_t^H, t \ge 0\}$ with $S_0^H = 0$ and

$$E\left[S_t^H S_s^H\right] = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |t-s|^{2H}\right] \tag{2.1}$$

for all $s, t \ge 0$. For $H = 1/2$, S^H coincides with the standard Brownian motion B. S^H is neither a semimartingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from stochastic analysis are not available when dealing with S^H. The subfBm has properties analogous to those of fractional Brownian motion (self-similarity, long-range dependence, Hölder paths). However, in comparison with fractional Brownian motion, the subfBm has non-stationary increments and the increments over non-overlapping intervals are more weakly correlated and their covariance decays polynomially as a higher rate in comparison with fractional Brownian motion (for this reason in Bojdecki *et al.* [6] it is called subfBm). The properties mentioned above make the subfBm a possible candidate for models which involve long-range dependence, self-similarity and non-stationary increment. More studies on the subfBm can be found in Bardina and Bascompte [2], Bojdecki *et al.* [7], Liu and Yan [18], Shen *et al.* [26, 27, 28], Yan and Shen [31] and the references therein.

Consider the integral representation of the subfBm S_t^H of the form

$$S_t^H = \int_0^t K_H(t,u) \mathrm{d}B_u, \quad 0 \le t \le T, \tag{2.2}$$

where $K_H(t,u)$ is the kernel

$$(2.3)\quad K_H(t,s)=\frac{c_H\sqrt{\pi}}{2^H\Gamma(H+\frac{1}{2})}s^{3/2-H}\left(\frac{(t^2-s^2)^{H-\frac{1}{2}}}{t}+\int_s^t\frac{(u^2-s^2)^{H-\frac{1}{2}}}{u^2}\mathrm{d}u\right)1_{(0,t)}(s).$$

In particular, when $\frac{1}{2}<H<1$, the kernel $K_H(t,s)$ can be written in a less complicated form:

$$(2.4)\qquad K_H(t,s)=\frac{c_H\sqrt{\pi}}{2^{H-1}\Gamma(H-\frac{1}{2})}s^{3/2-H}\int_s^t(u^2-s^2)^{H-\frac{3}{2}}\mathrm{d}u1_{(0,t)}(s),$$

where $c_H^2=\frac{\Gamma(1+2H)sin\pi H}{\pi}$. Using the idea in Hu [14], the kernel $K_H(t,s)$ defines an operator $\Gamma_{H,T}$ in $L^2([0,T])$ given by

$$\Gamma_{H,T}h(t)=\int_0^t K_H(t,u)h(u)\mathrm{d}u,\qquad h\in L^2([0,T]),$$

and the function $\Gamma_{H,T}h(t)$ is continuous and vanishes at zero. The transpose $\Gamma^*_{H,t}$ of $\Gamma_{H,T}$ restricted to the interval $[0,t](0\le t\le T)$ is

$$\begin{aligned}\Gamma^*_{H,t}g(s)=C_Hs^{3/2-H}[&(t^2-s^2)^{H-\frac{1}{2}}t^{-1}g(t)-\int_s^t(u^2-s^2)^{H-\frac{1}{2}}u^{-1}g'(u)\mathrm{d}u\\&+\int_s^t(u^2-s^2)^{H-\frac{1}{2}}u^{-2}g(u)\mathrm{d}u],\end{aligned}$$

for $g\in\mathbf{S}$, the set of all smooth functions on $[0,T]$ with bounded derivatives, where $C_H=\frac{c_H\sqrt{\pi}}{2^{H-1}\Gamma(H-\frac{1}{2})}$.

In particular, for $\frac{1}{2}<H<1$, we have

$$\Gamma^*_{H,t}g(s)=C_Hs^{\frac{3}{2}-H}\int_s^t(u^2-s^2)^{H-\frac{3}{2}}g(u)\mathrm{d}u.$$

Now, we recall the definition of the Wiener-Itô integral with respect to the subfBm, more work can be found in Nualart[21], Tudor[29].

2.1. Definition. Let

$$\Theta_H=\{f\in\mathbf{S}:||f||=\int_0^T[\Gamma^*_{H,T}f(t)]^2\mathrm{d}t<\infty\}.$$

For $f\in\Theta_H$, we define

$$\int_0^t f(u)\mathrm{d}S^H=\int_0^t\Gamma^*_{H,t}f(u)\mathrm{d}B_u,\quad 0\le t\le T,$$

where $B=\{B_t,0\le t\le T\}$ is a standard Brownian motion with $B_0=0$.

By applying the operator $\Gamma^*_{H,t}$, we can write the subfractional Ornstein-Uhlenbeck process $X^H=\{X^H_t,t\ge0\}$ starting from zero as

$$X^H_t=v\int_0^t F(t,u)\mathrm{d}B_u,\quad 0\le t\le T.$$

For $0<u<t$,

$$(2.5)\qquad F(t,u)=C_{H,T}e^{-t}u^{\frac{3}{2}-H}\int_u^t(m^2-u^2)^{H-\frac{3}{2}}e^m\mathrm{d}m,$$

with $\frac{1}{2} < H < 1$, and

$$F(t,u) = C_{H,T}u^{\frac{3}{2}-H}[-e^{-t}\int_u^t (m^2-u^2)^{H-\frac{1}{2}}m^{-1}e^m\mathrm{d}m \tag{2.6}$$
$$+ (t^2-u^2)^{H-\frac{1}{2}}t^{-1} + e^{-t}\int_u^t (m^2-u^2)^{H-\frac{1}{2}}m^{-2}e^m\mathrm{d}m],$$

with $0 < H < \frac{1}{2}$.

3. Some properties of subfractional Ornstein-Uhlenbeck process

In this section, we show that the subfractional Ornstein-Uhlenbeck process X^H is local nondeterministic and establish some estimates for the increments of the process.

The concept of local nondeterminism was first introduced by Berman[5] to unify and extend his methods for studying local times of real-valued Gaussion process. Define the relative prediction error:

$$V_n = \frac{\mathrm{Var}(X(t_n)-X(t_{n-1}))|X(t_1),...,X(t_{n-1}))}{\mathrm{Var}(X(t_n)-X(t_{n-1}))}$$

which is the ratio of the conditional to the unconditional variance. We consider this to be a measure of the relative predictability of the increment $X(t_n)-X(t_{n-1})$ based on the knowledge of the finite set of data $X(t_1),...,X(t_{n-1})$. It follows from the elementary property of conditional variance that $0 \leq V_n \leq 1$. If $V_n = 1$, then the increment is relatively completely unpredictable because the variance is not reduced by the information about $X(t_1),...,X(t_{n-1})$. On the other extreme, if $V_n = 0$, then the increment is relatively predictable. The process X is called locally nondeterministic on an interval $J \subset R_+$ if for every integer $n \geq 2$,

$$\lim_{\epsilon\to 0}\inf_{t_n-t_1\leq\epsilon} V_n > 0, \tag{3.1}$$

where the infimum in Eq. (3.1) is taken over all ordered points $t_1 < t_2 < ... < t_n$ in J with $t_n - t_1 \leq \epsilon$. This condition means that a small increment of the process X is not almost relatively predictable based on a finite number of observations from the immediate past.

It is well known that Eq.(3.1) is equivalent to the following property which says that X has locally approximately independent increments: for any positive integer $n \geq 2$, there exist positive constants C_n and δ (both may depend on n) such that

$$\mathrm{Var}\left(\sum_{j=1}^n u_j\left[X(t_j)-X(t_{j-1})\right]\right) \geq C_n\sum_{j=1}^n u_j^2\mathrm{Var}\left[X(t_j)-X(t_{j-1})\right] \tag{3.2}$$

for all ordered points $0 = t_0 < t_1 < t_2 < ... < t_n$ in J with $t_n - t_1 < \delta$ and all $u_j \in \mathbb{R}(1 \leq j \leq n)$. Xiao [34] give the properties of local nondeterminism of Gaussion and stable random fields.

By Berman[5], a process $X_t = \int_0^t K(t,u)\mathrm{d}B_u,\quad t \in J$ is local nondeterministic if and only if

$$\lim_{c\downarrow 0}\inf_{0<t-s<c:s,t\in J}\frac{\int_s^t K^2(t,u)\mathrm{d}u}{\int_0^s[K(t,u)-K(s,u)]^2\mathrm{d}u} > 0, \tag{3.3}$$

where K is a measurable function of (t,u) such that $\int_0^t K^2(t,u)\mathrm{d}u < \infty$ for all $t \in J$.

In order to prove that the subfractional Ornstein-Uhlenbeck process X^H is local nondeterministic, we firstly give the following Lemma.

3.1. Lemma. *Let $F(\cdot,\cdot)$ be given by* (2.5)*and* (2.6)*. Then we have*

$$\int_0^s [F(t,u)-F(s,u)]^2 du \le C_{H,T}(t-s)^{2H}, \quad 0\le s\le t,$$

for all $0<H<1$.

Proof. Firstly, for $\frac{1}{2}<H<1$ and $0<s<t<T$, we have

$$\begin{aligned}|F(t,u)-F(s,u)| \le C_{H,T}|e^{-t}-e^{-s}|u^{\frac{3}{2}-H}\int_u^s (m^2-u^2)^{H-\frac{3}{2}}e^m \mathrm{d}m \\ + C_{H,T}e^{-t}u^{\frac{3}{2}-H}\int_s^t (m^2-u^2)^{H-\frac{3}{2}}e^m \mathrm{d}m \\ := C_{H,T}u^{\frac{3}{2}-H}(I_1+I_2).\end{aligned}$$

It is obvious that $I_2 \le \int_s^t (m^2-u^2)^{H-\frac{3}{2}}\mathrm{d}m$, and

$$I_1 \le (t-s)\int_u^s (m^2-u^2)^{H-\frac{3}{2}}\mathrm{d}m \le C_H(t-s)u^{H-\frac{3}{2}}(s-u)^{H-\frac{1}{2}}.$$

So, we have

$$\begin{aligned}\int_0^s [F(t,u)-F(s,u)]^2\mathrm{d}u &\le C_H\int_0^s u^{3-2H}I_1^2\mathrm{d}u + C_H\int_0^s u^{3-2H}I_2^2\mathrm{d}u \\ &\le C_H(t-s)^2\int_0^s (s-u)^{2H-1}\mathrm{d}u \\ &\quad + C_H\int_s^t\int_s^t\int_0^s u^{3-2H}(m^2-u^2)^{H-\frac{3}{2}}(n^2-u^2)^{H-\frac{3}{2}}\mathrm{d}m\mathrm{d}n\mathrm{d}u \\ &\le C_H s^{2H}(t-s)^2 + C_{H,T}\int_s^t\int_s^t |m-n|^{2H-2}\mathrm{d}m\mathrm{d}n \\ &= C_H s^{2H}(t-s)^2 + C_{H,T}(t-s)^{2H} \le C_{H,T}(t-s)^{2H}.\end{aligned}$$

In the following, we consider the case $0<H<\frac{1}{2}$, we have

$$F(t,u)-F(s,u) = C_H u^{\frac{3}{2}-H}(M_1+M_2+M_3),$$

where

$$M_1 := (t^2-u^2)^{H-\frac{1}{2}}t^{-1} - (s^2-u^2)^{H-\frac{1}{2}}s^{-1},$$

$$M_2 := e^{-s}\int_u^s (m^2-u^2)^{H-\frac{1}{2}}m^{-1}e^m\mathrm{d}m - e^{-t}\int_u^t (m^2-u^2)^{H-\frac{1}{2}}m^{-1}e^m\mathrm{d}m,$$

$$M_3 := e^{-t}\int_u^t (m^2-u^2)^{H-\frac{1}{2}}m^{-2}e^m\mathrm{d}m - e^{-s}\int_u^s (m^2-u^2)^{H-\frac{1}{2}}m^{-2}e^m\mathrm{d}m.$$

Elementary calculus can show that

$$\int_0^s u^{3-2H}|M_1|^2\mathrm{d}u \le C_{H,T}(t-s)^{2H}. \tag{3.4}$$

For the term M_2, we have

$$\begin{aligned}|M_2| &\le (e^{-s}-e^{-t})\int_u^s (m^2-u^2)^{H-\frac{1}{2}}m^{-1}e^m\mathrm{d}m + e^{-t}\int_s^t (m^2-u^2)^{H-\frac{1}{2}}m^{-1}e^m\mathrm{d}m \\ &\le C_{H,T}u^{H-\frac{3}{2}}\left[(t-s)(s-u)^{H+\frac{1}{2}} + (t-s)^{H+\frac{1}{2}}\right].\end{aligned}$$

Hence,

$$\int_0^s u^{3-2H}|M_2|^2 du \le C_{H,T}(t-s)^{2H}. \tag{3.5}$$

For the term M_3. Noting that

$$\begin{aligned}|M_3| &\le (t-s)\int_u^s (m^2-u^2)^{H-\frac{1}{2}} m^{-2}\mathrm{d}m + \int_s^t (m^2-u^2)^{H-\frac{1}{2}} m^{-2}\mathrm{d}m\\ &:= (t-s)M_{3,1} + M_{3,2}\end{aligned}$$

On the one hand,

$$\begin{aligned}\int_0^s u^{3-2H}|M_{3,1}|^2\mathrm{d}u &= \int_0^s u^{3-2H}\mathrm{d}u\int_u^s\int_u^s (m^2-u^2)^{H-\frac{1}{2}}m^{-2}(n^2-u^2)^{H-\frac{1}{2}}n^{-2}\mathrm{d}m\mathrm{d}n\\ &\le \int_0^s \mathrm{d}u\int_u^s\int_u^s (m-u)^{H-\frac{3}{2}}(n-u)^{H-\frac{3}{2}}\mathrm{d}m\mathrm{d}n\\ &\le C_H\int_0^s (s-u)^{2H-1}\mathrm{d}u \le C_H s^{2H}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_0^s u^{3-2H}|M_{3,2}|^2\mathrm{d}u\\ &= \int_0^s u^{3-2H}\mathrm{d}u\int_s^t\int_s^t (m^2-u^2)^{H-\frac{1}{2}}m^{-2}(n^2-u^2)^{H-\frac{1}{2}}n^{-2}\mathrm{d}m\mathrm{d}n\\ &\le \int_s^t\int_s^t (mn)^{H-\frac{1}{2}}\mathrm{d}m\mathrm{d}n\int_0^{m\wedge n} u^{1-2(H+\frac{1}{2})}(m-u)^{(H+\frac{1}{2})-\frac{3}{2}}(n-u)^{(H+\frac{1}{2})-\frac{3}{2}}\mathrm{d}u\\ &\le C_{H,T}\int_s^t\int_s^t (mn)^{-\frac{1}{2}}|m-n|^{2H-1}\mathrm{d}m\mathrm{d}n \le C_{H,T}(t-s)^{2H}.\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^s u^{3-2H}|M_3|^2\mathrm{d}u &\le C_{H,T}(t-s)^{2H}\int_0^s u^{3-2H}|M_{3,1}|^2\mathrm{d}u + \int_0^s u^{3-2H}|M_{3,2}|^2\mathrm{d}u\\ &\le C_{H,T}(t-s)^{2H}.\end{aligned} \tag{3.6}$$

Combing with (3.4), (3.5) and (3.6), this completes the proof. □

3.2. Theorem. *The subfractional Ornstein-Uhlenbeck process X^H is local nondeterministic.*

Proof. Consider the integral representation of the subfractional Ornstein-Uhlenbeck process $X_t^H = v\int_0^t F(t,u)\mathrm{d}B_u, \quad 0\le t\le T.$

When $\frac{1}{2} < H < 1$, we get

$$\begin{aligned}F(t,u) &\ge C_H e^{-t+u}u^{\frac{3}{2}-H}\int_u^t (m^2-u^2)^{H-\frac{3}{2}}\mathrm{d}m\\ &\ge C_H e^{-t+u}u^{\frac{3}{2}-H}(t^2-u^2)^{H-\frac{3}{2}}(t-u)\\ &\ge C_{H,T}(t-u)^{H-\frac{1}{2}}.\end{aligned}$$

Hence, $\int_s^t F^2(t,u)\mathrm{d}u \ge C_{H,T}\int_s^t (t-u)^{2H-1}\mathrm{d}u \ge C_{H,T}(t-s)^{2H}$.

When $0 < H < \frac{1}{2}$, without loss of generality, one may assume $0 < T < 1$. By (2.6) we get that

$$F(t,u) \ge C_{H,T}(t^2-u^2)^{H-\frac{1}{2}}t^{-1}u^{\frac{3}{2}-H}.$$

Hence,

$$\int_s^t F^2(t,u)\mathrm{d}u \ge C_{H,T}\int_s^t u^{1-2H}(t^2-u^2)^{2H-1}\mathrm{d}u \ge C_{H,T}(t-s)^{2H}.$$

It follows from Lemma 3.1 and (3.3) that the Theorem3.2 holds. □

Next, we will study the variance of increment of the subfractional Ornstein-Uhlenbeck process. Let $X^H = \{X_t^H, 0 \le t \le T\}$ be the subfractional Ornstein-Uhlenbeck process starting from zero. Then we have

$$X_t^H = v \int_0^t e^{-t+u} \mathrm{d}S_u^H = v \int_u^t F(t,u) \mathrm{d}B_u.$$

Hence,

$$E[X_t^H X_s^H] = v^2 \int_0^{t\wedge s} F(t,u)F(s,u)\mathrm{d}u.$$

In particular, for $\frac{1}{2} < H < 1$ we have

$$EX_t^H X_s^H = v^2 e^{-t-s} \int_0^t \int_0^s e^{u+v} \phi(u,v) \mathrm{d}u\mathrm{d}v,$$

where $\phi(u,v) = H(2H-1)(|u-v|^{2H-2} - |u+v|^{2H-2})$.
First, we give the following Lemmas.

3.3. Lemma. *Let $0 < H < 1/2$. Then*

$$\int_0^s F(t,u)F(s,u)\,du \ge C_{H,T} G(t,s).$$

Proof. Without loss of generacity, one can assume that $0 < s < t < 1$. It follows from (2.6) that

$$F(t,u) \ge C_{H,T} u^{\frac{3}{2}-H} \int_u^t (m^2 - u^2)^{H-\frac{1}{2}} m^{-2} \mathrm{d}m.$$

So,

$$\begin{aligned}
&\int_0^s F(t,u)F(s,u)\mathrm{d}u \\
&\ge C_{H,T} \int_0^s \int_u^t \int_u^s (m^2-u^2)^{H-\frac{1}{2}} m^{-2} (n^2-u^2)^{H-\frac{1}{2}} n^{-2} u^{3-2H} \mathrm{d}m\mathrm{d}n\mathrm{d}u \\
&\ge C_{H,T} \int_0^s \int_0^s (mn)^{-2} \mathrm{d}m\mathrm{d}n \int_0^{m\wedge n} (m^2-u^2)^{H-\frac{1}{2}} (n^2-u^2)^{H-\frac{1}{2}} u^{3-2H} \mathrm{d}u \\
&\ge C_{H,T} \int_0^s m^{2H-4} \mathrm{d}m \int_0^m n^2 \mathrm{d}n = C_{H,T} s^{2H}.
\end{aligned}$$

Using the inequality $s^{2H} \ge t^{2H} - (t-s)^{2H}$, we get

$$\begin{aligned}
\int_0^s F(t,u)F(s,u)\,du &\ge C_{H,T}[s^{2H} + t^{2H} - (t-s)^{2H}] \\
&\ge C_{H,T}[s^{2H} + t^{2H} - \frac{1}{2}(t-s)^{2H} - \frac{1}{2}(t+s)^{2H}] \\
&= C_{H,T} G(t,s).
\end{aligned}$$

This completes the proof. □

3.4. Lemma. *Let $0 < H < \frac{1}{2}$. Then for all $0 < s \le t < T$, we have*

$$\int_0^s u^{3-2H}\,du \int_u^t \int_u^s (m^2-u^2)^{H-\frac{1}{2}} m^{-1} (n^2-u^2)^{H-\frac{1}{2}} n^{-1}\,dm\,dn \le C_{H,T} G(t,s),$$

$$\int_0^s u^{3-2H} (t^2-u^2)^{H-\frac{1}{2}} t^{-1} (s^2-u^2)^{H-\frac{1}{2}} s^{-1}\,du \le C_{H,T} G(t,s),$$

$$\int_0^s u^{3-2H} (t^2-u^2)^{H-\frac{1}{2}} t^{-1}\,du \int_u^s (m^2-u^2)^{H-\frac{1}{2}} m^{-1}\,dm \le C_{H,T} G(t,s),$$

$$\int_0^s u^{3-2H}(s^2-u^2)^{H-\frac{1}{2}}s^{-1}du\int_u^t(m^2-u^2)^{H-\frac{1}{2}}m^{-1}dm \le C_{H,T}G(t,s),$$

$$\int_0^s u^{3-2H}du\int_u^t\int_u^s(m^2-u^2)^{H-\frac{1}{2}}m^{-2}(n^2-u^2)^{H-\frac{1}{2}}n^{-2}dmdn \le C_{H,T}G(t,s).$$

Proof. We only prove the first and the third estimate, the other estimates can be proved similarily. On the one hand

$$\begin{aligned}
&\int_0^s u^{3-2H}\mathrm{d}u\int_u^t\int_u^s(m^2-u^2)^{H-\frac{1}{2}}m^{-1}(n^2-u^2)^{H-\frac{1}{2}}n^{-1}\mathrm{d}m\mathrm{d}n\\
&\le C_{H,T}\int_0^s \mathrm{d}u\int_u^t\int_u^s(m-u)^{H-\frac{1}{2}}(n-u)^{H-\frac{1}{2}}\mathrm{d}n\mathrm{d}m\\
&\le C_{H,T}\int_0^s[(t+u)^{2H+1}+(t-u)^{2H+1}]\mathrm{d}u\\
&\le C_{H,T}[(t+s)^{2H}-(t-s)^{2H}]\le C_{H,T}G(t,s).
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\int_0^s u^{3-2H}(t^2-u^2)^{H-\frac{1}{2}}t^{-1}\mathrm{d}u\int_u^s(m^2-u^2)^{H-\frac{1}{2}}m^{-1}\mathrm{d}m\\
&\le \int_0^s(t-u)^{H-\frac{1}{2}}\mathrm{d}u\int_u^s(m-u)^{H-\frac{1}{2}}\mathrm{d}m\\
&\le C_{H,T}\int_0^s(t-u)^{2H-1}du\le C_{H,T}G(t,s).
\end{aligned}$$

This completes the proof. □

3.5. Proposition. *Let $0<H<1$. Then for all $0<s<t<T$, we have*

$$c_{H,T}v^2G(t,s)\le E[X_t^HX_s^H]\le C_{H,T}v^2G(t,s). \tag{3.7}$$

Proof. For $0<H<1/2$, the left inequality in (3.7) follows from Lemma 3.3. Next, we prove the right estimate in (3.7) holds.

$$\begin{aligned}
E[X_t^HX_s^H]&=v^2\int_0^s F(t,u)F(s,u)\mathrm{d}u\\
&\le v^2\int_0^s u^{3-2H}\mathrm{d}u\int_u^t\int_u^s(m^2-u^2)^{H-\frac{1}{2}}m^{-1}(n^2-u^2)^{H-\frac{1}{2}}n^{-1}\mathrm{d}m\mathrm{d}n\\
&\quad+v^2\int_0^s u^{3-2H}(t^2-u^2)^{H-\frac{1}{2}}t^{-1}(s^2-u^2)^{H-\frac{1}{2}}s^{-1}\mathrm{d}u\\
&\quad+v^2\int_0^s u^{3-2H}(t^2-u^2)^{H-\frac{1}{2}}t^{-1}\mathrm{d}u\int_u^s(m^2-u^2)^{H-\frac{1}{2}}m^{-1}dm\\
&\quad+v^2\int_0^s u^{3-2H}(s^2-u^2)^{H-\frac{1}{2}}s^{-1}\mathrm{d}u\int_u^t(m^2-u^2)^{H-\frac{1}{2}}m^{-1}\mathrm{d}m\\
&\quad+v^2\int_0^s u^{3-2H}\mathrm{d}u\int_u^t\int_u^s(m^2-u^2)^{H-\frac{1}{2}}m^{-2}(n^2-u^2)^{H-\frac{1}{2}}n^{-2}\mathrm{d}m\mathrm{d}n.
\end{aligned}$$

Thus, Lemma 3.4 yields the right estimate in (3.7).
For $1/2<H<1$, by an elementary calculus we have

$$\frac{1}{2}e^{-t-s}v^2G(t,s)\le EX_t^HX_s^H\le\frac{1}{2}v^2G(t,s).$$

This completes the proof. □

3.6. Lemma. *Let $0 < H < 1$, then*

$$\int_s^t F^2(t,u)du \le C_{H,T}(t-s)^{2H}, \quad 0 \le s \le t.$$

Proof. Let $\frac{1}{2} < H < 1$, then

$$\begin{aligned}\int_s^t F^2(t,u)\mathrm{d}u &= C_{H,T}e^{-2t}\int_s^t u^{3-2H}\mathrm{d}u\int_u^t\int_u^t (m^2-u^2)^{H-\frac{3}{2}}(n^2-u^2)^{H-\frac{3}{2}}e^{m+n}\mathrm{d}m\mathrm{d}n\\ &\le C_{H,T}\int_s^t\int_s^t\int_s^{m\wedge n} u^{3-2H}(m^2-u^2)^{H-\frac{3}{2}}(n^2-u^2)^{H-\frac{3}{2}}\mathrm{d}u\mathrm{d}m\mathrm{d}n\\ &\le C_{H,T}\int_s^t\int_s^t\int_0^{m\wedge n} u^{3-2H}(m^2-u^2)^{H-\frac{3}{2}}(n^2-u^2)^{H-\frac{3}{2}}\mathrm{d}u\mathrm{d}m\mathrm{d}n\\ &\le C_{H,T}(t-s)^{2H}.\end{aligned}$$

Let $0 < H < \frac{1}{2}$, we have

$$\begin{aligned}|F(t,u)| &\le C_{H,T}u^{\frac{3}{2}-H}\left(\int_u^t (m^2-u^2)^{H-\frac{1}{2}}m^{-1}\mathrm{d}m + (t^2-u^2)^{H-\frac{1}{2}}t^{-1}\right.\\ &\quad\left.+\int_u^t (m^2-u^2)^{H-\frac{1}{2}}m^{-2}\mathrm{d}m\right) := C_{H,T}u^{\frac{3}{2}-H}(I+II+III).\end{aligned}$$

Since,

$$\begin{aligned}\int_s^t u^{3-2H}I^2\mathrm{d}u &= \int_s^t u^{3-2H}\int_u^t\int_u^t (m^2-u^2)^{H-\frac{1}{2}}m^{-1}(n^2-u^2)^{H-\frac{1}{2}}n^{-1}\mathrm{d}m\mathrm{d}n\mathrm{d}u\\ &\le \int_s^t \mathrm{d}u\left[\int_u^t (m-u)^{H-\frac{1}{2}}\mathrm{d}m\right]^2 = C_H(t-s)^{2H+2}.\end{aligned}$$

$$\int_s^t u^{3-2H}(t^2-u^2)^{2H-1}t^{-2}\mathrm{d}u \le \int_s^t (t-u)^{2H-1}\mathrm{d}u = C_H(t-s)^{2H}.$$

and

$$\begin{aligned}\int_s^t u^{3-2H}III^2\mathrm{d}u &= \int_s^t u^{3-2H}\int_u^t\int_u^t (m^2-u^2)^{H-\frac{1}{2}}m^{-2}(n^2-u^2)^{H-\frac{1}{2}}n^{-2}\mathrm{d}m\mathrm{d}n\mathrm{d}u\\ &\le \int_s^t u^{3-2H}\int_u^t\int_u^t (m-u)^{H-\frac{3}{2}}(n-u)^{H-\frac{3}{2}}u^{2H-3}\mathrm{d}m\mathrm{d}n\mathrm{d}u\\ &= \int_s^t\left[\int_u^t (m-u)^{H-\frac{3}{2}}\mathrm{d}m\right]^2\mathrm{d}u = C_H\int_s^t (t-u)^{2H-1}\mathrm{d}u = C_H(t-s)^{2H}.\end{aligned}$$

Hence, $\int_s^t F^2(t,u)du \le C_{H,T}(t-s)^{2H}$. This completes the proof. ☐

3.7. Theorem. *For all $0 \le s < t < T$. Let*

(3.8) $\quad \sigma_{t,s}^2 = E[(X_t^H - X_s^H)^2].$

Then,

(3.9) $\quad v^2 c_{H,T}(t-s)^{2H} \le \sigma_{t,s}^2 \le v^2 C_{H,T}(t-s)^{2H}.$

Proof. By Theorem 3.2, we have

$$\begin{aligned}\sigma_{t,s}^2 &= v^2 \int_0^t [F(t,u) - F(s,u)1_{[0,s]}(u)]^2 du \\ &= v^2 \int_0^s [F(t,u) - F(s,u)]^2 du + v^2 \int_s^t F^2(t,u) du \\ &\geq v^2 \int_s^t F^2(t,u) du \geq c_{H,T} v^2 (t-s)^{2H}.\end{aligned}$$

The right inequality follows from Lemma 3.1 and Lemma 3.6. This completes the proof. □

The following result show the subfractional Ornstein-Uhlenbeck process is not of long range dependence.

3.8. Proposition. *Let* $0 < H < 1$*, and let*

$$\rho_H(n) = E[X_1^H (X_{n+1}^H - X_n^H)],$$

for every positive integer n*. Then* $\sum_{n=1}^{\infty} |\rho_H(n)| < \infty$.

Proof. Let first consider $\frac{1}{2} < H < 1$. Clearly, we have

$$e^{-1} \int_u^{n+1} (m^2 - u^2)^{H-\frac{3}{2}} e^m dm - \int_u^n (m^2 - u^2)^{H-\frac{3}{2}} e^m dm \sim e^n n^{2H-3}, \qquad n \to \infty.$$

It follows that

$$|\rho_H(n)| = v^2 \left| \int_0^{n+1} F(1,u)[F(n+1,u) - F(n,u)] dB_u \right| \sim n^{2H-3}.$$

Thus,

$$\sum_{n=1}^{\infty} |\rho_H(n)| < \infty.$$

On the other hand, if $0 < H < 1/2$, we have

$$e^{-1} \int_u^{n+1} (m^2 - u^2)^{H-\frac{1}{2}} m^{-1} e^m dm - \int_u^n (m^2 - u^2)^{H-\frac{1}{2}} m^{-1} e^m dm \sim e^n n^{2H-2}, \qquad n \to \infty,$$

$$[(n+1)^2 - u^2]^{H-\frac{1}{2}} (n+1)^{-1} - (n^2 - u^2)^{H-\frac{1}{2}} n^{-1} \sim n^{2H-2}, \qquad n \to \infty,$$

$$e^{-1} \int_u^{n+1} (m^2 - u^2)^{H-\frac{1}{2}} m^{-2} e^m dm - \int_u^n (m^2 - u^2)^{H-\frac{1}{2}} m^{-2} e^m dm \sim e^n n^{2H-2}, \qquad n \to \infty.$$

So,

$$\begin{aligned}|\rho_H(n)| &= E[X_1^H (X_{n+1}^H - X_n^H)] \\ &= v^2 \left| \int_0^{n+1} F(1,u)[F(n+1,u) - F(n,u)] dB_u \right| \leq C_H v^2 n^{2H-2}.\end{aligned}$$

which leads to $\sum_{n=1}^{\infty} |\rho_H(n)| < \infty$. □

4. Existence of the intersection local time

The aim of this section is to prove the existence of the intersection local time of two independent subfractional Ornstein-Uhlenbeck process $X^H=\{X_t^H=(X_t^{H,1},\cdots,X_t^{H,d}),0\leq t\leq T\}$ and $\widetilde{X}^H=\{\widetilde{X}_t^H=(\widetilde{X}_t^{H,1},\cdots,\widetilde{X}_t^{H,d}),0\leq t\leq T\}$ on $\mathbb{R}^d, d\geq 2$ with the same index $H\in(0,1)$. The intersection local time is formally defined as : for every $T>0$

$$\ell_T=\int_0^T\int_0^T\delta(X_t^H-\widetilde{X}_s^H)\mathrm{d}s\mathrm{d}t, \tag{4.1}$$

where δ denotes the Dirac delta function. As we pointed out, this definition is only formal. In order to give a rigorous meaning to ℓ_T, we approximate the Dirac delta function by the heat kernel

$$p_\varepsilon(x)=(2\pi\varepsilon)^{-\frac{d}{2}}e^{-\frac{|x|^2}{2\varepsilon}},x\in\mathbb{R}^d.$$

Then, we consider the following family of random variables indexed by $\varepsilon>0$

$$\ell_{\varepsilon,T}=\int_0^T\int_0^T p_\varepsilon(X_t^H-\widetilde{X}_s^H)\mathrm{d}s\mathrm{d}t. \tag{4.2}$$

Using the following classical equality

$$p_\varepsilon(x)=\frac{1}{(2\pi\varepsilon)^{d/2}}e^{-\frac{|x|^2}{2\varepsilon}}=\frac{1}{(2\pi)^d}\int_{R^d}e^{i\langle\xi,x\rangle}e^{-\frac{|\xi|^2}{2}\varepsilon}\mathrm{d}\xi,$$

we have

$$\ell_{\varepsilon,T}=\int_0^T\int_0^T p_\varepsilon(X_t^H-\widetilde{X}_s^H)\mathrm{d}s\mathrm{d}t=\frac{1}{(2\pi)^d}\int_0^T\int_0^T\int_{R^d}e^{i\langle\xi,X_t^H-\widetilde{X}_s^H\rangle}e^{-\frac{|\xi|^2}{2}\varepsilon}\mathrm{d}\xi\mathrm{d}s\mathrm{d}t.$$

Let $\bar{\sigma}_{t,s}^2:=E(X_t^{H,i}-\tilde{X}_s^{H,i})^2,\sigma_t^2:=E(X_t^{H,i})^2,i=1,2.$ We have

$$\begin{aligned}E(\ell_{\varepsilon,T})&=\frac{1}{(2\pi)^d}\int_0^T\int_0^T\int_{R^d}E(e^{i\langle\xi,X_t^H-\widetilde{X}_s^H\rangle})e^{-\frac{|\xi|^2}{2}\varepsilon}\mathrm{d}\xi\mathrm{d}s\mathrm{d}t\\&=\frac{1}{(2\pi)^d}\int_0^T\int_0^T\int_{R^d}e^{-\frac{1}{2}(\varepsilon+\bar{\sigma}_{t,s}^2)|\xi|^2}\mathrm{d}\xi\mathrm{d}s\mathrm{d}t\\&=\frac{1}{(2\pi)^{d/2}}\int_0^T\int_0^T(\varepsilon+\bar{\sigma}_{t,s}^2)^{-\frac{d}{2}}\mathrm{d}s\mathrm{d}t,\end{aligned}$$

where we have used the fact that

$$\int_{R^d}e^{-\frac{1}{2}(\varepsilon+\bar{\sigma}_{t,s}^2)|\xi|^2}\mathrm{d}\xi=\left(\frac{2\pi}{\varepsilon+\bar{\sigma}_{t,s}^2}\right)^{d/2}.$$

We also have

$$E(\ell_{\varepsilon,T}^2)=\frac{1}{(2\pi)^{2d}}\int_{R^{2d}}E[e^{i\langle\xi,X_t^H-\tilde{X}_s^H\rangle+i\langle\eta,X_u^H-\tilde{X}_v^H\rangle}]\times e^{-\frac{\varepsilon(|\xi|^2+|\eta|^2)}{2}}\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}s\mathrm{d}t\mathrm{d}u\mathrm{d}v. \tag{4.3}$$

Let we introduce some notations that will be used throughout this paper

$$\lambda=Var(X_t^{H,1}-\tilde{X}_s^{H,2}),\quad \rho=Var(X_{t'}^{H,1}-\tilde{X}_{s'}^{H,2}),$$

and

$$\mu=Cov(X_t^{H,1}-\tilde{X}_s^{H,2},X_{t'}^{H,1}-\tilde{X}_{s'}^{H,2}).$$

Using the above notation, we can rewrite (4.3) as followings:

$$\begin{aligned} E(\ell^2_{\varepsilon,T}) &= \frac{1}{(2\pi)^{2d}} \int_{[0,T]^4} \int_{R^{2d}} exp\{-\frac{1}{2}[(\lambda+\varepsilon)|\xi|^2 + (\rho+\varepsilon)|\eta|^2 \\ &\quad + 2\mu\langle \xi, \eta\rangle]\} \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}s \mathrm{d}t \mathrm{d}s' \mathrm{d}t' \\ &= \frac{1}{(2\pi)^{d}} \int_{[0,T]^4} [(\lambda+\varepsilon)(\rho+\varepsilon) - \mu^2]^{-\frac{d}{2}} \mathrm{d}s \mathrm{d}t \mathrm{d}s' \mathrm{d}t'. \end{aligned} \tag{4.4}$$

Using the local nondeterminism of subfractional Ornstein-Uhlenbeck process, we have the follows Lemmas (see also Hu [13]).

4.1. Lemma. (1) *For $0 < s < s' < t < t' < T$, we have*

$$\lambda\rho - \mu^2 \geq kv^2[t^{2H} + s^{2H}][(t'-t)^{2H} + (s'-s)^{2H}],$$

(2) *For $0 < s' < s < t < t' < T$, we have*

$$\lambda\rho - \mu^2 \geq kv^2[(t^{2H} + s^{2H})(t'-t)^{2H} + (t'^{2H} + s'^{2H})(s-s')^{2H}],$$

(3) *For $0 < s < t < s' < t' < T$, we have*

$$\lambda\rho - \mu^2 \geq kv^2[(t^{2H} + s^{2H})(t'-t)^{2H} + (t'^{2H} + s'^{2H})(s-s')^{2H}].$$

where $k > 0$ is an enough small constant.

4.2. Lemma. *Let*

$$A_T := \int_{[0,T]^4} (\lambda\rho - \mu^2)^{-\frac{d}{2}} dsdtds'dt'.$$

Then $A_T < \infty$ if and only if $Hd < 2$.

Proof. First, we give the proof of sufficient condition. Let $Hd < 2$. We have

$$A_T = 2\left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right)(\lambda\rho - \mu^2)^{-\frac{d}{2}} \mathrm{d}s \mathrm{d}t \mathrm{d}s' \mathrm{d}t',$$

where

$$\begin{aligned} I_1 &= \{(s,t,s',t') : 0 < s < s' < t < t' < T\}, \\ I_2 &= \{(s,t,s',t') : 0 < s' < s < t < t' < T\}, \\ I_3 &= \{(s,t,s',t') : 0 < s < t < s' < t' < T\}. \end{aligned}$$

For $(s,t,s',t') \in I_1$, we have

$$\begin{aligned} &\int_{I_1} (\lambda\rho - \mu^2)^{-\frac{d}{2}} \mathrm{d}s \mathrm{d}t \mathrm{d}s' \mathrm{d}t' \\ &\leq C_{H,T} kv^2 \int_0^T \int_s^T \int_{s'}^T \int_t^T t^{-\frac{Hd}{2}} s^{-\frac{Hd}{2}} (t'-t)^{-\frac{Hd}{2}} (s'-s)^{-\frac{Hd}{2}} \mathrm{d}t' \mathrm{d}t \mathrm{d}s' \mathrm{d}s \\ &\leq C_{H,T} kv^2 \int_0^T \int_s^T \int_{s'}^T t^{-\frac{Hd}{2}} s^{-\frac{Hd}{2}} (s'-s)^{-\frac{Hd}{2}} \mathrm{d}t \mathrm{d}s' \mathrm{d}s \\ &\leq C_{H,T} kv^2 \int_0^T \int_0^{s'} s^{-\frac{Hd}{2}} (s'-s)^{-\frac{Hd}{2}} \mathrm{d}s \mathrm{d}s' \leq C_{H,T} kv^2 \int_0^T s^{1-Hd} \mathrm{d}s < \infty. \end{aligned}$$

By a similar way, we can prove that

$$\int_{I_2} (\lambda\rho - \mu^2)^{-\frac{d}{2}} \mathrm{d}s \mathrm{d}t \mathrm{d}s' \mathrm{d}t' < \infty, \quad \int_{I_3} (\lambda\rho - \mu^2)^{-\frac{d}{2}} \mathrm{d}s \mathrm{d}t \mathrm{d}s' \mathrm{d}t' < \infty.$$

Now, we turn to the proof of the necessary condition. By Proposition 3.5 one can get

$$\lambda \leq C_{H,T} v^2 (t^{2H} + s^{2H}), \quad \rho \leq C_{H,T} v^2 (t'^{2H} + s'^{2H}),$$

and

$$\mu^2 \geq c_{H,T}v^2[t^{2H} + s^{2H} + t'^{2H} + s'^{2H} - \frac{1}{2}((t+s)^{2H} + (t'+s')^{2H} + |t-s|^{2H} + |t'-s'|^{2H})].$$

So,

$$\lambda\rho - \mu^2 \leq C_{H,T}v^2\{(t^{2H}+s^{2H})(t'^{2H}+s'^{2H}) - [t^{2H} + s^{2H} + t'^{2H} + s'^{2H} - \frac{1}{2}((t+s)^{2H} + (t'+s')^{2H} + |t-s|^{2H} + |t'-s'|^{2H})]\}.$$

Hence, making a change to spherical coordinates, as the integrand is always positive, we have

$$\begin{aligned} A_T &= \int_{[0,T]^4} (\lambda\rho - \mu^2)^{-\frac{d}{2}} \mathrm{d}s\mathrm{d}t\mathrm{d}s'\mathrm{d}t' \geq \int_{D_T} (\lambda\rho - \mu^2)^{-\frac{d}{2}} \mathrm{d}s\mathrm{d}t\mathrm{d}s'\mathrm{d}t' \\ &\geq \int_0^T r^{3-2Hd} \int_\Theta \phi(\theta) d\theta, \end{aligned}$$

where $D_T := \{(s+t+s'+t') \in R_+^4 : s^2 + t^2 + s'^2 + t'^2 \leq \varepsilon^2\}$. Note that the angular integral is different from zero thanks to the positivity of the integrand. It follows that if $A_T < \infty$, then $Hd < 2$. Thus completes the proof. □

From Lemma 4.2, we get the following Theorem.

4.3. Theorem. *Let $H \in (0,1)$. Then $\ell_{\varepsilon,T}$ converges in $L^2(\Omega)$ as $\varepsilon \to 0$ if and only if $Hd < 2$. Morever, if the limits denoted by ℓ_T, then $\ell_T \in L^2(\Omega)$.*

Proof. A slight extension of (4.4) yields

$$E(\ell_{\varepsilon,T}\ell_{\eta,T}) = \frac{1}{(2\pi)^d} \int_{[0,T]^4} [(\lambda+\varepsilon)(\rho+\varepsilon) - \mu^2]^{-\frac{d}{2}} \mathrm{d}s\mathrm{d}t\mathrm{d}s'\mathrm{d}t'.$$

Consequently, a necessary and sufficient condition for the convergence in $L^2(\Omega)$ of $\ell_{\varepsilon,T}$ is that

$$A_T := \int_{[0,T]^4} (\lambda\rho - \mu^2)^{-\frac{d}{2}} \mathrm{d}s\mathrm{d}t\mathrm{d}s'\mathrm{d}t' < \infty.$$

Thus, it is sufficient to prove that $A_T < \infty$ if and only if $Hd < 2$. By Lemma 4.2, this complete the proof. □

Conclusions. In this paper, we discuss and analyze the subfractional Ornstein-Uhlenbeck process and show that this process is local nondeterministic. At the same time, we establish several estimates for the increments of the process, and give the sufficient and necessary conditions for the existence of the intersection local time of two independent subfractional Ornstein-Uhlenbeck process. In a sequel of this paper we will study the Ornstein-Uhlenbeck process driven by general Gaussian process.

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